Faraday effect: A field theoretical point of view

Avijit K. Ganguly*
Saha Institute of Nuclear Physics, 1/AF, Bidhan-Nagar, Calcutta 700064, India

Sushan Konar†
Inter-University Centre for Astronomy and Astrophysics, Pune 411007, India

Palash B. Pal‡
Saha Institute of Nuclear Physics, 1/AF, Bidhan-Nagar, Calcutta 700064, India

(Received 10 May 1999; published 20 October 1999)

We analyze the structure of the vacuum polarization tensor in the presence of a background electromagnetic field in a medium. We use various discrete symmetries and crossing symmetry to constrain the form factors obtained for the most general case. From these symmetry arguments, we show why the vacuum polarization tensor has to be even in the background field when there is no background medium. Taking then the background field to be purely magnetic, we evaluate the vacuum polarization to linear order in it. The result shows the phenomenon of Faraday rotation, i.e., the rotation of the plane of polarization of a plane polarized light passing through this background. We find that the usual expression for Faraday rotation, which is derived for a non-degenerate plasma in the non-relativistic approximation, undergoes substantial modification if the background is degenerate and/or relativistic. We give explicit expressions for Faraday rotation in completely degenerate and ultra-relativistic media. [S0556-2821(99)07618-3]

PACS number(s): 12.20.−m

I. SCOPE AND OUTLINE OF THE PAPER

It is well known that electromagnetic wave propagating through a medium in an ambient magnetic field suffers Faraday rotation; i.e., the plane of a plane polarized light rotates as it travels through the medium in the magnetic field. The amount of this rotation is derived in various texts on electromagnetic theory [1] and plasma physics [2] with the assumptions that the medium consists of non-relativistic and non-degenerate electrons and nucleons.

Faraday rotation is extensively used in a variety of situations, including astrophysical and cosmological ones [2,3]. In such situations, either of the aforesaid assumptions about the medium may not be valid. For example, for compact stars, the plasma is likely to be degenerate. In the very early universe, when the temperature was high, the assumption of non-relativistic plasma is bound to break down. Motivated by such situations, we reinvestigate this problem. For a general framework, the formalism of quantum field theory proves to be helpful. The aim of this paper is to use a quantum field theoretical formalism to calculate Faraday rotation in different kinds of media.

The paper is organized as follows. In Sec. II, we introduce the vacuum polarization tensor and find its most general form in a background medium and in the presence of a general electromagnetic field, consistent with Lorentz and gauge invariances. Calculation of the vacuum polarization tensor requires the electron propagator, which is discussed in Sec. III. We summarize there how Schwinger's proper-time propagator in a constant magnetic field is modified in the presence of a background medium. In Sec. IV, we show how some discrete symmetries help us constraining some of the form factors appearing in the vacuum polarization. Following this, we set up the calculation in Sec. V. Starting from the basic Feynman rules, we arrive at an expression for the polarization tensor that is explicitly gauge invariant. In Sec. VI, we derive the expression for Faraday rotation per unit length in terms of the components of the polarization tensor. Then in Sec. VII, we provide explicit results for different kinds of backgrounds. This is the section which contains the essential results of the paper. The non-relativistic and non-degenerate case is shown in Sec. VII C, where we obtain the result usually quoted in textbooks. In Secs. VII D and VII E, we find results for a completely degenerate medium and an ultra-relativistic one. Finally, we present our conclusions.

*Email address: avijit@tnp.saha.ernet.in
†Email address: sushan@iucaa.ernet.in
‡Email address: pbpal@tnp.saha.ernet.in

II. FORM FACTORS IN THE POLARIZATION TENSOR

The classical action of a free electromagnetic field is given by

$$\mathcal{A} = -\frac{1}{4}\int d^4x F_{\lambda\rho}(x) F^{\lambda\rho}(x). \tag{2.1}$$

In the momentum space, this can be written as

$$\mathcal{A} = \int \frac{d^4k}{(2\pi)^4}\mathcal{L}, \tag{2.2}$$

where $\mathcal{L}$ is the momentum-space Lagrangian, which can be obtained by taking Fourier transforms in Eq. (2.1):

$$\mathcal{L}=-\frac{1}{2}k^2\tilde{g}_{\lambda\rho}A^\lambda(k)A^\rho(-k), \tag{2.3}$$

where

$$\tilde{g}_{\lambda\rho}=g_{\lambda\rho}-\frac{k_\lambda k_\rho}{k^2}. \tag{2.4}$$

Once quantum corrections are added, one obtains more quadratic terms in the Lagrangian. These are represented by the vacuum polarization tensor $\Pi_{\lambda\rho}$. In other words, after the quantum corrections are put in, the quadratic part of the Lagrangian becomes

$$\mathcal{L}=\frac{1}{2}[-k^2\tilde{g}_{\lambda\rho}+\Pi_{\lambda\rho}(k)]A^\lambda(k)A^\rho(-k). \tag{2.5}$$

Owing to gauge invariance, $\Pi_{\lambda\rho}$ satisfies the conditions

$$k^\lambda\Pi_{\lambda\rho}(k)=0, \quad k^\rho\Pi_{\lambda\rho}(k)=0. \tag{2.6}$$

In addition, Bose symmetry implies

$$\Pi_{\lambda\rho}(k)=\Pi_{\rho\lambda}(-k). \tag{2.7}$$

In the vacuum, the tensor $\Pi_{\lambda\rho}$ depends only on the momentum vector k. Thus, the most general form for $\Pi_{\lambda\rho}$ is given by

$$\Pi_{\lambda\rho}=\Pi_0[k^2g_{\lambda\rho}-k_\lambda k_\rho], \tag{2.8}$$

where Π_0 is a Lorentz-invariant form factor, which can therefore depend only on k^2. The important point is that the tensor structure for $\Pi_{\lambda\rho}$ is exactly the same as the tensor appearing in the classical Lagrangian. Thus, this correction term can be done away with by a redefinition of the photon field A^λ.

In a nontrivial background, this is no more the case. Although $\Pi_{\lambda\rho}$ still has to satisfy Eq. (2.6), the form given in Eq. (2.8) does not follow. This is because $\Pi_{\lambda\rho}$ can now depend, apart from the momentum vector k^λ, on various vectors or tensors which characterize the background medium. Even for a homogeneous and isotropic medium, there is an extra vector in the form of the velocity of its center of mass, u^λ. The most general form for $\Pi_{\lambda\rho}$ in the presence of these two vectors has been discussed in the literature [4].

Our interest lies in a more complicated background where in addition to the medium there is also an external electromagnetic field $B_{\lambda\rho}$. We will work in the weak field limit throughout. This means that the background field will be considered feeble, and we will keep only linear terms in it. For the moment, we will not specialize to magnetic fields. We will keep the discussion general, with a medium characterized by the vector u^λ and a background electromagnetic field $B_{\lambda\rho}$ in arbitrary direction.

There are many independent tensors constructed out of k^λ, u^λ and $B_{\lambda\rho}$ which satisfy Eq. (2.6). For future convenience, we categorize them into several groups. In the first group, there is only one tensor which depends only on the vector k^λ, viz.,

$$P^{(0)}_{\lambda\rho}\equiv\tilde{g}_{\lambda\rho}=g_{\lambda\rho}-\frac{k_\lambda k_\rho}{k^2}. \tag{2.9}$$

This, of course, is the same tensor which appears in Eq. (2.8).

In the second group, we include tensors constructed only of k and u. These are [4]

$$P^{(1)}_{\lambda\rho}=\tilde{u}_\lambda\tilde{u}_\rho/\tilde{u}^2, \tag{2.10}$$

$$P^{(2)}_{\lambda\rho}=\varepsilon_{\lambda\rho\sigma\tau}k^\sigma u^\tau, \tag{2.11}$$

where

$$\tilde{u}_\lambda=\tilde{g}_{\lambda\sigma}u^\sigma. \tag{2.12}$$

Next, we bring in tensors constructed from k and B only, without any occurrence of u. Using the shorthand

$$(k\cdot B)_\lambda=k^\sigma B_{\sigma\lambda}, \tag{2.13}$$

we can write these as

$$P'^{(1)}_{\lambda\rho}=k^2B_{\lambda\rho}-k_\lambda(k\cdot B)_\rho+k_\rho(k\cdot B)_\lambda, \tag{2.14}$$

$$P'^{(2)}_{\lambda\rho}=\varepsilon_{\lambda\rho\sigma\tau}k^\sigma(k\cdot B)^\tau. \tag{2.15}$$

One might think that there might be additional terms obtained by replacing $B_{\lambda\rho}$ by $\tilde{B}_{\lambda\rho}$, where

$$\tilde{B}_{\lambda\rho}=\frac{1}{2}\varepsilon_{\lambda\rho\sigma\tau}B^{\sigma\tau}. \tag{2.16}$$

But it is straightforward to show that no other independent term arises this way.

Finally, to write down the tensors where all three of k, u and B occur, we employ a notation $(u\cdot B)_\lambda$ defined in a way similar to $(k\cdot B)_\lambda$. Then the tensors are

$$P''^{(1)}_{\lambda\rho}=k\cdot uB_{\lambda\rho}-u_\lambda(k\cdot B)_\rho+u_\rho(k\cdot B)_\lambda \tag{2.17}$$

$$P''^{(2)}_{\lambda\rho}=\varepsilon_{\lambda\rho\sigma\tau}k^\sigma(u\cdot B)^\tau \tag{2.18}$$

$$P''^{(3)}_{\lambda\rho}=\tilde{u}_\lambda(k\cdot B)_\rho-\tilde{u}_\rho(k\cdot B)_\lambda \tag{2.19}$$

$$P''^{(4)}_{\lambda\rho}=\tilde{u}_\lambda(k\cdot B)_\rho+\tilde{u}_\rho(k\cdot B)_\lambda \tag{2.20}$$

$$P''^{(5)}_{\lambda\rho}=\tilde{u}_\lambda\tilde{g}_{\rho\tau}(u\cdot B)^\tau-\tilde{u}_\rho\tilde{g}_{\lambda\tau}(u\cdot B)^\tau \tag{2.21}$$

$$P''^{(6)}_{\lambda\rho}=\tilde{u}_\lambda\tilde{g}_{\rho\tau}(u\cdot B)^\tau+\tilde{u}_\rho\tilde{g}_{\lambda\tau}(u\cdot B)^\tau \tag{2.22}$$

$$P''^{(7)}_{\lambda\rho}=\tilde{u}_\lambda(k\cdot\tilde{B})_\rho-\tilde{u}_\rho(k\cdot\tilde{B})_\lambda \tag{2.23}$$

$$P''^{(8)}_{\lambda\rho}=\tilde{u}_\lambda(k\cdot\tilde{B})_\rho+\tilde{u}_\rho(k\cdot\tilde{B})_\lambda \tag{2.24}$$

$$P''^{(9)}_{\lambda\rho}=\tilde{u}_\lambda\tilde{g}_{\rho\tau}(u\cdot\tilde{B})^\tau-\tilde{u}_\rho\tilde{g}_{\lambda\tau}(u\cdot\tilde{B})^\tau \tag{2.25}$$

$$P''^{(10)}_{\lambda\rho}=\tilde{u}_\lambda\tilde{g}_{\rho\tau}(u\cdot\tilde{B})^\tau+\tilde{u}_\rho\tilde{g}_{\lambda\tau}(u\cdot\tilde{B})^\tau. \tag{2.26}$$

A collection of gauge invariant tensors which can appear in the vacuum polarization were listed by Pérez Rojas and Shabad [5]. They have tensors involving more than one power of the background field, which we are not interested in. As for the tensors linear in $B_{\lambda\rho}$, they list what we call $P'^{(1)}_{\lambda\rho}$, $P''^{(1)}_{\lambda\rho}$ and $P''^{(8)}_{\lambda\rho}$, but none of the rest. We conclude that the most general form for $\Pi_{\lambda\rho}$ consistent with Eq. (2.6) and keeping only terms linear in the external electromagnetic field is given by:

$$\Pi_{\lambda\rho}=\sum_i \Pi^{(i)}P^{(i)}_{\lambda\rho}+\sum_i \Pi'^{(i)}P'^{(i)}_{\lambda\rho}+\sum_i \Pi''^{(i)}P''^{(i)}_{\lambda\rho}, \tag{2.27}$$

where in each case, the sum over i runs over the appropriate set of values. The coefficients of the tensors are form factors, which we discuss now.

First, notice that the tensors $P^{(0)}_{\lambda\rho}$ and $P^{(1)}_{\lambda\rho}$ do not depend on the background electromagnetic field. The form factors associated with these terms are related to the dielectric constant and the magnetic permeability of the medium [6]. The tensor $P^{(2)}_{\lambda\rho}$ also does not involve the background electromagnetic field. It is, however, parity asymmetric, and accounts for natural optical activity [4]. Since our aim is to discuss Faraday rotation which is also a type of optical activity, we will disregard any natural optical activity. Thus, we assume that the form factor associated with $P^{(2)}_{\lambda\rho}$ is zero for our medium.

The form factors are Lorentz invariant quantities. Thus, they can depend only on the Lorentz invariant combinations of k^λ, u^λ and $B_{\lambda\rho}$. Since $u^2=1$, we can obtain the following invariant parameters, keeping at most one factor of the background field:

$$\omega\equiv k\cdot u, \tag{2.28}$$

$$K\equiv\sqrt{\omega^2-k^2}, \tag{2.29}$$

$$b\equiv k^\lambda u^\rho B_{\lambda\rho}, \tag{2.30}$$

$$\tilde{b}\equiv k^\lambda u^\rho \tilde{B}_{\lambda\rho}. \tag{2.31}$$

In addition, of course, the form factors can depend on the Lorentz scalars which define the background medium, e.g., the chemical potential μ and the temperature $1/\beta$.

Since we are interested only in linear terms in the background field, we have discarded higher order invariants in $B_{\mu\nu}$. Moreover, notice that the tensors $P'^{(i)}_{\lambda\rho}$ and $P''^{(i)}_{\lambda\rho}$ are linear in the background field. Thus, for their co-efficients, we can neglect the field dependence for the sake of consistency. Thus, for our purpose, the form factors $\Pi'^{(i)}$ and $\Pi''^{(i)}$ should be treated as functions of ω and K only, and possibly of μ and β. We summarize this statement as

$$\hat{\Pi}^{(i)}=\hat{\Pi}^{(i)}(\omega,K,\mu,\beta), \tag{2.32}$$

where $\hat{\Pi}^{(i)}$ stands for the $\Pi'^{(i)}$'s and the $\Pi''^{(i)}$'s. In Sec. IV, we will see how arguments about various discrete symmetries restrict the form factors in significant ways. The constraint of hermiticity of the action implies

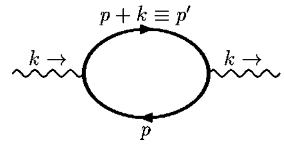

FIG. 1. One-loop diagram for vacuum polarization.

$$\Pi_{\lambda\rho}(k)=\Pi^*_{\rho\lambda}(k), \tag{2.33}$$

whose consequences will be mentioned at the end of Sec. IV.

III. THE ELECTRON PROPAGATOR

At the 1-loop level, the vacuum polarization tensor arises from the diagram in Fig. 1. The dominant contribution to the vacuum polarization comes from the electron line in the loop. To evaluate this diagram, one needs to use the electron propagator within a thermal medium in the presence of a background electromagnetic field. Rather than working with the complicated expression for a general background field, we will specialize to the case of a purely magnetic field. Once this is assumed, the field can be taken in the z-direction without any further loss of generality. We will denote the magnitude of this field by $\mathcal{B}$.

Ignoring at first the presence of the medium, the electron propagator in such a field can be written down following Schwinger's approach [7–9]:

$$iS^V_B(p)=\int_0^\infty ds e^{\Phi(p,s)}G(p,s), \tag{3.1}$$

where Φ and G are defined below. To write these in a compact notation, we decompose the metric tensor into two parts:

$$g_{\alpha\beta}=g^{\parallel}_{\alpha\beta}-g^{\perp}_{\alpha\beta}, \tag{3.2}$$

where

$$g^{\parallel}_{\alpha\beta}=\mathrm{diag}(1,0,0,-1),$$

$$g^{\perp}_{\alpha\beta}=\mathrm{diag}(0,1,1,0). \tag{3.3}$$

This allows us to write, for any two objects q and q' (including the γ-matrices) carrying Lorentz indices,

$$q\cdot q'_{\parallel}=q_0q'_0-q_3q'_3, \tag{3.4}$$

$$q\cdot q'_{\perp}=q_1q'_1+q_2q'_2. \tag{3.5}$$

Using these notations, we can write

$$\Phi(p,s)\equiv is\left(p^2_{\parallel}-\frac{\tan(e\mathcal{B}s)}{e\mathcal{B}s}p^2_{\perp}-m^2\right)-\epsilon|s|, \tag{3.6}$$

$$G(p,s) \equiv \frac{e^{ie\mathcal{B}s\sigma_z}}{\cos(e\mathcal{B}s)}\left(\not{p}_\parallel - \frac{e^{-ie\mathcal{B}s\sigma_z}}{\cos(e\mathcal{B}s)}\not{p}_\perp + m\right)$$
$$= (1 + i\sigma_z \tan e\mathcal{B}s)(\not{p}_\parallel + m) - (\sec^2 e\mathcal{B}s)\not{p}_\perp, \tag{3.7}$$

where

$$\sigma_z = i\gamma_1\gamma_2 = -\gamma_0\gamma_3\gamma_5, \tag{3.8}$$

and we have used

$$e^{ie\mathcal{B}s\sigma_z} = \cos e\mathcal{B}s + i\sigma_z \sin e\mathcal{B}s. \tag{3.9}$$

The expression for Φ can have an additional gauge dependent phase factor, but it does not contribute to the polarization tensor. Usually, one writes s instead of $|s|$ in Eq. (3.6). It is equivalent since in the range of integration indicated in Eq. (3.1) s is never negative. However, the definition of $\Phi(p,s)$ is useful in this form for what follows next.

In the presence of a background medium, the above propagator is modified to [10]

$$iS(p) = iS_B^V(p) - \eta_F(p)[iS_B^V(p) - i\overline{S}_B^V(p)], \tag{3.10}$$

where

$$\overline{S}_B^V(p) \equiv \gamma_0 S_V^\dagger(p)\gamma_0 \tag{3.11}$$

for a fermion propagator, and $\eta_F(p)$ contains the distribution function for particles and antiparticles:

$$\eta_F(p) = \Theta(p\cdot u) f_F(p,\mu,\beta) + \Theta(-p\cdot u) f_F(-p,-\mu,\beta). \tag{3.12}$$

Here, Θ is the step function, which takes the value $+1$ for positive values of its argument and vanishes for negative values of the argument, and f_F denotes the Fermi-Dirac distribution function

$$f_F(p,\mu,\beta) = \frac{1}{e^{\beta(p\cdot u - \mu)} + 1}. \tag{3.13}$$

Putting in the form of $S_B^V(p)$ from Eq. (3.1), we obtain the additional term in the propagator to be

$$S_B^\eta(p) \equiv -i\,\eta_F(p)[S_B^V(p) - \overline{S}_B^V(p)]$$
$$= -\eta_F(p)\int_{-\infty}^{\infty} ds\, e^{\Phi(p,s)} G(p,s), \tag{3.14}$$

with $\Phi(p,s)$ and $G(p,s)$ defined in Eqs. (3.6) and (3.7).

It is straightforward to see that when $\mathcal{B}=0$, the propagator in Eq. (3.1) reduces to

$$iS_0^V(p) = \int_0^\infty ds\, \exp[is(p^2 - m^2 + i\epsilon)](\not{p} + m)$$
$$= i\frac{\not{p} + m}{p^2 - m^2 + i\epsilon}, \tag{3.15}$$

which is the vacuum propagator. In the same limit, the background dependent part reduces to

$$S_0^\eta(p) = -2\pi\delta(p^2 - m^2)\,\eta_F(p)(\not{p} + m). \tag{3.16}$$

IV. DISCRETE SYMMETRIES AND THE FORM FACTORS

Before embarking on diagram calculations, let us discuss some of the symmetries of the problem, which will help us constraining some of the form factors.

A. Bose symmetry

This symmetry was already discussed in Eq. (2.7), viz., under the operation

$$k \to -k, \quad \lambda \leftrightarrow \rho, \tag{4.1}$$

the vacuum polarization tensor must be invariant.

Since the tensors $P'^{(i)}_{\lambda\rho}$ change sign under the operation of Eq. (4.1), this implies that the associated form factors should satisfy the condition

$$\Pi'^{(i)}(\omega,K,\mu,\beta) = -\Pi'^{(i)}(-\omega,K,\mu,\beta). \tag{4.2}$$

On the form factors denoted by Π'', the effect is more complicated since some of the tensors $P''_{\lambda\rho}$ are symmetric under the operation of Eq. (2.7) and some are antisymmetric. In general, let us write

$$P''^{(i)}_{\rho\lambda}(-k) = n_i P''^{(i)}_{\lambda\rho}(k) \tag{4.3}$$

where, by inspection, we see that

$$n_i = \begin{cases} +1 & \text{for } i = 1,2,3,6,7,10, \\ -1 & \text{for } i = 4,5,8,9. \end{cases} \tag{4.4}$$

The associated form factors should then satisfy the relation

$$\Pi''^{(i)}(\omega,K,\mu,\beta) = n_i \Pi''^{(i)}(-\omega,K,\mu,\beta). \tag{4.5}$$

To apply this symmetry on the unprimed form factors, one has to take into account the dependence of these form factors on b and $\tilde{b}$ defined in Eq. (2.31). Thus, this symmetry implies

$$\Pi^{(i)}(\omega,K,\mu,\beta,b,\tilde{b}) = +\Pi^{(i)}(-\omega,K,\mu,\beta,-b,-\tilde{b}) \quad \text{for } i=0,1 \tag{4.6}$$

$$\Pi^{(2)}(\omega,K,\mu,\beta,b,\tilde{b}) = -\Pi^{(2)}(-\omega,K,\mu,\beta,-b,-\tilde{b}). \tag{4.7}$$

B. Charge conjugation symmetry

In calculating the form factors, we are neglecting any corrections coming from weak interactions. In fact, these corrections occur only at the 2-loop level, and therefore are anyway irrelevant for the 1-loop calculation that we will be performing. In this case, the interactions are all purely electromagnetic, and so they obey charge conjugation (or C) symmetry. The conclusion of this symmetry is that $\Pi_{\lambda\rho}$ should be invariant under the substitutions

$$\mu \rightarrow -\mu, \quad B_{\sigma\tau} \rightarrow -B_{\sigma\tau}. \tag{4.8}$$

Said in words, it means that if we calculate the vacuum polarization in a medium with a certain background field, it should be the same as that obtained in a charged conjugated medium with an opposite background field. This means that, in the vacuum polarization, there are terms even in the background field and even in μ, or odd in both. The terms linear in B, which we will calculate, should therefore be odd in μ. This implies that the primed form factors, which are independent of the background medium, i.e., do not contain μ, must vanish. This is known from the direct calculations of the polarization tensor in absence of a medium [8,11].

C. A symmetry of the propagator

Lastly, notice that in the calculation of $\Pi_{\lambda\rho}$, the center-of-mass velocity u^λ and the chemical potential μ can enter only through the function η_F in the propagator. Further, from Eq. (3.12), notice that η_F is invariant under the following transformation:

$$u \rightarrow -u, \quad \mu \rightarrow -\mu. \tag{4.9}$$

So the vacuum polarization must also obey this symmetry.

For the form factors, this fact has an interesting consequence. Some of the tensors $P''^{(i)}_{\lambda\rho}$ are even in u, some are odd. Accordingly, the form factors would satisfy

$$\Pi''^{(i)}(\omega,K,\mu,\beta) = n'_i \Pi''^{(i)}(-\omega,K,-\mu,\beta), \tag{4.10}$$

where

$$n'_i = \begin{cases} -1 & \text{for } i=1,2,3,4,7,8, \\ +1 & \text{for } i=5,6,9,10. \end{cases} \tag{4.11}$$

Using the consequence of C-symmetry, we can conclude that, for terms linear in B,

$$\Pi''^{(i)}(\omega,K,\mu,\beta) = -n'_i \Pi''^{(i)}(-\omega,K,\mu,\beta). \tag{4.12}$$

In order that this is consistent with Eq. (4.5), we must have

$$n_i n'_i = -1 \quad (\text{no sum on } i). \tag{4.13}$$

Using Eqs. (4.4) and (4.11), then, we obtain that the doubly-primed form factors $\Pi''^{(4)}$, $\Pi''^{(6)}$, $\Pi''^{(8)}$ and $\Pi''^{(10)}$ vanish.

The analysis performed in this section is valid for general electromagnetic fields in the weak limit. Although we have used the propagator in the presence of a purely magnetic field, substituting the more general form does not affect the arguments. We can now discuss how this analysis can be simplified if the background field is a purely magnetic field in the rest frame of the background medium, i.e., if $u^\sigma B_{\sigma\tau} = 0$. Among the field-dependent tensors, $P''^{(2)}_{\lambda\rho}$, $P''^{(5)}_{\lambda\rho}$ and $P''^{(6)}_{\lambda\rho}$ vanish in this case. Therefore, in the final count, we will have only four form factors associated with the field-dependent tensors, viz., $\Pi''^{(1)}$, $\Pi''^{(3)}$, $\Pi''^{(7)}$ and $\Pi''^{(9)}$. Since all of these are antisymmetric tensors, the hermiticity condition of Eq. (2.33) implies that the corresponding form factors must be purely imaginary in the dispersive part.

In addition, of course, we can have the form factors $\Pi^{(0)}$ and $\Pi^{(1)}$, whereas $\Pi^{(2)}$ vanishes because of our assumption of vanishing natural optical activity.

V. CALCULATION OF THE 1-LOOP VACUUM POLARIZATION

A. Identifying the relevant terms

The amplitude of the 1-loop diagram of Fig. 1 can be written as

$$i\Pi_{\lambda\rho}(k) = -\int \frac{d^4p}{(2\pi)^4}(ie)^2 \, \mathrm{tr}[\gamma_\lambda iS(p)\gamma_\rho iS(p+k)], \tag{5.1}$$

where the minus sign on the right side is for a closed fermion loop, and $S(p)$ is the propagator given in Eq. (3.10). This implies

$$\Pi_{\lambda\rho}(k) = -ie^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}[\gamma_\lambda iS(p)\gamma_\rho iS(p+k)]. \tag{5.2}$$

From Eq. (3.10) we see that there are two terms in the propagator — the vacuum part $S_B^V(p)$ and the other part which involves the background matter distribution. If we insert two such propagators in Eq. (5.2), we will obtain four terms.

The term obtained from the S_B^V factor in both propagators is the contribution in the vacuum. It has no importance to our discussion of background effects. The terms with the distribution function factor from both propagators contributes only to the absorptive part of the vacuum polarization, which we do not discuss in this article. Thus we are left with the terms in which we use the vacuum part of one propagator and the background dependent part of the other. These terms contribute to the part $\Pi''_{\lambda\rho}$ in the notation of Sec. II. Thus

$$\Pi''_{\lambda\rho}(k) = -ie^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\,[\gamma_\lambda S_B^\eta(p)\gamma_\rho iS_B^V(p') + \gamma_\lambda iS_B^V(p)\gamma_\rho S_B^\eta(p')], \tag{5.3}$$

where, for the sake of notational simplicity, we have used

$$p' = p + k. \tag{5.4}$$

Substituting p by $-p'$ in the second term and using the cyclic property of traces, we can write Eq. (5.3) as

$$\Pi''_{\lambda\rho}(k) = -ie^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}[\gamma_\lambda S_B^\eta(p)\gamma_\rho i S_B^V(p') + \gamma_\rho S_B^\eta(-p)\gamma_\lambda i S_B^V(-p')]. \tag{5.5}$$

Using now the form of the propagators from Eqs. (3.1) and (3.14), we obtain

$$\Pi''_{\lambda\rho}(k) = ie^2 \int \frac{d^4p}{(2\pi)^4} \int_{-\infty}^{\infty} ds\, e^{\Phi(p,s)} \int_0^\infty ds'\, e^{\Phi(p',s')} \times [\eta_F(p)\mathrm{tr}(\gamma_\lambda G(p,s)\gamma_\rho G(p',s')) + \eta_F(-p)\mathrm{tr}(\gamma_\rho G(-p,s)\gamma_\lambda G(-p',s'))]. \tag{5.6}$$

B. Extracting the gauge invariant piece

In order to discuss Faraday effect, we need only the terms in the vacuum polarization tensor which are odd in $\mathcal{B}$. Notice that the phase factors appearing in Eq. (5.6) are even in $\mathcal{B}$. Thus, we need only the odd terms from the traces. Performing the traces is straightforward, and the odd terms come out to be

$$O_{\lambda\rho}(k) = 4ie^2 \int \frac{d^4p}{(2\pi)^4} \eta_-(p) \times \int_{-\infty}^{\infty} ds\, e^{\Phi(p,s)} \int_0^\infty ds'\, e^{\Phi(p',s')} R_{\lambda\rho} \tag{5.7}$$

where we have introduced the notation

$$\eta_-(p) \equiv \eta_F(p) - \eta_F(-p), \tag{5.8}$$

and

$$\begin{aligned} R_{\lambda\rho} = {} & \varepsilon_{\lambda\rho 03} m^2 (\tan e\mathcal{B}s - \tan e\mathcal{B}s') \\ & + \varepsilon_{\lambda\rho\alpha_\parallel\beta_\parallel}(p^{\tilde{\alpha}_\parallel} p'^{\beta_\parallel} \tan e\mathcal{B}s - p'^{\tilde{\alpha}_\parallel} p^{\beta_\parallel} \tan e\mathcal{B}s') \\ & + \varepsilon_{\lambda\rho\alpha_\parallel\beta_\perp}(p^{\tilde{\alpha}_\parallel} p'^{\beta_\perp} \tan e\mathcal{B}s \sec^2 e\mathcal{B}s' \\ & - p'^{\tilde{\alpha}_\parallel} p^{\beta_\perp} \tan e\mathcal{B}s' \sec^2 e\mathcal{B}s). \end{aligned} \tag{5.9}$$

In writing this expression, we have used the notation of $p^{\tilde{\alpha}_\parallel}$, for example. This signifies a component of p which can take only the ''parallel'' indices, i.e., 0 and 3, and is moreover different from the index α appearing elsewhere in the expression.

Using now the definition of p' from Eq. (5.4), we can write

$$R_{\lambda\rho} = R^{(1)}_{\lambda\rho} + R^{(2)}_{\lambda\rho}, \tag{5.10}$$

where

$$R^{(1)}_{\lambda\rho} = \varepsilon_{\lambda\rho\alpha_\parallel\beta}[p^{\tilde{\alpha}_\parallel} \tan e\mathcal{B}s + p'^{\tilde{\alpha}_\parallel} \tan e\mathcal{B}s']k^\beta \tag{5.11}$$

and

$$\begin{aligned} R^{(2)}_{\lambda\rho} = {} & \tan e\mathcal{B}s[m^2 \varepsilon_{\lambda\rho 03} + \varepsilon_{\lambda\rho\alpha_\parallel\beta_\parallel} p^{\tilde{\alpha}_\parallel} p^{\beta_\parallel} \\ & + \varepsilon_{\lambda\rho\alpha_\parallel\beta_\perp}(p^{\tilde{\alpha}_\parallel} p^{\beta_\perp} + p^{\tilde{\alpha}_\parallel} p'^{\beta_\perp} \tan^2 e\mathcal{B}s')] \\ & - \tan e\mathcal{B}s'[m^2 \varepsilon_{\lambda\rho 03} + \varepsilon_{\lambda\rho\alpha_\parallel\beta_\parallel} p'^{\tilde{\alpha}_\parallel} p'^{\beta_\parallel} \\ & + \varepsilon_{\lambda\rho\alpha_\parallel\beta_\perp}(p'^{\tilde{\alpha}_\parallel} p'^{\beta_\perp} + p'^{\tilde{\alpha}_\parallel} p^{\beta_\perp} \tan^2 e\mathcal{B}s)]. \end{aligned} \tag{5.12}$$

Obviously, $R^{(1)}_{\lambda\rho}$ is gauge invariant, i.e., $k^\lambda R^{(1)}_{\lambda\rho} = k^\rho R^{(1)}_{\lambda\rho} = 0$. To simplify the other term, we first note that the combinations in which the parallel components of p and p' appear in Eq. (5.12) can be simplified by using the following identity:

$$\varepsilon_{\lambda\rho\alpha_\parallel\beta_\parallel} a^{\tilde{\alpha}_\parallel} b^{\beta_\parallel} = -\varepsilon_{\lambda\rho 03} a \cdot b_\parallel, \tag{5.13}$$

which holds for any two vectors a and b. For the terms involving the transverse compoents, we make an important observation. We will be performing the calculations in the rest frame of the medium where $p \cdot u = p_0$. Thus, the distribution function does not depend on the spatial components of p. In the last term of each square bracket of Eq. (5.12), the integral over the transverse components of p has the following generic structure:

$$\int d^2p_\perp e^{\Phi(p,s)} e^{\Phi(p',s')} \times (p^{\beta_\perp} \text{ or } p'^{\beta_\perp}). \tag{5.14}$$

Notice now that

$$\frac{\partial}{\partial p_{\beta_\perp}}[e^{\Phi(p,s)} e^{\Phi(p',s')}] = \frac{2i}{e\mathcal{B}}(\tan e\mathcal{B}s\ p^{\beta_\perp} + \tan e\mathcal{B}s'\, p'^{\beta_\perp}) \times e^{\Phi(p,s)} e^{\Phi(p',s')}. \tag{5.15}$$

However, this expression, being a total derivative, should integrate to zero. Thus we obtain that

$$\tan e\mathcal{B}s\ p^{\beta_\perp} \doteq -\tan e\mathcal{B}s'\, p'^{\beta_\perp}, \tag{5.16}$$

where the sign ''$\doteq$'' means that the expressions on both sides of it, though not necessarily equal algebraically, yield the same integral. This gives

$$p^{\beta_\perp} \doteq -\frac{\tan e\mathcal{B}s'}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} k^{\beta_\perp},$$

$$p'^{\beta_\perp} \doteq \frac{\tan e\mathcal{B}s}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} k^{\beta_\perp}. \tag{5.17}$$

Using these identities, we can rewrite Eq. (5.12) in the following form:

$$R^{(2)}_{\lambda\rho} = \varepsilon_{\lambda\rho 03}[(m^2-p_\parallel^2)\tan e\mathcal{B}s-(m^2-p_\parallel'^2)\tan e\mathcal{B}s'] - \varepsilon_{\lambda\rho\alpha_\parallel\beta_\perp}\frac{\tan e\mathcal{B}s\tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')}(p+p')^{\tilde{\alpha}_\parallel}k^{\beta_\perp} = R^{(2a)}_{\lambda\rho}+\varepsilon_{\lambda\rho 03}R^{(2b)}, \tag{5.18}$$

where

$$R^{(2a)}_{\lambda\rho} = -\varepsilon_{\lambda\rho\alpha_\parallel\beta}\frac{\tan e\mathcal{B}s\tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')}(p+p')^{\tilde{\alpha}_\parallel}k^{\beta}, \tag{5.19}$$

$$R^{(2b)} = (m^2-p_\parallel^2)\tan e\mathcal{B}s-(m^2-p_\parallel'^2)\tan e\mathcal{B}s' - \frac{\tan e\mathcal{B}s\tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')}(p+p')\cdot k_\parallel. \tag{5.20}$$

The term called $R^{(2b)}$ does not vanish on contraction with arbitrary k^λ. This term is not gauge invariant, and therefore must vanish on integration. In the Appendix, we show that this is indeed true, so that the contribution to the vacuum polarization tensor which is odd in $\mathcal{B}$ is given by

$$O_{\lambda\rho}(k) = 4ie^2\int\frac{d^4p}{(2\pi)^4}\eta_-(p)\times\int_{-\infty}^{\infty}ds\,e^{\Phi(p,s)}\int_0^\infty ds'\,e^{\Phi(p',s')}[R^{(1)}_{\lambda\rho}+R^{(2a)}_{\lambda\rho}]$$
$$= 4ie^2\varepsilon_{\lambda\rho\alpha_\parallel\beta}k^\beta\int\frac{d^4p}{(2\pi)^4}\eta_-(p)\times\int_{-\infty}^{\infty}ds\,e^{\Phi(p,s)}\int_0^\infty ds'\,e^{\Phi(p',s')}\left[p^{\tilde{\alpha}_\parallel}\tan e\mathcal{B}s + p'^{\tilde{\alpha}_\parallel}\tan e\mathcal{B}s' - \frac{\tan e\mathcal{B}s\tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')}(p+p')^{\tilde{\alpha}_\parallel}\right]. \tag{5.21}$$

In order to perform this integration, we need to introduce further assumptions, which will be done in Sec. VII.

VI. DISPERSION RELATIONS

A. Magnetic field-independent terms in the vacuum polarization

The contributions to the vacuum polarization tensor determines the equation of motion of a photon through the medium. We have already found the magnetic field-dependent terms in the vacuum polarization. To obtain the dispersion relations, however, we need also the terms which do not depend on the background magnetic field. These terms are necessarily even in $\mathcal{B}$ and therefore did not appear in $O_{\lambda\rho}$. Here we outline the calculation of these terms.

Rather than going back to Eq. (5.6) which contains also the even terms in $\mathcal{B}$, we use directly the propagators at $\mathcal{B}=0$ given in Eqs. (3.15) and (3.16) to write the background dependent dispersive terms as

$$\Pi'_{\lambda\rho}(k) = -ie^2\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}[\gamma_\lambda S_0^\eta(p)\gamma_\rho iS_0^V(p') + \gamma_\lambda iS_0^V(p)\gamma_\rho S_0^\eta(p')]. \tag{6.1}$$

Changing, as before, the integration variable in the second term, we obtain

$$\Pi'_{\lambda\rho}(k) = -e^2\int\frac{d^4p}{(2\pi)^3}\frac{\delta(p^2-m^2)}{(p+k)^2-m^2}\times\mathrm{tr}[\gamma_\lambda(\not p+m)\eta_F(p)\gamma_\rho(\not p+\not k+m) + \gamma_\rho(\not p-m)\eta_F(-p)\gamma_\lambda(\not p+\not k-m)]$$
$$= -4e^2\int\frac{d^4p}{(2\pi)^3}\frac{\delta(p^2-m^2)}{k^2+2p\cdot k}\times[2p_\lambda p_\rho+p_\lambda k_\rho+k_\lambda p_\rho-g_{\lambda\rho}p\cdot k][f_++f_-]. \tag{6.2}$$

In writing the last form, we have put $p^2=m^2$ in the denominator and in the trace, in view of the presence of the δ-function, and used

$$\eta_F(p)+\eta_F(-p) = f_++f_-, \tag{6.3}$$

where we introduce the notations

$$f_\pm = f_F(|p_0|,\mp\mu). \tag{6.4}$$

The expression presented in Eq. (6.2) has a particularly simple form in the long wavelength limit, i.e., in the limit of $K=0$. In this case, one can show that the Π_{00} component vanishes, whereas the Π_{ij} components are proportional to the unit matrix. Since the same is true for the tensor $\tilde{g}_{\mu\nu}$, we can summarize all this information by writing

$$\Pi'_{\lambda\rho}(k) = \omega_0^2\tilde{g}_{\lambda\rho}, \tag{6.5}$$

where ω_0 is called the plasma frequency, and is given by

$$\omega_0^2 = 4e^2\int\frac{d^3p}{(2\pi)^3 2E_p}\left(1-\frac{P^2}{3E_p^2}\right)[f_++f_-], \tag{6.6}$$

where $P=|\vec{p}|$.

B. Dispersion relations and Faraday rotation

We have thus obtained expressions for the vacuum polarization tensor. For the rest of this paper, we will consider only photon propagation along the direction of the magnetic field. Thus, in Eq. (5.21), the index β can only take the values 0 or 3. Since the index α appearing in that equation had also parallel components only, the antisymmetric tensor now implies that $\Pi''_{\lambda\rho}$ vanishes unless both λ and ρ are transverse, i.e., have values 1 or 2. Thus, the only non-vanishing components of $\Pi''_{\lambda\rho}$ are

$$\Pi''_{12}(k) = -\Pi''_{21}(k) = -ia, \tag{6.7}$$

where a has to be determined by evaluating the integral in Eq. (5.21). The contributions which come from the medium even without the magnetic field have been given in Eq. (6.5).

To obtain the dispersion relations, we go back to the Lagrangian given in Eq. (2.5). The equation of motion obtained from this Lagrangian is

$$[(-k^2+\omega_0^2)\tilde{g}_{\lambda\rho} + \Pi''_{\lambda\rho}]A^\rho = 0. \tag{6.8}$$

In view of the Lorentz gauge condition $k_\rho A^\rho = 0$, this can also be written as

$$[(-k^2+\omega_0^2)g_{\lambda\rho} + \Pi''_{\lambda\rho}]A^\rho = 0. \tag{6.9}$$

For the transverse components of the photon field A^ρ, the above equation implies the following condition:

$$\begin{pmatrix} -k^2+\omega_0^2 & -ia \\ ia & -k^2+\omega_0^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0. \tag{6.10}$$

The eigenvalues of the matrix give the dispersion relations

$$k^2 = \omega_0^2 \pm a \tag{6.11}$$

for the normalized eigenmodes

$$(A_1 \pm iA_2)/\sqrt{2}, \tag{6.12}$$

which describe circularly polarized states of the photon.

Writing k^2 as $\omega^2 - K^2$, we obtain the following solutions for K:

$$K_\pm = \sqrt{\omega^2-\omega_0^2}\left[1 \mp \frac{a}{\omega^2-\omega_0^2}\right]^{1/2}. \tag{6.13}$$

For small magnetic fields, a will be small, and then we can write

$$K_\pm = \sqrt{\omega^2-\omega_0^2}\left[1 \mp \frac{a}{2(\omega^2-\omega_0^2)}\right], \tag{6.14}$$

which gives, for the difference of the two solutions,

$$\Delta K = \frac{a}{\sqrt{\omega^2-\omega_0^2}}. \tag{6.15}$$

For a plane polarized electromagnetic wave propagating with a frequency ω, this means that, after travelling a distance l, the plane of propagation will be rotated by an amount $l\Delta K$. Thus, the rate of rotation of the polarization angle Φ is given by

$$\frac{d\Phi}{dl} = \Delta K = \frac{a}{\sqrt{\omega^2-\omega_0^2}}. \tag{6.16}$$

This is the Faraday rotation per unit length. The magnitude of this quantity is thus determined once we determine a and ω_0.[1]

In what follows, we find out what ω_0 and a are for different types of backgrounds, and consequently what is the amount of Faraday rotation sufferred by plane polarized light in such backgrounds. We will do this for three different kinds of backgrounds, depending on the relative importance of the temperature $T = 1/\beta$, the chemical potential μ, and the electron mass m_e.

VII. RESULTS FOR DIFFERENT BACKGROUNDS

A. General observations and assumptions

Before starting with any of the specific cases, let us note some general features and some common assumptions in the calculations. We will perform all the calculations assuming that the background medium is at rest in the frame in which we have a purely magnetic field. In other words, for the 4-vector u, the only non-zero component is the time component, which has the value unity. All other components are zero.

As already mentioned, we will consider photon propagation along the z-direction (positive or negative). In addition, we will take the long wavelength limit, i.e., $K \ll \omega$. This implies that in Eq. (5.21), the term with the external factor of $k^0 = \omega$ dominates over the one with $k^3 = K$.

Finally, we will assume the magnetic field to be small, so that we can use only the linear terms in $\mathcal{B}$. To this order, then, the dominant contribution of Eq. (5.21) is given by

$$\Pi''_{\lambda\rho}(k) = 8ie^3\mathcal{B}\varepsilon_{\lambda\rho 30}\omega I, \tag{7.1}$$

where

$$I = \int \frac{d^4p}{(2\pi)^4}\,\eta_-(p)p_0 \int_{-\infty}^{\infty} ds\; e^{is(p^2-m^2)-\epsilon|s|} \times \int_0^\infty ds'\, e^{is'(p'^2-m^2)-\epsilon|s'|}\left[s+s'-\frac{ss'}{s+s'}\right]. \tag{7.2}$$

Here, since the other factor is already linear in $\mathcal{B}$, we have used $\mathcal{B}=0$ in the exponents. Moreover, we have made the further assumption that $\omega \ll m_e$, which enables us to neglect k_0 compared to p_0 in the factor inside the square bracket. In the notation introduced in Eq. (6.7), we can write

$$a = 8e^3\mathcal{B}\omega I. \tag{7.3}$$

The expression for I can be put in a convenenient form. For this, we first define the integral

[1]Instead of ω_0, one can also use the index of refraction r, defined by the relation $r = K/\omega$. In absence of the magnetic field, i.e., when $a=0$, Eq. (6.11) gives $r^2 = 1-\omega_0^2/\omega^2$. We can use this relation to eliminate ω_0 from the formulas above and express everything in terms of r, i.e., the refractive index in absence of the magnetic field.

$$J_n = \int \frac{d^4p}{(2\pi)^4}\,\eta_-(p)p_0 \int_{-\infty}^{\infty} ds\; e^{is(p^2-m^2)-\epsilon|s|} \times \int_0^\infty ds' e^{is'(p'^2-m^2)-\epsilon|s'|} s'^n. \tag{7.4}$$

If we now rewrite the factor in the square brackets in Eq. (7.2) in the following form,

$$s+s' - \frac{ss'}{s+s'} = (s+s') - s' + \frac{s'^2}{s+s'}, \tag{7.5}$$

it is easily seen that

$$I = i\frac{\partial}{\partial(m^2)} J_0 - J_1 + i\int d(m^2) J_2. \tag{7.6}$$

The task is now to evaluate J_n for $n=0,1,2$. The s integral in J_n gives

$$\int_{-\infty}^{\infty} ds\; e^{is(p^2-m^2)-\epsilon|s|} = 2\pi\delta(p^2-m^2), \tag{7.7}$$

whereas the s' integral gives

$$\int_0^\infty ds' e^{is'(p'^2-m^2)-\epsilon|s'|} s'^n = \frac{i^{n+1} n!}{(p'^2-m^2)^{n+1}}. \tag{7.8}$$

Writing now

$$\delta(p^2-m^2) = \frac{1}{2E_p}[\delta(p_0-E_p)+\delta(p_0+E_p)], \tag{7.9}$$

and using

$$\eta_-(p) = \mathrm{sgn}(p_0)[f_+(p) - f_-(p)] \tag{7.10}$$

which follows from the definitions in Eqs. (3.12) and (5.8), we obtain

$$J_n = i^{n+1} n! \int \frac{d^4p}{(2\pi)^3} \frac{p_0\,\mathrm{sgn}(p_0)}{2E_p}[\delta(p_0-E_p)+\delta(p_0+E_p)] \times \frac{f_+ - f_-}{(k^2+2p_0\omega - 2PK\cos\theta')^{n+1}}. \tag{7.11}$$

Here, $k^\mu \equiv (\omega, \vec{K})$, $P \equiv |\vec{p}|$ and θ' is the angle between $\vec{K}$ and $\vec{p}$. We have denoted this angle by θ' in order to emphasize that, for a general direction of propagation, it can be different from the angle θ which is measured from the z axis, i.e., from the direction of the magnetic field which we have already specified. For our specific case of photon propagation along the magnetic field direction, however, we will put $\theta' = \theta$.

We further notice that we can neglect the term k^2 because of our assumptions stated earlier. Thus,

$$J_n = \frac{i^{n+1} n!}{8} \int \frac{d^3p}{(2\pi)^3}(f_+ - f_-)\left[\frac{1}{(E_p\omega - PK\cos\theta)^{n+1}} + \frac{1}{(-E_p\omega - PK\cos\theta)^{n+1}}\right]. \tag{7.12}$$

The azimuthal integration gives a factor 2π, and the θ integration can be exactly performed here. This shows that $J_n = 0$ for even values of n. This conclusion can be avoided only if ω and/or K becomes comparable to m_e. Since we have already assumed otherwise, we obtain

$$I = -J_1, \tag{7.13}$$

i.e.,

$$I = \frac{1}{8}\int \frac{d^3p}{(2\pi)^3}(f_+ - f_-)\left[\frac{1}{(E_p\omega - PK\cos\theta)^2} + \frac{1}{(-E_p\omega - PK\cos\theta)^2}\right]. \tag{7.14}$$

In general, however, even this integral cannot be performed analytically. So, in order to discuss the amount of Faraday rotation caused by this term, we need to take recourse to some specific limits.

B. Connection with the form factors

It is of interest to see how our final result for $\Pi''_{\lambda\rho}$ conforms to the general form obtained on the basis of gauge and Lorentz invariance, a subject that was discussed in Sec. II. At the end of Sec. IV, we remarked that in our case, we can get at most four independent form factors, viz, those associated with the field-dependent tensors $P''^{(1)}_{\lambda\rho}$, $P''^{(3)}_{\lambda\rho}$, $P''^{(7)}_{\lambda\rho}$ and $P''^{(9)}_{\lambda\rho}$. However, the simplifying assumptions made above imply that all components of $\tilde{u}_\lambda$ vanishes to the leading order. Thus, only $P''^{(1)}_{\lambda\rho}$ survives in this case. Moreover, since we choose the direction of propagation to be along the magnetic field, $(k\cdot B)_\lambda = 0$ as well. Thus, from Eq. (2.17), we find that the tensor $P''^{(1)}_{\lambda\rho}$, in the case of our choice, is simply proportional to $B_{\lambda\rho}$. This is what the explicit calculation of Eq. (7.1) tells us as well.

C. A non-relativistic background

Suppose we have a gas of electrons and positrons where all the particles are non-relativistic. In this case, we can put $E_p \approx m_e$ within the integral, and neglect all occurrences of P since it is small compared to E_p. Then we obtain

$$I = \frac{1}{4m_e^2\omega^2}\int \frac{d^3p}{(2\pi)^3}(f_+ - f_-) = \frac{1}{8m_e^2\omega^2}(n_e - n_{\bar{e}}). \tag{7.15}$$

Using Eqs. (7.3) and (6.16) now, the Faraday rotation per unit length is obtained to be

$$\frac{d\Phi}{dl}=\frac{e^3\mathcal{B}}{m_e^2\omega\sqrt{\omega^2-\omega_0^2}}(n_e-n_{\bar{e}}), \tag{7.16}$$

where ω_0, in this limit, can be simplified by using the general formula in Eq. (6.6):

$$\omega_0^2=\frac{2e^2}{m_e}\int\frac{d^3p}{(2\pi)^3}[f_+ + f_-]=\frac{e^2}{m_e}(n_e+n_{\bar{e}}). \tag{7.17}$$

If the background contains no positrons, the expression for Faraday rotation can be written as

$$\frac{d\Phi}{dl}=\frac{\omega_0^2\omega_c}{\omega\sqrt{\omega^2-\omega_0^2}}, \tag{7.18}$$

where $\omega_c\equiv e\mathcal{B}/m_e$ is called the cyclotron frequency.

D. A degenerate background

We now consider a degenerate electron background at zero temperature. The distribution functions are now given by

$$f_+=\begin{cases}1 & \text{for } P\leqslant P_F,\\ 0 & \text{for } P>P_F,\end{cases}$$
$$f_-=0, \tag{7.19}$$

where P_F is called the Fermi momentum. As we know, although the temperature is zero, the electrons need not be non-relativistic in this case, since Pauli exclusion principle would require all electrons to be in different states, and so some of them can be at very large momentum. The number density of electrons in this case is given by

$$n_e=2\int\frac{d^3p}{(2\pi)^3}f_+=\frac{P_F^3}{3\pi^2}. \tag{7.20}$$

In this case, we first calculate the plasma frequency. Performing the angular integrations of Eq. (6.6), this can be written in the form

$$\omega_0^2=\frac{e^2m_e^2}{\pi^2}\int_0^{x_F}dx\left(\frac{x^2}{(1+x^2)^{1/2}}-\frac{x^4}{3(1+x^2)^{3/2}}\right), \tag{7.21}$$

where x is the integration variable defined by P/m_e, and $x_F=P_F/m_e$. The integration can be performed in a straightforward manner by substituting $x=\sinh\zeta$, and the result is

$$\omega_0^2=\frac{e^2m_e^2}{3\pi^2}\frac{x_F^3}{\sqrt{1+x_F^2}}=\frac{e^2}{3\pi^2}\frac{P_F^3}{E_F} \tag{7.22}$$

where E_F is the Fermi energy,

$$E_F=\sqrt{P_F^2+m_e^2}. \tag{7.23}$$

We now evaluate the integral I. Starting from the expression in Eq. (7.14) for the general case, we perform the angular integrations to obtain

$$I=\frac{1}{16\pi^2K}\int_0^\infty dP\,P(f_+-f_-)\left[\frac{1}{E_p\omega-PK}-\frac{1}{E_p\omega+PK}\right]$$
$$=\frac{1}{8\pi^2\omega^2}\int_0^\infty dP\frac{P^2}{E_p^2}(f_+-f_-), \tag{7.24}$$

for $K\to 0$.

Using now the distribution functions appropriate for this case from Eq. (7.19), we obtain

$$I=\frac{1}{8\pi^2\omega^2}[P_F-m_e\tan^{-1}(P_F/m_e)], \tag{7.25}$$

where the result of the arctan function is restricted within the domain 0 to $\pi/2$. From this, we obtain the Faraday rotation per unit length to be

$$\frac{d\Phi}{dl}=\frac{\omega_0^2\omega_c}{\omega\sqrt{\omega^2-\omega_0^2}}\cdot\frac{3m_eE_F}{P_F^3}[P_F-m_e\tan^{-1}(P_F/m_e)]. \tag{7.26}$$

It can be easily checked that if $P_F\ll m_e$, in which case the background is non-relativistic, the formulas derived for this case reduce to those derived in Sec. VII C.

E. An ultra-relativistic background

Let us now discuss the case where the temperature T of the background is much higher than the electron mass. In this case, we can put $E_p\approx P$. Then, using the dimensionless integration variable $y=P/T$, the plasma frequency can be expressed as

$$\omega_0^2=\frac{2e^2}{3\pi^2\beta^2}\int_0^\infty dy\;y\left(\frac{1}{\exp(y-\beta\mu)+1}+\frac{1}{\exp(y+\beta\mu)+1}\right). \tag{7.27}$$

This integration can in fact be performed exactly. In the first integration, use the new integration variable $y'=y-\beta\mu$. In the second one, use $y'=y+\beta\mu$. The resulting integrations can then be written in the form

$$\omega_0^2=\frac{2e^2}{3\pi^2\beta^2}\left[\int_0^\infty dy'\frac{2y'}{e^{y'}+1}+\int_{-\beta\mu}^0 dy'\frac{y'+\beta\mu}{e^{y'}+1}-\int_0^{\beta\mu}dy'\frac{y'-\beta\mu}{e^{y'}+1}\right]. \tag{7.28}$$

The first integration can now be performed by expressing the denominator as a geometric series. The other two can be combined after substituting $y'\to -y'$ in the second integration, and the final result is

$$\omega_0^2=e^2\left[\frac{1}{9\beta^2}+\frac{\mu^2}{3\pi^2}\right]. \tag{7.29}$$

For the integration I, we start from the general expression in Eq. (7.24). Using similar substitutions as before, we obtain

$$I=\frac{1}{8\pi^2\beta\omega^2}\int_0^\infty dy\left(\frac{1}{\exp(y-\beta\mu)+1}-\frac{1}{\exp(y+\beta\mu)+1}\right)$$
$$=\frac{\mu}{8\pi^2\omega^2}. \tag{7.30}$$

The Faraday rotation is obtained from Eqs. (6.16) and (7.3):

$$\frac{d\Phi}{dl}=\frac{e^3\mathcal{B}\mu}{\pi^2\omega\sqrt{\omega^2-\omega_0^2}}. \tag{7.31}$$

If we put $\beta\to\infty$, the results obtained for this case reduce to those obtained in Sec. VII D with $m_e=0$.

As for the previous cases, one may want to express these results in terms of the number densities of electrons and positrons in the medium rather than in terms of the chemical potential μ. The connection comes from the relation

$$n_e-n_{\bar{e}}=2\int\frac{d^3p}{(2\pi)^3}(f_+-f_-). \tag{7.32}$$

Again, the integration can be performed exactly, following the steps described above, and the result is

$$n_e-n_{\bar{e}}=\frac{\mu}{3\beta^2}+\frac{\mu^3}{3\pi^2}. \tag{7.33}$$

One can use this to express μ in terms of $n_e-n_{\bar{e}}$.

VIII. CONCLUSIONS

We have thus shown that the amount of Faraday rotation depends very significantly on the characteristics of the medium in which the background magnetic field rests. If the medium consists of non-relativistic particles only, we obtain the formula given in Eq. (7.18). Usually, we assume that this formula should be applicable for low-temperature media, since the particles should be non-relativistic in this case. However, we show that if the medium is strongly degenerate, the formula changes. For $P_F\ll m_e$ it still agrees with the non-relativistic result as it should. But for $P_F\gg m_e$ the change is drastic, and Faraday rotation becomes very small, as can be seen from Eq. (7.26). Similarly, if the medium is so hot that the kinetic energies of the particles are much larger than their masses, we obtain a different result, as shown in Eq. (7.31). But it is interesting to note that in all the cases discussed, the quantity a has the same dependence on ω, viz., that it is inversely proportional to $\omega\sqrt{\omega^2-\omega_0^2}$.

APPENDIX: PROOF OF GAUGE INVARIANCE

In the text, we claimed that the contribution coming from $R^{(2b)}$ must vanish in order that the vacuum polarization tensor is gauge invariant. Here, we justify this claim. This contribution is proportional to the following integral:

$$C=\int\frac{d^4p}{(2\pi)^4}\eta_-(p)\int_{-\infty}^{\infty}ds\ e^{\Phi(p,s)}\int_0^\infty ds'\,e^{\Phi(p',s')}R^{(2b)}$$
$$=\int\frac{d^4p}{(2\pi)^4}\eta_-(p)\int_{-\infty}^{\infty}ds e^{\Phi(p,s)}\int_0^\infty ds'\,e^{\Phi(p',s')}$$
$$\times\left[(m^2-p_\parallel^2)\tan e\mathcal{B}s-(m^2-p_\parallel'^2)\tan e\mathcal{B}s'-\frac{\tan e\mathcal{B}s\tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')}(p+p')\cdot k_\parallel\right]. \tag{A1}$$

Using the definition of the exponential factor $\Phi(p,s)$ from Eq. (3.6), we notice that

$$m^2\tan e\mathcal{B}s e^{\Phi(p,s)}e^{\Phi(p',s')}=\tan e\mathcal{B}s\left(i\frac{d}{ds'}+(p_\parallel'^2-\sec^2 e\mathcal{B}s'p_\perp'^2)\right)e^{\Phi(p,s)}e^{\Phi(p',s')}, \tag{A2}$$

$$m^2\tan e\mathcal{B}s' e^{\Phi(p,s)}e^{\Phi(p',s')}=\tan e\mathcal{B}s'\left(i\frac{d}{ds}+(p_\parallel^2-\sec^2 e\mathcal{B}sp_\perp^2)\right)e^{\Phi(p,s)}e^{\Phi(p',s')}. \tag{A3}$$

This implies that we can write

$$C=C_1+iC_2, \tag{A4}$$

where

$$C_1 = \int \frac{d^4p}{(2\pi)^4}\,\eta_-(p)\int_{-\infty}^{\infty} ds\, e^{\Phi(p,s)}\int_0^{\infty} ds'\, e^{\Phi(p',s')}$$
$$\times\left[(p_\parallel'^2 - \sec^2 e\mathcal{B}s'\, p_\perp'^2 - p_\parallel^2)\tan e\mathcal{B}s - (p_\parallel^2 - \sec^2 e\mathcal{B}s\, p_\perp^2 - p_\parallel'^2)\tan e\mathcal{B}s' - \frac{\tan e\mathcal{B}s \tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')}(p+p')\cdot k_\parallel\right], \quad \text{(A5)}$$

$$C_2 = \int \frac{d^4p}{(2\pi)^4}\,\eta_-(p)\int_{-\infty}^{\infty} ds\int_0^{\infty} ds'\left(\tan e\mathcal{B}s\frac{d}{ds'} - \tan e\mathcal{B}s'\frac{d}{ds}\right)e^{\Phi(p,s)}e^{\Phi(p',s')}. \quad \text{(A6)}$$

Let us first consider the contribution C_2. Performing the s' integration in the first term and the s integration in the second, it can be written as

$$C_2 = \int \frac{d^4p}{(2\pi)^4}\,\eta_-(p)\left[e^{\Phi(p',s')}\Big|_0^{\infty}\int_{-\infty}^{\infty} ds \tan e\mathcal{B}s\; e^{\Phi(p,s)} - e^{\Phi(p,s)}\Big|_{-\infty}^{\infty}\int_0^{\infty} ds' \tan e\mathcal{B}s'\, e^{\Phi(p',s')}\right]. \quad \text{(A7)}$$

The second term vanishes since $e^{\Phi(p,s)}$ vanishes at both limits due to the term $-\epsilon|s|$ in it. The other exponential survives only at the limit $s'=0$, and gives

$$C_2 = \int \frac{d^4p}{(2\pi)^4}\,\eta_-(p)\int_{-\infty}^{\infty} ds \tan e\mathcal{B}s\; e^{\Phi(p,s)}$$
$$=0, \quad \text{(A8)}$$

where the last step follows on performing the integration over p, since $\Phi(p,s)$ is an even function of p and $\eta_-(p)$ is odd.

Let us now look at the other contribution, C_1. Separating out the terms involving parallel components from those involving transverse components, we write

$$C_1 = \int \frac{d^4p}{(2\pi)^4}\,\eta_-(p)\int_{-\infty}^{\infty} ds\int_0^{\infty} ds' \exp[\Phi(p,s)+\Phi(p',s')]$$
$$\times\left[(p_\parallel'^2 - p_\parallel^2)\left(\tan e\mathcal{B}s + \tan e\mathcal{B}s' - \frac{\tan e\mathcal{B}s \tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')}\right) + p_\perp^2 \tan, e\mathcal{B}s' \sec^2 e\mathcal{B}s - p_\perp'^2 \tan e\mathcal{B}s \sec^2 e\mathcal{B}s'\right]. \quad \text{(A9)}$$

From the definition of Φ, it follows that, apart from the small convergence factors,

$$\Phi(p,s)+\Phi(p',s') = \frac{i}{2}(s+s')(p_\parallel'^2 + p_\parallel^2 - 2m^2) - \frac{i}{2}(s-s')(p_\parallel'^2 - p_\parallel^2) - \frac{i}{e\mathcal{B}}(\tan e\mathcal{B}s'\, p_\perp'^2 + \tan e\mathcal{B}s\; p_\perp^2)$$
$$= \frac{i}{e\mathcal{B}}[(p_\parallel'^2 + p_\parallel^2 - 2m^2)\xi - (p_\parallel'^2 - p_\parallel^2)\zeta - p_\perp'^2 \tan(\xi-\zeta) - p_\perp^2 \tan(\xi+\zeta)], \quad \text{(A10)}$$

where we have defined two new parameters

$$\xi = \frac{1}{2}e\mathcal{B}(s+s'),$$
$$\zeta = \frac{1}{2}e\mathcal{B}(s-s'). \quad \text{(A11)}$$

Thus,

$$ie\mathcal{B}\frac{d}{d\zeta}e^{\Phi(p,s)+\Phi(p',s')} = e^{\Phi(p,s)+\Phi(p',s')}[p_\parallel'^2 - p_\parallel^2 - p_\perp'^2 \sec^2(\xi-\zeta) + p_\perp^2 \sec^2(\xi+\zeta)]. \quad \text{(A12)}$$

Using this, we can rewrite Eq. (A9) as

$$C_1=\int \frac{d^4p}{(2\pi)^4}\,\eta_-(p)\int_{-\infty}^{\infty}ds\int_0^{\infty}ds'\Bigg[\left(\tan e\mathcal{B}s+\tan e\mathcal{B}s'-\frac{\tan e\mathcal{B}s\,\tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')}\right)ie\mathcal{B}\frac{d}{d\zeta}e^{\Phi(p,s)+\Phi(p',s')}$$
$$+e^{\Phi(p,s)+\Phi(p',s')}\left\{p_\perp'^2\tan e\mathcal{B}s'\sec^2 e\mathcal{B}s'\left(1-\frac{\tan e\mathcal{B}s}{\tan e\mathcal{B}(s+s')}\right)-p_\perp^2\tan e\mathcal{B}s\sec^2 e\mathcal{B}s\left(1-\frac{\tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')}\right)\right\}\Bigg]. \tag{A13}$$

We are now left with only the transverse components everywhere except in the exponents. To write them in a useful form, we turn to Eq. (5.15) and take another derivative with respect to $p^{\alpha_\perp}$. From the fact that this derivative should also vanish on p integration, we find

$$p_\perp^\alpha p_\perp^\beta \doteq \frac{1}{\tan e\mathcal{B}s+\tan e\mathcal{B}s'}\left[-\frac{ie\mathcal{B}}{2}g_\perp^{\alpha\beta}+\frac{\tan^2 e\mathcal{B}s'}{\tan e\mathcal{B}s+\tan e\mathcal{B}s'}k_\perp^\alpha k_\perp^\beta\right]. \tag{A14}$$

In particular, then,

$$p_\perp^2 \doteq \frac{1}{\tan e\mathcal{B}s+\tan e\mathcal{B}s'}\left[-ie\mathcal{B}+\frac{\tan^2 e\mathcal{B}s'}{\tan e\mathcal{B}s+\tan e\mathcal{B}s'}k_\perp^2\right]. \tag{A15}$$

It then simply follows that

$$p_\perp'^2 \doteq \frac{1}{\tan e\mathcal{B}s+\tan e\mathcal{B}s'}\left[-ie\mathcal{B}+\frac{\tan^2 e\mathcal{B}s}{\tan e\mathcal{B}s+\tan e\mathcal{B}s'}k_\perp^2\right]. \tag{A16}$$

We now put these into Eq. (A13). After some straightforward but cumbersome algebra, it is found that the terms involving k cancel out, and we are left with

$$C_1=ie\mathcal{B}\int \frac{d^4p}{(2\pi)^4}\,\eta_-(p)\int_{-\infty}^{\infty}ds\int_0^{\infty}ds'\Bigg[\left(\tan e\mathcal{B}s+\tan e\mathcal{B}s'-\frac{\tan e\mathcal{B}s\,\tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')}\right)\frac{d}{d\zeta}e^{\Phi(p,s)+\Phi(p',s')}$$
$$-\frac{e^{\Phi(p,s)+\Phi(p',s')}}{\tan e\mathcal{B}s+\tan e\mathcal{B}s'}\left\{\tan e\mathcal{B}s'\sec^2 e\mathcal{B}s'\left(1-\frac{\tan e\mathcal{B}s}{\tan e\mathcal{B}(s+s')}\right)-\tan e\mathcal{B}s\sec^2 e\mathcal{B}s\left(1-\frac{\tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')}\right)\right\}\Bigg]. \tag{A17}$$

It is straightforward to show that this can be written in the following form:

$$C_1=ie\mathcal{B}\int \frac{d^4p}{(2\pi)^4}\,\eta_-(p)\int_{-\infty}^{\infty}ds\int_0^{\infty}ds'\frac{d}{d\zeta}\mathcal{F}(\xi,\zeta), \tag{A18}$$

where

$$\mathcal{F}(\xi,\zeta)=\left(\tan e\mathcal{B}s+\tan e\mathcal{B}s'-\frac{\tan e\mathcal{B}s\,\tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')}\right)e^{\Phi(p,s)+\Phi(p',s')}, \tag{A19}$$

with s and s' related to ξ and ζ through Eq. (A11).

We can now change the integration variables to ξ and ζ. This gives

$$C_1 = \frac{2i}{e\mathcal{B}} \int \frac{d^4p}{(2\pi)^4} \eta_-(p) \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \; \Theta(\xi-\zeta) \frac{d}{d\zeta} \mathcal{F}(\xi,\zeta)$$

$$= \frac{2i}{e\mathcal{B}} \int \frac{d^4p}{(2\pi)^4} \eta_-(p) \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \left[\frac{d}{d\zeta} \{\Theta(\xi-\zeta)\mathcal{F}(\xi,\zeta)\} - \delta(\xi-\zeta)\mathcal{F}(\xi,\zeta) \right]$$

$$= -\frac{2i}{e\mathcal{B}} \int \frac{d^4p}{(2\pi)^4} \eta_-(p) \int_{-\infty}^{\infty} d\xi \; \mathcal{F}(\xi,\xi), \tag{A20}$$

since the other term vanishes at the limits. In this integrand, $\zeta = \xi$, which means $s' = 0$. Looking back at the definition of $\mathcal{F}$, we find

$$\mathcal{F}(\xi,\xi) = \exp\left\{ \Phi\left(p, \frac{2\xi}{e\mathcal{B}} \right) \right\} \tan 2\xi. \tag{A21}$$

This is an even function of p, whereas $\eta_-(p)$ is odd. Thus, the expression vanishes on integrating over p.

[1] See, e.g., A.M. Portis, *Electromagnetic Fields: Sources and Media* (Wiley, New York, 1978), see Chap. 12.

[2] See, e.g., A. Rai Choudhuri, *The Physics of Fluids and Plasmas: An Introduction for Astrophysicists* (Cambridge University Press, Cambridge, England, 1999), see Chap. 12.

[3] M. Giovannini, Phys. Rev. D **56**, 3198 (1997); M. Giovannini and M. Shaposhnikov, *ibid.* **57**, 2186 (1998); M.J. Rees, Q. J. R. Astron. Soc. **28**, 197 (1987); P.P. Kronberg, Rep. Prog. Phys. **57**, 325 (1994); T. Kolatt, astro-ph/9704243; A. Loeb and A. Kosowsky, Astrophys. J. **469**, 1 (1996).

[4] J.F. Nieves and P.B. Pal, Phys. Rev. D **39**, 652 (1989); **40**, 2148(E) (1989).

[5] H. Pérez Rojas and A.E. Shabad, Ann. Phys. (N.Y.) **121**, 432 (1979).

[6] M.B. Kislinger and P.D. Morley, Phys. Rev. D **13**, 2765 (1976); **13**, 2771 (1976); H.A. Weldon, *ibid.* **26**, 1394 (1982).

[7] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[8] W.Y. Tsai, Phys. Rev. D **10**, 1342 (1974); **10**, 2699 (1974).

[9] W. Dittrich, Phys. Rev. D **19**, 2385 (1979).

[10] P. Elmfors, D. Grasso, and G. Raffelt, Nucl. Phys. **B479**, 3 (1996).

[11] L.F. Urrutia, Phys. Rev. D **17**, 1977 (1978).