Non-Bogomol'nyi SU(N) BPS monopoles

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For $N>2$ we present static monopole solutions of the second order SU(N) BPS Yang-Mills-Higgs equations which are not solutions of the first order Bogomol'nyi equations. These spherically symmetric solutions may be interpreted as monopole anti-monopole configurations and their construction involves harmonic maps into complex projective spaces. [S0556-2821(99)10318-7]

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I. INTRODUCTION

In this paper we study static monopole solutions of the SU(N) Yang-Mills-Higgs equations in $\mathbb{R}^3$, in the Bogomol'nyi-Prasad-Sommerfield (BPS) limit of vanishing Higgs potential. It is well known that the solutions which correspond to the global minima of the Yang-Mills-Higgs energy functional are all given by solutions of the simpler first order Bogomol'nyi equations [1]. However, as proven by Taubes [2] using infinite dimensional Morse theory, there are more solutions to the Yang-Mills-Higgs equations than just the Bogomol'nyi ones. These solutions are saddle points of the energy functional and correspond to monopole anti-monopole configurations, which have an instability to annihilation.

We construct a class of spherically symmetric SU(N) non-Bogomol'nyi monopoles and calculate some of their properties, such as magnetic charges and energies. It is interesting that this construction involves harmonic maps of the plane into $\mathbb{CP}^{N-1}$. Atiyah [3] has described a relationship between instantons in two and four dimensions, and by reduction a correspondence between SU(N) hyperbolic Bogomol'nyi monopoles and instantons of the two-dimensional $\mathbb{CP}^{N-1}$ sigma model. Our non-Bogomol'nyi solutions are obtained by using the non-instanton solutions of the $\mathbb{CP}^{N-1}$ sigma model and so our results appear to suggest that Atiyah's connection between Bogomol'nyi monopoles and sigma model instantons may have some form of an extension outside the self-dual sector. In any case, it is clear that we have found some monopole analogues of the sigma model non-instanton solutions.

II. SU(N) MONOPOLES

Static monopoles are solutions of the SU(N) Yang-Mills-Higgs equations in $\mathbb{R}^3$ which, in the BPS limit of a massless Higgs boson, are derived from the energy functional

$$E=-\frac{1}{4\pi}\int \mathrm{tr}\{(D_i\Phi)^2+\tfrac{1}{2}F_{ij}^2\}d^3x \tag{2.1}$$

where A_i, for $i=1,2,3$, is the su(N)-valued gauge potential, with field strength $F_{ij}=\partial_iA_j-\partial_jA_i+[A_i,A_j]$, and Φ is the su(N)-valued Higgs field. Variation of the energy (2.1) gives the second order Yang-Mills-Higgs equations

$$D_iD_i\Phi=0,\quad D_iF_{ij}=[D_j\Phi,\Phi]. \tag{2.2}$$

The boundary conditions are that the energy is finite and that, in a chosen direction (say along the x_3-axis), the Higgs field at infinity is a given constant diagonal matrix, $\Phi(0,0,\infty)=i\Phi_0$ where

$$\Phi_0=\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_N). \tag{2.3}$$

Here we choose the ordering such that $\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_N$ and because $\Phi\in\mathrm{su}(N)$ we have that $\Sigma_{i=1}^N\lambda_i=0$.

At large radius, the magnetic field, $B_k=\frac{1}{2}\varepsilon_{kij}F_{ij}$, has the leading order behavior

$$B_k\sim\frac{i\hat{x}_k}{2r^2}G(\hat{x}_1,\hat{x}_2,\hat{x}_3) \tag{2.4}$$

where the matrix G contains $N-1$ integers, n_i, which are the magnetic charges [4] and provide a topological characterization of the monopole solution. Since we impose the framing condition (2.3) along the x_3-axis, then along this axis we have that

$$G_0=G(0,0,1)=\mathrm{diag}(n_1,n_2-n_1,\ldots,n_{N-1}-n_{N-2},-n_{N-1}). \tag{2.5}$$

By completing the square in the energy density (2.1) the Bogomol'nyi bound [1] is obtained as

$$E=-\frac{1}{4\pi}\int \mathrm{tr}\{(D_i\Phi\pm B_i)^2\mp 2B_iD_i\Phi\}d^3x \tag{2.6}$$

$$\geqslant|(\lambda_1-\lambda_2)n_1+(\lambda_2-\lambda_3)n_2+\ldots+(\lambda_{N-1}-\lambda_N)n_{N-1}|. \tag{2.7}$$

The inequality (2.7) is obtained by noting that the final term in Eq. (2.6) can be written as a total derivative and as such can be expressed in terms of the magnetic charges and the eigenvalues of the Higgs field at infinity.

Clearly, within each magnetic charge sector the minimal energy solutions are obtained by solving the Bogomol'nyi equations

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$$D_i\Phi = \mp B_i \tag{2.8}$$

whose solutions saturate the energy bound. The upper sign corresponds to monopoles, that is, $n_i \geqslant 0$, whereas choosing the lower sign, which we shall refer to as the anti-Bogomol'nyi equations, results in anti-monopoles with $n_i \leqslant 0$.

From the energy bound (2.7), which gives the energy for Bogomol'nyi monopoles, it can be seen that the difference $m_j = \lambda_j - \lambda_{j+1}$ determines the mass of the monopole of type j, of which there are n_j in total. If the eigenvalues of the Higgs field are not distinct, say $\lambda_j = \lambda_{j+1}$, then $m_j = 0$, that is, the monopole of type j is massless and the integer n_j does not appear in the Bogomol'nyi bound (2.7). This reflects the fact that n_j is no longer a topological quantity. In this case the residual symmetry group is non-Abelian, rather than being the maximal torus $\mathrm{U}(1)^{N-1}$, and the integer n_j is not a magnetic charge. To distinguish these cases such integers are referred to as magnetic weights [4].

Although the global minima of the energy functional (2.1) are all obtained as Bogomol'nyi monopoles, that is, solutions of the Bogomol'nyi equations (2.8), Taubes [2] has proven the existence of other critical points. In other words, there are solutions of the full second order Yang-Mills-Higgs equations (2.2) which are not solutions of the first order Bogomol'nyi equations (2.8). In the following sections we construct examples of spherically symmetric non-Bogomol'nyi monopoles and investigate some of their properties. In general it is a much more difficult task to solve the full Yang-Mills-Higgs equations than the Bogomol'nyi equations, not just because the equations are second order instead of first order, but because the Bogomol'nyi equations are integrable and so a variety of techniques from integrable systems can be applied, whereas this property is lost for the Yang-Mills-Higgs equations.

III. THE HARMONIC MAP ANSATZ

The starting point for our investigation is the introduction of the coordinates $r, z, \bar{z}$ on $\mathbb{R}^3$. In terms of the usual spherical coordinates r, θ, φ the Riemann sphere variable is $z = e^{i\varphi}\tan(\theta/2)$. Using these coordinates the Yang-Mills-Higgs equations (2.2) take the form

$$[D_r\Phi, \Phi] = \frac{(1+|z|^2)^2}{2r^2}(D_z F_{r\bar{z}} + D_{\bar{z}} F_{rz}) \tag{3.1}$$

$$D_r(r^2 D_r\Phi) = -\frac{(1+|z|^2)^2}{2}(D_z D_{\bar{z}}\Phi + D_{\bar{z}} D_z \Phi) \tag{3.2}$$

$$[D_z\Phi, \Phi] + D_r F_{rz} = \frac{1}{2r^2} D_z[(1+|z|^2)^2 F_{z\bar{z}}] \tag{3.3}$$

and the Bogomol'nyi equations (2.8) become

$$iD_z\Phi = F_{rz}, \quad iD_r\Phi = \frac{(1+|z|^2)^2}{2r^2} F_{z\bar{z}}. \tag{3.4}$$

From Eq. (2.4) the matrix of magnetic charges, G, is given by

$$G = (1+|z|^2)^2 F_{z\bar{z}} \tag{3.5}$$

where the right hand side of the above is evaluated on the two-sphere at $r = \infty$.

Our ansatz for SU(N) monopoles is to set

$$\Phi = i\sum_{j=0}^{N-2} h_j\left(P_j - \frac{1}{N}\right), \quad A_z = \sum_{j=0}^{N-2} g_j[P_j, \partial_z P_j], \quad A_r = 0 \tag{3.6}$$

where $h_j(r), g_j(r)$ are real functions depending only on the radial coordinate r, and $P_j(z,\bar{z})$ are $N\times N$ Hermitian projectors, that is, $P_j = P_j^\dagger = P_j^2$, which are independent of the radius r. The set of $N-1$ projectors are taken to be orthogonal, so that $P_i P_j = 0$ for $i \neq j$. Note that we are working in a real gauge, so that $A_{\bar{z}} = -A_z^\dagger$. In Eq. (3.6), and for the remainder of the paper, we drop the summation convention.

The above ansatz is motivated by our recent study [5] of Bogomol'nyi monopoles and their construction in terms of harmonic maps. In the case of Bogomol'nyi monopoles it was convenient to a choose a particular complex gauge, but the existence of this complex gauge choice relies on a solution of the Bogomol'nyi equations and in general is not valid for the Yang-Mills-Higgs equations. However, after converting these Bogomol'nyi solutions to a real gauge they have the above form, although the ansatz (3.6) is more general.

Substituting the ansatz (3.6) into the Yang-Mills-Higgs equations one finds that the left hand side of Eq. (3.1) is identically zero. This follows from the fact that the projectors are independent of r and form an orthogonal set. The requirement that the right hand side of Eq. (3.1) is zero gives the following condition

$$\sum_{j=0}^{N-2} g_j'[P_j, \partial_z\partial_{\bar{z}} P_j] = 0. \tag{3.7}$$

The equation

$$[P, \partial_z\partial_{\bar{z}} P] = 0 \tag{3.8}$$

is the harmonic map equation of the two-dimensional $\mathbb{CP}^{N-1}$ sigma model (see for example Ref. [6]). Thus we take each P_j to be a harmonic map and then the first Yang-Mills-Higgs equation (3.1) is automatically satisfied. It is satisfying that the harmonic map equation emerges naturally from the Yang-Mills-Higgs equations since in the study of Bogomol'nyi monopoles it was found [5] to be useful to introduce harmonic maps but the equations themselves did not appear.

To proceed further we need to briefly recall some results about harmonic maps of the two-dimensional $\mathbb{CP}^{N-1}$ sigma model. See Zakrzewski [6] for a more detailed account of two-dimensional sigma models and their solutions.

If we regard the second order harmonic map equation (3.8) as a lower dimensional analogue of the Yang-Mills-

Higgs equations (2.2), then the analogue of the first order Bogomol'nyi equations (2.8) is the instanton equation

$$P\partial_z P=0 \tag{3.9}$$

whose solutions automatically satisfy the harmonic map equation (3.8). The instanton equation (3.9) is easy to solve, with the general solution being given by

$$P(f)=\frac{ff^\dagger}{|f|^2} \tag{3.10}$$

where $f(z)$ is an N-component column vector which is a holomorphic function of z and whose degree is equal to the instanton number of the sigma model. Another set of solutions are the anti-instantons, which satisfy the equation $P\partial_{\bar z}P=0$, and have the same form as the instanton solutions but this time f is an anti-holomorphic function. For anti-instantons the sigma model instanton number is minus the degree of f.

For $N=2$ these are all the finite action solutions to Eq. (3.8), but for $N>2$ there are other non-instanton solutions. These can be obtained from the instanton solutions by a process of differentiation and Gram-Schmidt orthogonalization. Explicitly, introduce the operator Δ defined by its action on any vector $f\in\mathbb{C}^N$ as

$$\Delta f=\partial_z f-\frac{f(f^\dagger\partial_z f)}{|f|^2} \tag{3.11}$$

and then define further vectors $\Delta^k f$ by induction as $\Delta^k f=\Delta(\Delta^{k-1}f)$.

When calculating with these objects it is useful to be aware of the following properties [6] of $\Delta^k f$ when f is holomorphic:

$$(\Delta^k f)^\dagger\Delta^l f=0,\quad k\neq l \tag{3.12}$$

$$\partial_{\bar z}(\Delta^k f)=-\Delta^{k-1}f\,\frac{|\Delta^k f|^2}{|\Delta^{k-1}f|^2},\qquad \partial_z\left(\frac{\Delta^{k-1}f}{|\Delta^{k-1}f|^2}\right)=\frac{\Delta^k f}{|\Delta^{k-1}f|^2}. \tag{3.13}$$

Defining the projectors P_k corresponding to the family of vectors $\Delta^k f$, that is,

$$P_k=P(\Delta^k f),\quad k=0,\ldots,N-1 \tag{3.14}$$

gives our required set of orthogonal harmonic maps. Since the projectors obtained from this sequence always satisfy the relation $\Sigma_{k=0}^{N-1}P_k=1$ and we are going to be taking arbitrary linear combinations, then we can neglect the final projector P_{N-1}.

Note that applying Δ a total of $N-1$ times to a holomorphic vector gives an anti-holomorphic vector, so that a further application of Δ gives the zero vector and hence no corresponding projector. In the $\mathbb{CP}^1$ case the operator Δ converts a holomorphic vector to an anti-holomorphic vector, that is, instantons to anti-instantons and these are all the solutions in this case.

In order for our ansatz (3.6) to give solutions to the two remaining Yang-Mills-Higgs equations, (3.2) and (3.3), the harmonic maps used must have spherical symmetry—essentially the factors of $(1+|z|^2)^2$ which appear in the Yang-Mills-Higgs equations must be cancelled. The required harmonic maps are obtained by applying the above procedure to the initial holomorphic vector

$$f=(f_0,\ldots,f_j,\ldots,f_{N-1})^t,\quad \text{where}\ \ f_j=z^j\sqrt{\binom{N-1}{j}} \tag{3.15}$$

and $\binom{N-1}{j}$ denote the binomial coefficients. For a discussion of the spherical symmetry of these maps see Ref. [5]. Here we merely point out that it is at least plausible that the required factors do indeed cancel since $|f|^2=(1+|z|^2)^{N-1}$.

In the following sections we shall describe the non-Bogomol'nyi monopoles obtained from our harmonic map ansatz in some detail for the simplest cases of SU(3) and SU(4). The situation for general SU(N) will then become clear.

IV. SPHERICAL MONOPOLES

In dealing with the equations which arise from the harmonic map ansatz (3.6) it is convenient to exchange the profile functions $h_j(r),g_j(r)$ for the functions $b_j(r),c_j(r)$ which are defined as the following linear combinations

$$h_j=\sum_{k=j}^{N-2}b_k,\quad c_j=1-g_j-g_{j+1},\quad \text{for}\ j=0,\ldots,N-2. \tag{4.1}$$

In the above we have defined $g_{N-1}=0$. Provided the eigenvalues of the Higgs field at infinity are correctly ordered, as in Eq. (2.3) [which corresponds to $b_j(\infty)\geqslant 0$], then the monopole masses are simply given by the asymptotic values of the functions $b_j(r)$, that is, $m_j=b_{j-1}(\infty)$ for $j=1,\ldots,N-1$. Thus if $b_j(\infty)=0$ this signals a change of symmetry breaking to a non-maximal case. This will be an important point in what follows. For the ansatz to be well-defined at the origin the boundary conditions $b_j(0)=0$ and $c_j(0)=1$ for all $j=0,\ldots,N-2$, must be imposed.

A. SU(2)

As we have mentioned in Sec. III, there are no non-instanton solutions of the $\mathbb{CP}^1$ sigma model and hence we cannot employ our ansatz to obtain non-Bogomol'nyi monopoles for gauge group SU(2). There are only two profile functions b_0,c_0, and the only solution is the standard spherically symmetric Bogomol'nyi 1-monopole. The non-Bogomol'nyi SU(2) monopole of Taubes [2] is shown to exist by making use of an axially symmetric ansatz. Furthermore, numerical evidence suggests [7] that the solution has only an axial symmetry and is not spherically symmetric.

This is consistent with the fact that this solution does not fall into the class of solutions which we obtain here.

B. SU(3)

For $N=3$ there are four profile functions, b_0,b_1,c_0,c_1, and our ansatz (3.6) reduces the Yang-Mills-Higgs equations to the following set of second order nonlinear ordinary differential equations

$$(b_j' r^2)' = 2(2b_j c_j^2 - b_k c_k^2)$$
$$r^2 c_j'' = c_j(2c_j^2 - c_k^2 - 1 + b_j^2 r^2). \tag{4.2}$$

Here the indices are chosen from the set {0,1} and $k \neq j$. Recall that the summation convention is no longer used in this paper. It is immediately clear that there is a symmetry under the interchange of indices, $0 \leftrightarrow 1$, when applied simultaneously to both the b_j and c_j functions; we shall make use of this symmetry later.

The corresponding energy, Eq. (2.1), is given by

$$E = 2\int_0^\infty \frac{r^2}{3}(b_0'^2 + b_1'^2 + b_0' b_1') + c_0'^2 + c_1'^2 + c_0^2 b_0^2 + c_1^2 b_1^2 + \frac{1}{2r^2}[(1-c_0^2)^2 + (1-c_1^2)^2 + (c_0^2 - c_1^2)^2]dr. \tag{4.3}$$

From this expression it can be seen that the energy is finite providing the functions approach their asymptotic values at least as fast as $1/r$, and if in addition the constraints that $c_j(\infty)b_j(\infty)=0$ are imposed for $j=0,1$.

Before studying the second order equations (4.2) it is first useful to examine the first order Bogomol'nyi equations, which in this formalism become

$$r^2 b_j' = -(2c_j^2 - c_k^2 - 1)$$
$$c_j' = -c_j b_j \tag{4.4}$$

where the notation is as above. Integrating the last equation gives the asymptotic behavior for c_j as

$$c_j \sim \exp[-r b_j(\infty) + O(1)]. \tag{4.5}$$

Now since c_j must be finite as $r \to \infty$ this gives that $b_j(\infty) \geqslant 0$. Thus we can characterize Bogomol'nyi monopoles by the fact that the asymptotic values of the b_j's are all nonnegative. If we consider the anti-Bogomol'nyi equations then they are given by Eqs. (4.4) but in which the minus signs are removed from the right hand side of the equations. In this case the requirement that $c_j(\infty)$ is finite implies that $b_j(\infty) \leqslant 0$. Thus we see that Bogomol'nyi monopoles have the property that all the asymptotic values of the b_j's have the same sign, positive for monopoles and negative for anti-monopoles. It is then natural to look for non-Bogomol'nyi solutions of the second order equations in which the asymptotic values $b_0(\infty)$ and $b_1(\infty)$ have opposite sign, and to interpret these as monopole anti-monopole solutions.

In order to read off the properties of a given solution we need to compute the Higgs field and magnetic charge matrix at $\mathbf{x}=(0,0,\infty)$ (which corresponds to the direction $z=0$). Explicitly, these are given by

$$\Phi_0 = \frac{1}{3}\mathrm{diag}(2b_0 + b_1, -b_0 + b_1, -b_0 - 2b_1) \tag{4.6}$$

$$G_0 = \mathrm{diag}(2(1-c_0^2), 2(c_0^2 - c_1^2), 2(c_1^2 - 1)). \tag{4.7}$$

As an example, consider the Bogomol'nyi monopole with maximal symmetry breaking and equal monopole masses given by $b_0(\infty)=b_1(\infty)=2$. Then $\Phi_0=\mathrm{diag}(2,0,-2)$ and the boundary conditions force that $c_0(\infty)=c_1(\infty)=0$ so that $G_0=\mathrm{diag}(2,0,-2)$. Comparing with Eq. (2.5) we see that the magnetic charges are $(n_1,n_2)=(2,2)$. For Bogomol'nyi monopoles the solutions can be obtained explicitly and the monopole charges are understood in terms of the degrees of the harmonic map projectors from which they are constructed [5]. At first sight it might appear from Eq. (4.7) that the magnetic charges are determined only by the boundary values $c_j(\infty)$ and are independent of the values of $b_j(\infty)$. However, this naive view is incorrect as is easily seen by considering the simple case of the anti-Bogomol'nyi solution with $b_0(\infty)=b_1(\infty)=-2$. Again the boundary conditions imply that $c_0(\infty)=c_1(\infty)=0$ and hence we obtain the same matrix $G_0=\mathrm{diag}(2,0,-2)$. But now we must be aware that in this case we have $\Phi_0=\mathrm{diag}(-2,0,2)$ so that the entries are not correctly ordered from the largest to the smallest. A constant gauge transformation permutes the entries to obtain $\Phi_0=\mathrm{diag}(2,0,-2)$ but this acts in the same way on the magnetic charge matrix so that after this gauge transformation we are left with the charge matrix $G_0=\mathrm{diag}(-2,0,2)$. Now that Φ_0 has the correct order we can compare this charge matrix with Eq. (2.5) and conclude that $(n_1,n_2)=(-2,-2)$. Although this example of computing the magnetic charges is trivial it illustrates the important point that the asymptotic values $b_j(\infty)$ are required in order to determine the magnetic charges. We shall see more interesting consequences of this fact in what follows.

For the moment we shall consider the case for which $b_j(\infty)\neq 0$, so that the boundary conditions are $c_j(\infty)=0$ for $j=0,1$. The Bogomol'nyi equations (4.4) are integrable and allow explicit solutions to be found for any choice of the positive parameters $b_0(\infty),b_1(\infty)$, which give the monopole masses m_1,m_2. However, it seems unlikely that explicit non-Bogomol'nyi solutions to the second order equations (4.2) can be found in closed form. Therefore we resort to a numerical solution of these equations. We apply a gradient flow algorithm with a finite difference scheme to compute the solution with a given set of boundary values $b_0(\infty),b_1(\infty)$. For all choices of these parameters we were able to find a numerical solution. As a test on the accuracy of the code we computed the charge (2,2) Bogomol'nyi solution with $b_0(\infty)=b_1(\infty)=2$. In this case, since the monopole masses are equal, $m_1=m_2=2$, and the total number of

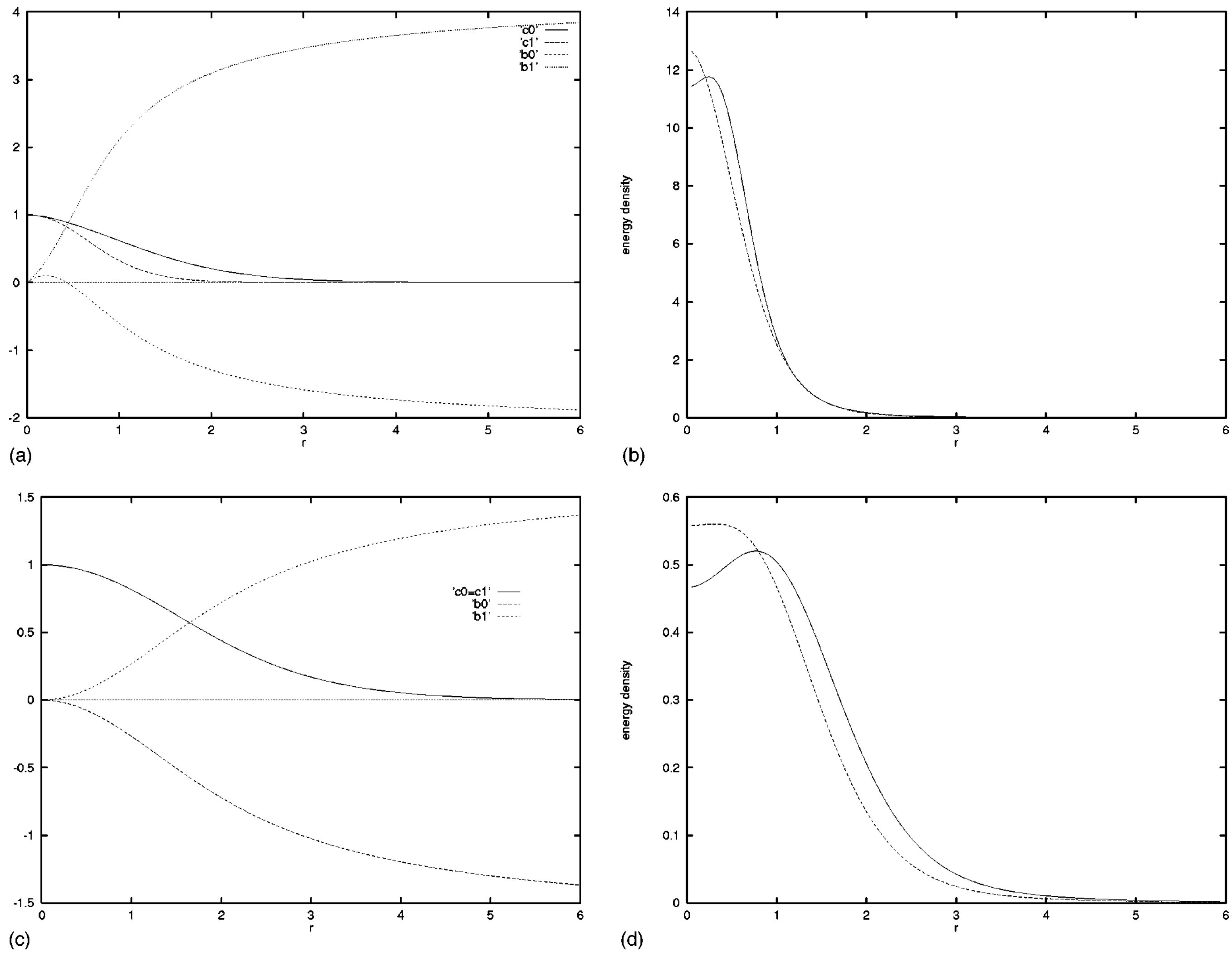

FIG. 1. (a) The profile functions for the SU(3) non-Bogomol'nyi monopole with maximal symmetry breaking. (b) Energy density for SU(3) monopoles with maximal symmetry breaking; non-Bogomol'nyi solution (solid line); Bogomol'nyi solution (dashed line). (c) The profile functions for the SU(3) non-Bogomol'nyi monopole with minimal symmetry breaking. (d) Energy density for SU(3) monopoles with minimal symmetry breaking; non-Bogomol'nyi solution (solid line); Bogomol'nyi solution (dashed line).

monopoles is four then the energy is $E=8$. This value of the energy was obtained from our numerical code to within an accuracy of three decimal places.

In order to consider the non-Bogomol'nyi analogue of this solution we want to fix the monopole masses in the same way as $m_1=m_2=2$. Thus the eigenvalues of Φ_0 must again be 0 and ± 2, though this time their order will not be correct. For example, consider the choice of ordering $\Phi_0=\mathrm{diag}(0,2,-2)$, which by Eq. (4.6) corresponds to the boundary values $b_0(\infty)=-2$ and $b_1(\infty)=4$. Thus, since $b_0(\infty)$ and $b_1(\infty)$ have opposite sign, this gives a non-Bogomol'nyi solution. In Fig. 1(a) we plot the functions b_0, b_1, c_0, c_1 obtained from the numerical solution in this case. In Fig. 1(b) we plot the energy density of this solution (solid line) and the energy density of the corresponding Bogomol'nyi solution (dashed line). Note the dip in the energy density at the origin for the non-Bogomol'nyi solution, so that some energy density isosurfaces will be shell-like. We compute the energy of this solution to be $E=9.0$, so that it is larger than the Bogomol'nyi solution. As discussed above, the entries of the magnetic charge matrix must also be permuted (in accordance with the permutation of the entries in the Higgs field to obtain the correct ordering) and this results in $G_0=\mathrm{diag}(0,2,-2)$. Comparison with Eq. (2.5) then gives the charges as $(n_1,n_2)=(0,2)$. Clearly the energy of this solution has little to do with the Bogomol'nyi bound (2.7), and it would be nice to understand its value. With this aim in mind we now attempt some phenomenology to interpret the charge and energy of this monopole.

As mentioned in Sec. III the projectors used in the harmonic map ansatz have a sigma model interpretation in terms of instanton anti-instanton configurations. However, the Bogomol'nyi monopole solutions clearly have no anti-monopoles, so it appears that in this case the profile functions are such that the monopole does not see any anti-soliton content. Nevertheless, when the profile functions are modified to a non-Bogomol'nyi solution some of the anti-soliton content becomes visible — we have already seen a signature for this in terms of the signs of the asymptotic values of the profile functions. This suggests that we should think of the charge (0,2) solution as the composite (0,2) $=(+2-2,+2-0)$, where the plus signs denote monopoles and the minus signs anti-monopoles (the positive monopole content is taken from the Bogomol'nyi solution). With this interpretation the (0,2) solution contains two monopoles and two monopole anti-monopole pairs. Since the energy of the solution is $E=9.0$ and the monopole mass is 2, this phenomenology gives an approximate value for the energy of a monopole anti-monopole pair as $E_{m\bar{m}}=2.5$. This value is at least reasonable, since (having normalized the monopole mass to 2) the energy of a monopole anti-monopole pair should be something less than 4; the precise value depends on the details of the monopole anti-monopole interaction. Some non-trivial tests of the above interpretation will arise later when we consider SU(4) monopoles. Note that although we know that these solutions are not global minima and we expect them to be unstable (to annihilation of the monopole anti-monopole pairs) we have not proved that they are unstable. A stability analysis would need to be undertaken to prove that the solutions are not local minima, though it would be extremely surprising if they were stable.

In SU(3) there are six possible ways to order the eigenvalues of the Higgs field and the corresponding monopole charges are $\pm(2,2),(0,\pm 2),(\pm 2,0)$, thus all the other monopole solutions are trivially related to the two examples we have discussed.

Let us now turn to minimal symmetry breaking, which we may take to be given by $\Phi_0=\mathrm{diag}(1,-\frac{1}{2},-\frac{1}{2})$. The Bogomol'nyi monopole has boundary values given by $b_0(\infty)=\frac{3}{2}$ and $b_1(\infty)=0$. Since the second profile function vanishes at infinity then the boundary conditions now allow that $c_1(\infty)\neq 0$. As this is a Bogomol'nyi solution then it can be found explicitly and the appropriate boundary condition turns out to be [5] $c_1(\infty)=1/\sqrt{2}$, which gives the charge (2,[1]). Here the notation is that magnetic weights are denoted by square brackets. The monopole masses are $m_1=\frac{3}{2},m_2=0$, so the energy is $E=3$.

Recall that the Bogomol'nyi solutions have the property that all the b_j's at infinity have the same sign, whereas for non-Bogomol'nyi solutions this is not the case. Thus one may wonder whether these two types of solution are smoothly connected as the values of (one or more) $b_j(\infty)$ are varied to change sign. The answer is that they are not, since this smooth variation must pass through the point $b_j(\infty)=0$ where (as we see above) the symmetry breaking pattern changes and the boundary condition on $c_j(\infty)$ suffers a discontinuous jump.

For Bogomol'nyi monopoles a non-maximal symmetry breaking pattern requires the vanishing of at least one $b_j(\infty)$, but for non-Bogomol'nyi monopoles this is not the case. For example, permuting the eigenvalues into the order $\Phi_0=\mathrm{diag}(-\frac{1}{2},1,-\frac{1}{2})$ corresponds to the choice $b_0(\infty)=-b_1(\infty)=-\frac{3}{2}$ and hence the charge (0,[2]). The symmetry of equations (4.2) together with the symmetry of the boundary conditions in this case force the reduction $c_0=c_1$ and $b_0=-b_1$. The minimally broken SU(3) non-Bogomol'nyi monopole that arises from this specific reduction has been obtained previously by Burzlaff [8], using a hedgehog-like ansatz and a group theoretic approach in which the gauge potential involves the principle SU(2) triplet in SU(3) but the Higgs field involves the associated 5-plet. Burzlaff [8] proved the existence of a solution to the equations which result in this case. An important feature of the proof is the fact that the equations arise as the variation of an energy functional and therefore it only remains to show that a minimizer exists, for which standard methods can be employed. In our general case we have the variational formulation which comes from the Yang-Mills-Higgs energy and so it should be possible to use similar techniques to prove the existence of solutions. In this paper we are content with numerical solutions of the profile function equations and these are shown in Fig. 1(c) for this case of charge (0,[2]). In Fig. 1(d) we plot the energy density for this solution (solid line) and the energy density of the corresponding Bogomol'nyi charge (2,[1]) solution (dashed line). Again note the dip in energy density near the origin of the non-Bogomol'nyi solution. The energy of the non-Bogomol'nyi solution is found to be $E=4.3$, which should be compared with the energy $E=3$ of the Bogomol'nyi solution.

C. SU(4)

For $N=4$ there are six profile functions, b_j,c_j, $j=0,1,2$, and the Yang-Mills-Higgs equations reduce to

$$\begin{aligned}
(r^2b_0')' &= 6c_0^2b_0-4c_1^2b_1\\
(r^2b_1')' &= 8c_1^2b_1-3c_0^2b_0-3c_2^2b_2\\
(r^2b_2')' &= 6c_2^2b_2-4c_1^2b_1\\
r^2c_0'' &= c_0(3c_0^2-2c_1^2-1+b_0^2r^2)\\
r^2c_1'' &= c_1\left(4c_1^2-\frac{3}{2}c_0^2-\frac{3}{2}c_1^2-1+b_1^2r^2\right)\\
r^2c_2'' &= c_2(3c_2^2-2c_1^2-1+b_2^2r^2).
\end{aligned} \tag{4.8}$$

If we regard the indices on the profile functions as labelling sites on a linear lattice then we see that the equations involve only a nearest neighbor coupling. Also, there is again a symmetry of reflecting the lattice about its midpoint, which in this case is the interchange $0\leftrightarrow 2$.

A consideration of the associated energy shows that the boundary conditions again require that $c_j(\infty)b_j(\infty)=0$, for all j. In this section we shall mainly be concerned with maximal symmetry breaking, in which case $c_j(\infty)=0$.

The eigenvalues and charges are given by

$$\begin{aligned}
\Phi_0=\frac{1}{4}\mathrm{diag}(&3b_0+2b_1+b_2,2b_1+b_2-b_0,\\
&b_2-b_0-2b_1,-b_0-2b_1-3b_2)
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
G_0=\mathrm{diag}(&3(1-c_0^2),(1+3c_0^2-4c_1^2),\\
&-(1+3c_2^2-4c_1^2),-3(1-c_2^2)).
\end{aligned} \tag{4.10}$$

TABLE I. Boundary conditions, charges and energies for SU(4) monopoles.

$b_0(\infty)$	$b_1(\infty)$	$b_2(\infty)$	(n_1,n_2,n_3)	E	$\tilde{E}$
2	2	2	(3,4,3)	20.0	20.0
2	4	−2	(3,4,1)	21.6	21.0
4	−2	4	(3,2,3)	22.0	21.0
−2	6	−2	(1,4,1)	22.9	22.0
6	−2	−2	(3,0,−1)	22.9	25.5
4	2	−4	(3,0,1)	23.3	23.0
6	−4	2	(3,2,−1)	24.1	24.5
4	−6	4	(−1,2,−1)	27.0	28.0

Normalizing the monopole mass to two, the Bogomol'nyi solution with $b_0(\infty)=b_1(\infty)=b_2(\infty)=2$ has $\Phi_0=\mathrm{diag}(3,1,-1,-3)$. From (4.10) this solution has $G_0=\mathrm{diag}(3,1,-1,-3)$ and hence by comparison with Eq. (2.5) the charge is (3,4,3), with corresponding energy $E=20$.

There are 24 different orderings of the eigenvalues ± 3, ± 1 and of these there are 8 which give fundamentally different monopoles. In Table I we list the values of $b_j(\infty)$ and the corresponding magnetic charges. As in the SU(3) case, the magnetic charges are computed by first finding the permutation required so that Φ_0 has the correct ordering, that is, $\Phi_0=\mathrm{diag}(3,1,-1,-3)$, then applying this permutation to the elements of $G_0=\mathrm{diag}(3,1,-1,-3)$ which can then be compared with the definition (2.5).

In Table I we also list the computed energies, E, of the solutions and the approximate values, $\tilde{E}$, calculated using the monopole anti-monopole interpretation discussed in the previous section. Thus for example, we write $(3,0,1)=(3-0,4-4,3-2)$, where the positive monopole content is taken from the Bogomol'nyi solution. The interpretation of this solution is therefore that it contains six monopole anti-monopole pairs and four monopoles. The approximate energy is then $\tilde{E}=4\times 2+6E_{m\bar{m}}=23$, where we have used the result of the previous section that $E_{m\bar{m}}=2.5$. The true energy of this solution is $E=23.3$ which is in good agreement with the approximate value. A glance at Table I reveals that the approximate energies are in reasonable agreement with the calculated values, which adds support to our interpretation of the solutions in terms of monopole anti-monopole configurations. Only one case, the charge $(3,0,-1)$ solution, shows a significant discrepancy and it is easy to imagine that the finer details of the arrangement of the monopoles/anti-monopoles needs to be taken into account to get a better agreement. Indeed the surprising fact is that our naive counting produces results which are remarkably accurate.

We have only discussed the case of maximal symmetry breaking in this section but solutions for all symmetry breaking patterns can also be found. As an example, the case of minimal symmetry breaking can be obtained by exploiting the symmetry of the equations to set $b_0=-b_2$, $b_1=0$, $c_0=c_2$, $c_1=0$. This gives the SU(4) analogue of Burzlaff's SU(3) solution, and indeed the equations in the two cases have the same structure.

V. CONCLUSION

In this paper we have presented some static spherically symmetric monopole solutions of the SU(N) Yang-Mills-Higgs equations which are not solutions of the Bogomol'nyi equation. Some of the properties of these solutions have been calculated and their interpretation in terms of monopole anti-monopole configurations discussed. A crucial tool in the investigation was the connection with the two-dimensional $\mathbb{CP}^{N-1}$ sigma model, and indeed our solutions may be considered to be the monopole analogues of the sigma model non-instanton solutions. Finally, it is perhaps interesting to note that in the context of string theory Sen [9] has revealed new perspectives on D-branes and other aspects of string theory by consideration of the dynamics of unstable D-brane anti-D-brane configurations. In particular D-branes themselves appear as topological defects in the worldvolume of higher dimensional unstable brane configurations. It remains to be seen whether monopole anti-monopole solutions have any important role to play in this light, but if so then our solutions are perhaps the simplest examples upon which any further studies might be based.

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