High temperature resummation in the linear δ expansion

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The optimized linear δ expansion is applied to the $\lambda\phi^4$ theory at high temperature. Using the imaginary time formalism the thermal mass is evaluated perturbatively up to order δ^2. A variational procedure associated with the method generates nonperturbative results which are used to obtain the critical temperature for the phase transition. Our results are compared with the ones given by propagator dressing methods.
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I. INTRODUCTION

The breakdown of perturbation expansion in high temperature quantum field theory is a well known problem [1,2] whose solution is still a matter of study and discussion today, with different authors using different methods [3–11]. High temperature perturbation expansion breaks down due to the appearance of infrared divergences close to critical temperatures (in field theories displaying a second order phase transition or a weakly first order transition [12,13]), or for massless field theories, such as QCD. In particular, there are parameter regimes where conventional perturbation schemes become unreliable at high temperature when powers of the coupling constants become surmounted by powers of the temperature.

In general, these problems are treated with resummation techniques which try to account, in a self-consistent way, for the leading contributions in the infrared region. Among these schemes are the popular daisy and super-daisy schemes [3,4], composite operator method [6] and field propagator dressing methods [7,8]. Some of these resummation methods have been compared in Ref. [9], where their difficulties and possible caveats have also been discussed. The majority of approaches used within this subject have a potential drawback concerning the achievement of self-consistency as higher order diagrams are resummed. This happens because the dressed propagator changes order by order. Therefore, special care must be taken when selecting the correct order in the coupling constants. Another problem associated with some of these methods is related to the implementation of renormalization, as discussed in Ref. [11].

In this paper, we apply the optimized linear δ expansion [14,15] (for earlier references see, e.g., [16]) to the $\lambda\phi^4$ theory obtaining the thermal mass to second order in the perturbative parameter δ. Our results show that the use of a proper optimization scheme is equivalent to solve the gap equation for the thermal mass, where leading and higher order infrared regularizing contributions are nonperturbatively taken into account. An advantage of the linear δ expansion is that the same simple propagator is used in the evaluation of any diagram, avoiding the potential bookkeeping problems mentioned above. This makes the method particularly useful and simple to use in different applications, including the study of nonperturbative high temperature effects.

This work is organized as follows. In Sec. II we briefly describe the linear δ-expansion technique and then use it to evaluate the thermal mass up to order δ^2 in the $3+1d$ $\lambda\phi^4$ theory. Details of the renormalization up to this order are given in Sec. III where we also discuss renormalization at higher orders in δ. We show that it does not present any additional difficulty when compared to renormalization in the usual perturbative or loop expansions. In Sec. IV we present our results for the thermal mass, including the critical temperature for the phase transition, and compare them with other results found in the literature. In Sec. V concluding remarks are given.

II. THE LINEAR δ EXPANSION APPLIED TO THE EVALUATION OF THE THERMAL MASS IN THE $\lambda\phi^4$ THEORY

A. The linear δ expansion

The optimized linear δ expansion is an alternative nonperturbative approximation which has been successfully used in a plethora of different problems in particle theory [15,17–19], quantum mechanics [20,21] statistical physics [22], nuclear matter [23] and lattice field theory [24]. One advantage of this method is that the selection and evaluation (including renormalization) of Feynman diagrams are done exactly as in ordinary perturbation theory using a very simple modified propagator which depends on an arbitrary mass parameter. Nonperturbative results are then obtained by fixing this parameter. An interesting result obtained with this method in the finite temperature domain is given in Ref. [18] where the critical temperature value for the Gross-Neveu

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model in 1+1 dimensions nicely converges, order by order, towards the exact result set by Landau's theorem ($T_c=0$).[1]

The standard application of the linear δ expansion to a theory described by a Lagrangian density $\mathcal{L}$ starts with an interpolation defined by

$$\mathcal{L}^{\delta}=(1-\delta)\mathcal{L}_0(\eta)+\delta\mathcal{L}=\mathcal{L}_0(\eta)+\delta[\mathcal{L}-\mathcal{L}_0(\eta)], \tag{2.1}$$

where $\mathcal{L}_0(\eta)$ is the Lagrangian density of a solvable theory which can contain an arbitrary mass parameter (η). The Lagrangian density $\mathcal{L}^{\delta}$ interpolates between the solvable $\mathcal{L}_0(\eta)$ (when $\delta=0$) and the original $\mathcal{L}$ (when $\delta=1$). In this work we consider the $\lambda\phi^4$ model described by

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4!}\phi^4+\mathcal{L}_{\rm ct}, \tag{2.2}$$

where

$$\mathcal{L}_{\rm ct}=A\frac{1}{2}(\partial_\mu\phi)^2-B\frac{1}{2}m^2\phi^2-\frac{\lambda}{4!}C\phi^4 \tag{2.3}$$

represents the counterterms needed to render the model finite. Note that $\mathcal{L}_{\rm ct}$ requires an extra piece if one attempts to evaluate the thermal effective potential [11], which is not the case here. Choosing

$$\mathcal{L}_0(\eta)=\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}m^2\phi^2-\frac{1}{2}\eta^2\phi^2 \tag{2.4}$$

and following the general prescription one can write

$$\mathcal{L}^{\delta}=\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}m^2\phi^2-\frac{\delta\lambda}{4!}\phi^4-\frac{1}{2}(1-\delta)\eta^2\phi^2+\mathcal{L}_{\rm ct}^{\delta}, \tag{2.5}$$

or

$$\mathcal{L}^{\delta}=\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}\Omega^2\phi^2-\frac{\delta\lambda}{4!}\phi^4+\delta\frac{1}{2}\eta^2\phi^2+\mathcal{L}_{\rm ct}^{\delta}, \tag{2.6}$$

where $\Omega^2=m^2+\eta^2$ and

$$\mathcal{L}_{\rm ct}^{\delta}=A^{\delta}\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}\Omega^2B^{\delta}\phi^2-\frac{\delta\lambda}{4!}C^{\delta}\phi^4+\delta\frac{1}{2}\eta^2B^{\delta}\phi^2. \tag{2.7}$$

One should note that the δ-expansion interpolation introduces only ''new'' quadratic terms not altering the renormalizability of the original theory. That is, the counterterms contained in $\mathcal{L}_{\rm ct}^{\delta}$, as well as in the original $\mathcal{L}_{\rm ct}$, have the same polynomial structure.

The general way the method works becomes clear by looking at the Feynman rules generated by $\mathcal{L}^{\delta}$. First, the original ϕ^4 vertex has its original Feynman rule $-i\lambda$ modified to $-i\delta\lambda$. This minor modification is just a reminder that one is really expanding in orders of the artificial parameter δ. Most importantly, let us look at the modifications implied by the addition of the arbitrary quadratic part. The original bare propagator

$$S(k)=i(k^2-m^2+i\epsilon)^{-1}, \tag{2.8}$$

becomes

$$S(k)=i(k^2-\Omega^2+i\epsilon)^{-1}$$
$$=\frac{i}{k^2-m^2+i\epsilon}\left[1-\frac{i}{k^2-m^2+i\epsilon}(-i\eta^2)\right]^{-1}, \tag{2.9}$$

indicating that the term proportional to $\eta^2\phi^2$ contained in $\mathcal{L}_0$ is entering the theory in a nonperturbative way. On the other hand, the piece proportional to $\delta\eta^2\phi^2$ is only being treated perturbatively as a quadratic vertex (of weight $i\delta\eta^2$). Since only an infinite order calculation would be able to compensate for the infinite number of $(-i\eta^2)$ insertions contained in Eq. (2.9), one always ends up with a η dependence in any quantity calculated to finite order in δ. Then, at the end of the calculation one sets $\delta=1$ (the value at which the original theory is retrieved) and fixes η with the variational procedure known as the principle of minimal sensitivity (PMS) [27][2]

$$\left.\frac{\partial P(\eta)}{\partial\eta}\right|_{\bar{\eta}}=0, \tag{2.10}$$

where P represents a physical quantity calculated *perturbatively* in powers of δ.

This optimization procedure, together with the convergence problem, has been discussed in detail for simple cases in low dimensions in Refs. [20] and [21] where possible implications to more realistic theories have also been investigated. Both references provide proofs of convergence. Using the anharmonic oscillator, Bellet *et al.* [28] have also studied the convergence of an alternative version of the linear δ expansion. Their method has been later extended to the Gross-Neveu model where the optimization procedure was studied in conjunction with the renormalization group [29]. The convergence of the δ expansion in quantum field theory is still a subject deserving further investigation. In principle, it seems that if the convergence proof conditions for the anharmonic oscillator could be extended to quantum field theory in $d<4$ [21], one could pursue a similar investigation for the $\lambda\phi^4$ theory in $3+1d$ at finite temperature. This is due to the fact that at very high temperatures this model gets

[1]Consistent results in the $\lambda\phi^4$ theory at finite temperature have also been obtained with a variant of the linear δ expansion, the nonlinear δ expansion [25,26]. However, beyond first order, this latter version presents cumbersome technical problems associated with the evaluation of graphs.

[2]The third reference in [20] discusses alternative conditions for fixing η.

FIG. 1. Diagrams contributing to the self-energy at first order in δ.

dimensionally reduced to an effective $3d$ theory for the zero Matsubara field modes [30]. However, we shall not pursue this discussion.

As far as renormalization is concerned it is important to note that in general, as a result of the optimization procedure, the arbitrary η turns out to be a function of the original model parameters, scales introduced through regularization as well as external parameters such as the temperature and/or density. Therefore, in order to get physically acceptable results the optimization procedure must be carried out after all divergences have been eliminated. The renormalization problem, in the large-N limit, has been addressed in Ref. [31]. The way renormalization will be carried out here is well discussed in Ref. [11].

B. The evaluation of the thermal mass at order δ^2

We can now start our evaluation of the thermal mass, defined by

$$M_T^2 = \Omega^2 + \Sigma_T^{\delta}(p), \tag{2.11}$$

where $\Sigma_T^{\delta}(p)$ is the thermal self-energy. At lowest order (first order in δ) the relevant contributions, which are momentum independent, are shown in Fig. 1 and given by

$$\Sigma_T^{\delta^1}(p) = -\delta\eta^2 + \delta\frac{\lambda}{2}\int_T \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\Omega^2+i\epsilon}. \tag{2.12}$$

The temperature dependence can be readily obtained by using the imaginary time formalism prescription (see, e.g. [1])

$$p_0 \to i\omega_n, \quad \int_T \frac{d^4k}{(2\pi)^4} \to iT\sum \int \frac{d^3\mathbf{k}}{(2\pi)^3}. \tag{2.13}$$

Then, the self-energy becomes

$$\Sigma_T^{\delta^1}(p) = -\delta\eta^2 + \delta T\frac{\lambda}{2}\sum \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\omega_n^2+E^2}, \tag{2.14}$$

where $E^2 = \mathbf{k}^2 + \Omega^2$. Summing over Matsubara's frequencies one gets

$$\Sigma_T^{\delta^1}(p) = -\delta\eta^2 + \delta\frac{\lambda}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left\{\frac{1}{2E} - \frac{1}{E[1-\exp(E/T)]}\right\}. \tag{2.15}$$

Then, using dimensional regularization [32] one obtains the thermal mass

$$M_T^2 = \Omega^2 - \delta\eta^2 + \delta\frac{\lambda}{32\pi^2}\Omega^2\left[-\frac{1}{\epsilon} + \ln\left(\frac{\Omega^2}{4\pi\mu^2}\right) + \gamma_E - 1\right] + \delta\lambda T^2 h\left(\frac{\Omega}{T}\right), \tag{2.16}$$

where μ is a mass scale introduced by dimensional regularization and

$$h(y) = \frac{1}{4\pi^2}\int_0^\infty dx \frac{x^2}{[x^2+y^2]^{1/2}[\exp(x^2+y^2)^{1/2}-1]}, \tag{2.17}$$

where $x = k/T$. Note that the temperature independent term diverges and must be renormalized. In this paper we chose the minimal subtraction (MS) scheme where the counterterms eliminate the poles only. At this order the only divergence is

$$\Sigma_{\text{div}}^{\delta^1}(\Omega^2) = -\delta\frac{\lambda\Omega^2}{32\pi^2\epsilon}, \tag{2.18}$$

which is easily eliminated by the $\mathcal{O}(\delta)$ mass counterterm

$$\Sigma_{\text{ct}}^{\delta^1}(\Omega^2) = B^{\delta^1}\Omega^2 = \left(\delta\frac{\lambda}{32\pi^2\epsilon}\right)\Omega^2. \tag{2.19}$$

By looking at Eq. (2.16) one can see that the terms proportional to $\delta\lambda$ represent exactly the same diagram which appears at $\mathcal{O}(\lambda)$ in ordinary $\lambda\phi^4$ theory, excepted that we now have Ω^2 instead of m^2 and $\delta\lambda$ instead of λ. Therefore, it is not surprising that to this order the renormalization procedure implied by the interpolated theory is identical to the procedure implied by the original theory at $\mathcal{O}(\lambda)$.

Let us now analyze the temperature dependent integral which is expressed, in the high temperature limit ($\Omega/T \ll 1$), as [33]

$$h(y) = \frac{1}{24} - \frac{1}{8\pi}y - \frac{1}{16\pi^2}y^2\left[\ln\left(\frac{y}{4\pi}\right) + \gamma_E - \frac{1}{2}\right] + \cdots. \tag{2.20}$$

In principle, since η is arbitrary, one could be reluctant in taking the limit $\Omega/T \ll 1$. Therefore, to be sure that the PMS can be safely applied to the thermal mass in the high temperature limit, we have performed numerical calculations using both forms for the integral $h(y)$ finding that the optimization results do not lead to any significant numerical changes. Then, taking the integral $h(y)$ in the high temperature limit, one obtains the $\mathcal{O}(\delta)$ thermal mass:

$$M_T^2 = \Omega^2 - \delta\eta^2 + \delta\frac{\lambda T^2}{24} - \delta\frac{\lambda T\Omega}{8\pi} + \delta\frac{\lambda\Omega^2}{32\pi^2}\left[\ln\left(\frac{4\pi T^2}{\mu^2}\right) - \gamma_E\right]. \tag{2.21}$$

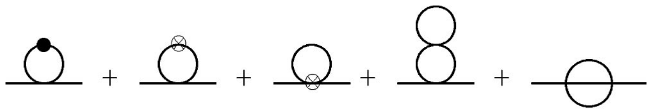

FIG. 2. Diagrams which are order δ^2 contributing to the self-energy.

At $\mathcal{O}(\delta^2)$ the self-energy receives contributions from momentum independent as well as momentum dependent diagrams. At this order there are five diagrams contributing, which are shown in Fig. 2. Let us first consider the momentum independent diagram given by the first diagram in Fig. 2, which we call $\Sigma_1^{\delta^2}$. In the high temperature approximation it is given by

$$\Sigma_1^{\delta^2} \simeq \delta^2 \frac{\lambda T \eta^2}{16\pi\Omega} - \delta^2 \frac{\lambda \eta^2}{32\pi^2}\left[-\frac{1}{\epsilon} + \ln\left(\frac{4\pi T^2}{\mu^2}\right) - \gamma_E\right]. \tag{2.22}$$

As will be shortly seen, this contribution can be rendered finite using a mass type counterterm contained in $\mathcal{L}_{\rm ct}^{\delta}$ which is tailored to account for divergences arising from the extra quadratic vertex introduced during the interpolation process.

Considering the $\mathcal{O}(\delta)$ mass counterterm used to eliminate the divergence in $\Sigma_{\rm div}^{\delta^1}$ [see Eq. (2.19)], one is able to build a one loop $\mathcal{O}(\delta^2)$ diagram whose contribution is given by the second diagram in Fig. 2. In the high temperature approximation one obtains

$$\Sigma_2^{\delta^2} \simeq -\delta^2 \frac{\lambda^2\Omega^2}{(32\pi^2)^2\epsilon^2} + \delta^2 \frac{\lambda^2}{32\pi^2\epsilon}\left\{-\frac{T\Omega}{16\pi} + \frac{\Omega^2}{32\pi^2}\left[\ln\left(\frac{4\pi T^2}{\mu^2}\right) - \gamma_E\right]\right\} - \delta^2 \frac{\lambda^2\Omega^2}{2(32\pi^2)^2}\left\{\left[\ln\left(\frac{\Omega^2}{4\pi\mu^2}\right) + \gamma_E\right]^2 + \frac{\pi^2}{6}\right\}. \tag{2.23}$$

Next, one considers the vertex counterterm, whose Feynman rule, $-3i\delta^2\lambda^2/(32\pi^2\epsilon)$, can be obtained, as in Ref. [32], by evaluating the order-δ^2 contribution to the four point function. The one loop graph evaluated with this counterterm, given by the third diagram in Fig. 2, gives

$$\Sigma_3^{\delta^2} \simeq -\delta^2 \frac{3\lambda^2\Omega^2}{(32\pi^2)^2\epsilon^2} + \delta^2 \frac{3\lambda^2}{32\pi^2\epsilon}\left\{\frac{T^2}{24} - \frac{T\Omega}{8\pi} + \frac{\Omega^2}{32\pi^2}\left[\ln\left(\frac{4\pi T^2}{\mu^2}\right) - \gamma_E\right]\right\} - \delta^2 \frac{3\lambda^2\Omega^2}{2(32\pi^2)^2}\left\{\left[\ln\left(\frac{\Omega^2}{4\pi\mu^2}\right) - \gamma_E - 1\right]^2 + 1 + \frac{\pi^2}{6}\right\}. \tag{2.24}$$

The next momentum independent contribution is given by the first two loop diagram shown in Fig. 2:

$$\begin{aligned}\Sigma_4^{\delta^2} \simeq\, & \delta^2 \frac{\lambda^2\Omega^2}{(32\pi^2)^2}\frac{1}{\epsilon^2} - \delta^2\frac{\lambda^2}{32\pi^2}\frac{1}{\epsilon}\left\{\frac{T^2}{24} - \frac{3T\Omega}{16\pi} + \frac{\Omega^2}{16\pi^2}\left[\ln\left(\frac{4\pi T^2}{\mu^2}\right) - \gamma_E\right]\right\} \\ & - \delta^2\lambda^2\frac{T^3}{384\pi\Omega} + \delta^2\lambda^2\frac{T^2}{128\pi^2} + \delta^2\frac{\lambda^2}{(16\pi)^2}\left\{\frac{T^2}{3} - \frac{3T\Omega}{2\pi} + \frac{\Omega^2}{4\pi^2}\left[\ln\left(\frac{4\pi T^2}{\mu^2}\right) - \gamma_E\right]\right\} \\ & \times\left[\ln\left(\frac{4\pi T^2}{\mu^2}\right) - \gamma_E\right] + \delta^2\frac{\lambda^2\Omega^2}{(32\pi^2)^2}\left[\ln^2\left(\frac{\Omega^2}{4\pi\mu^2}\right) + (2\gamma_E - 1)\ln\left(\frac{\Omega^2}{4\pi\mu^2}\right) + 2.4\right].\end{aligned} \tag{2.25}$$

To render this diagram finite one needs mass and vertex counterterms [32].

The final contribution to the self-energy at $\mathcal{O}(\delta^2)$ comes from the two-loop ''setting sun'' diagram shown by the last term in Fig. 2. This is a momentum dependent contribution which is given by the real part of

$$\Sigma_5^{\delta^2} = -\delta^2\frac{\lambda^2}{6}(G_0 + G_1 + G_2), \tag{2.26}$$

where G_0 is the zero temperature part (in Euclidean time) of the diagram and G_1 and G_2 are the finite temperature ones (with one and two Bose factors, respectively). $\mathrm{Re}[G_0]$ is given by $(d = 4 - 2\epsilon)$

$$\mathrm{Re}[G_0(p)] = \mu^{4\epsilon}\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dq}{(2\pi)^d}\frac{1}{k^2+\Omega^2}\frac{1}{q^2+\Omega^2}\frac{1}{(p-k-q)^2+\Omega^2}. \tag{2.27}$$

This contribution has been evaluated in details in Ref. [34] where the quoted result is

$$\mathrm{Re}[G_0(p)]=\frac{\mu^{4\epsilon}}{(2\pi)^{2d}}\frac{\pi^{d+1}(\Omega^2)^{d-3}}{\sin\pi\left(\frac{d}{2}-2\right)}\sum_{k,n=0}^{\infty}(-1)^n\left(\frac{p}{\Omega}\right)^{2n}\frac{1}{n!\,\Gamma\left(\frac{d}{2}+n\right)}$$
$$\times\left[\frac{\Gamma\left(\frac{d}{2}+k\right)B(1+k,1+k)\Gamma\left(2-\frac{d}{2}+k\right)}{(k-n)!\,\Gamma\left(\frac{d}{2}+k-n\right)}-\frac{\Gamma(2+k)B\left(3-\frac{d}{2}+k,3-\frac{d}{2}+k\right)\Gamma(4-d+k)}{(k-n+1)!\,\Gamma\left(3+k-n-\frac{d}{2}\right)}\right]$$
$$+\frac{\mu^{4\epsilon}}{(2\pi)^{2d}}\pi^d(\Omega^2)^{d-3}\Gamma\left(\frac{d}{2}-1\right)\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(3-d+n)}{\Gamma\left(\frac{d}{2}+n\right)}\left(\frac{p}{\Omega}\right)^{2n}B\left(2-\frac{d}{2}+n,2-\frac{d}{2}+n\right), \tag{2.28}$$

where $B(x,y)$ is the Beta function: $B(x,y)=[\Gamma(x)\Gamma(y)]/\Gamma(x+y)$. In what follows we evaluate the self-energy on-shell ($\vec{p}=\mathbf{0},\ p_0=-i\Omega$). For $\epsilon\to 0$, we obtain the following result for the above expression:

$$\frac{\delta^2\lambda^2}{6}\mathrm{Re}[G_0(-i\Omega,\mathbf{0})]=\frac{\delta^2\lambda^2\Omega^2}{4(4\pi)^4}\left[\frac{1}{\epsilon^2}+\frac{3-2\gamma_E}{\epsilon}-\frac{2}{\epsilon}\ln\left(\frac{\Omega^2}{4\pi\mu^2}\right)\right]+\frac{\delta^2\lambda^2p^2}{4(4\pi)^4}\frac{1}{6\epsilon}$$
$$+\frac{\delta^2\lambda^2\Omega^2}{2(4\pi)^4}\left[\ln^2\left(\frac{\Omega^2}{4\pi\mu^2}\right)+\left(2\gamma_E-\frac{17}{6}\right)\ln\left(\frac{\Omega^2}{4\pi\mu^2}\right)+\gamma_E^2-\frac{17\gamma_E}{6}+3.5140\right], \tag{2.29}$$

where we purposefully left the momentum dependence in the relevant divergent term to make explicit the need for a wave-function renormalization counterterm.

The finite temperature contributions G_1 and G_2 are given, as in Ref. [8], by

$$-\delta^2\frac{\lambda^2}{6}\mathrm{Re}[G_1(-i\Omega,\mathbf{0})]=F_0+F_1+F_2, \tag{2.30}$$

where

$$F_0=-\delta^2\frac{\lambda^2T^2}{(4\pi)^2}\frac{1}{\epsilon}h\left(\frac{\Omega}{T}\right), \tag{2.31}$$

$$F_1=-\delta^2\frac{\lambda^2T^2}{2(4\pi)^2}h\left(\frac{\Omega}{T}\right)\left[-\ln\left(\frac{\Omega^2}{4\pi\mu^2}\right)+2-\gamma_E\right] \tag{2.32}$$

and

$$F_2=-\delta^2\frac{\lambda^2}{8(2\pi)^4}\int_0^\infty dk\frac{k}{E_k(e^{\beta E_k}-1)}\int_0^\infty\frac{dq}{E_q}\left[q\ln\left|\frac{X_+}{X_-}\right|-4k\right], \tag{2.33}$$

$$X_\pm=[\Omega^2-(E_k+E_q+E_{k\pm q})^2][\Omega^2-(-E_k+E_q+E_{k\pm q})^2]. \tag{2.34}$$

For G_2, one has [8]

$$-\frac{\delta^2\lambda^2}{6}\mathrm{Re}[G_2(-i\Omega,\mathbf{0})]=H(\Omega)=-\frac{\delta^2\lambda^2\Omega^2}{4(2\pi)^4}\int_1^\infty\frac{dx}{e^{\beta\Omega x}-1}\int_1^\infty\frac{dy}{e^{\beta\Omega y}-1}\ln\left|\frac{\sqrt{x^2-1}+\sqrt{y^2-1}}{\sqrt{x^2-1}-\sqrt{y^2-1}}\right|. \tag{2.35}$$

In the high temperature limit, one can show [8] that

$$F_2+H(\Omega)\simeq\delta^2\frac{\lambda^2T^2}{24(4\pi)^2}\left[\ln\left(\frac{\Omega^2}{T^2}\right)+5.0669\right]. \tag{2.36}$$

Finally, using the high temperature approximation for $h(y)$ and putting all together one gets

$$\mathrm{Re}[\Sigma_5^{\delta^2}(p)] \simeq \delta^2 \frac{\lambda^2\Omega^2}{(32\pi^2)^2}\frac{1}{\epsilon^2} + \delta^2 \frac{\lambda^2\Omega^2}{(32\pi^2)^2}\frac{1}{\epsilon} + \delta^2 \frac{\lambda^2 p^2}{(32\pi^2)^2}\frac{1}{6\epsilon} - \delta^2 \frac{\lambda^2}{16\pi^2\epsilon} \times\left\{\frac{T^2}{24} - \frac{T\Omega}{8\pi} + \frac{\Omega^2}{32\pi^2}\left[\ln\left(\frac{4\pi T^2}{\mu^2}\right) - \gamma_E\right]\right\} + \delta^2 \frac{\lambda^2\Omega^2}{2(4\pi)^4}\left[\ln^2\left(\frac{\Omega^2}{4\pi\mu^2}\right) + \left(2\gamma_E - \frac{17}{6}\right)\ln\left(\frac{\Omega^2}{4\pi\mu^2}\right) + \gamma_E^2 - \frac{17\gamma_E}{6} + 3.5140\right] - \delta^2 \frac{\lambda^2}{32\pi^2}\left[-\ln\left(\frac{\Omega^2}{4\pi\mu^2}\right) + 2 - \gamma_E\right] \times\left\{\frac{T^2}{24} - \frac{T\Omega}{8\pi} - \frac{\Omega^2}{16\pi^2}\left[\ln\left(\frac{\Omega}{4\pi T}\right) + \gamma_E - \frac{1}{2}\right]\right\} + \delta^2 \frac{\lambda^2 T^2}{24(4\pi)^2}\left[\ln\left(\frac{\Omega^2}{T^2}\right) + 5.0669\right]. \tag{2.37}$$

III. ON THE RENORMALIZATION AT ORDER δ^2 AND AT ORDER δ^n

To obtain the total finite order δ^2 contribution one can add all divergences appearing in Eqs. (2.22)–(2.25) and (2.37). As it can be easily seem all the nonrenormalizable temperature dependent divergences cancel exactly and one is left with

$$\Sigma_{\rm div}^{\delta^2} = \Sigma_{\rm div}^{\delta^2}(\Omega^2) + \Sigma_{\rm div}^{\delta^2}(p^2) + \Sigma_{\rm div}^{\delta^2}(\eta^2), \tag{3.1}$$

where

$$\Sigma_{\rm div}^{\delta^2}(\Omega^2) = \delta^2 \frac{\lambda^2\Omega^2}{(32\pi^2)^2}\left(-\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\right), \tag{3.2}$$

$$\Sigma_{\rm div}^{\delta^2}(p^2) = \delta^2 \frac{\lambda^2}{(32\pi^2)^2}\frac{p^2}{6\epsilon}, \tag{3.3}$$

and

$$\Sigma_{\rm div}^{\delta^2}(\eta^2) = \delta^2 \frac{\lambda\eta^2}{(32\pi^2)}\frac{1}{\epsilon}. \tag{3.4}$$

By looking at all diagrams which contribute to this order one can identify two classes. The first is composed by diagrams such as the ones described by Eqs. (2.23)–(2.26). All of them are analogous to the diagrams which appear at $\mathcal{O}(\lambda^2)$ in the original theory and can be rendered finite by similar mass and wave-function counterterms, which are respectively

$$\Sigma_{\rm ct}^{\delta^2}(\Omega^2) = B^{\delta^2}\Omega^2 = -\delta^2 \frac{\lambda^2}{(32\pi^2)^2}\left(-\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\right)\Omega^2, \tag{3.5}$$

and

$$\Sigma_{\rm ct}^{\delta^2}(p^2) = A^{\delta^2}p^2 = -\delta^2 \frac{\lambda^2}{(32\pi^2)^2}\frac{p^2}{6\epsilon}. \tag{3.6}$$

The second kind of diagram is exclusive of the interpolated theory and carries at least one $\delta\eta^2$ vertex. At $\mathcal{O}(\delta^2)$ this diagram is described by Eq. (2.22) which displays the divergent term $\Sigma_{\rm div}^{\delta^2}(\eta^2)$. Looking at $\mathcal{L}_{\rm ct}^{\delta}$ one identifies a η^2 counterterm whose Feynman rule is $i\delta B^{\delta}\eta^2$. Since the actual pole is of order δ^2 one then identifies the coefficient as being B^{δ^1}, displayed in Eq. (2.18). Then

$$\Sigma_{\rm ct}^{\delta^2}(\eta^2) = -\delta B^{\delta^1}\eta^2 = -\delta^2 \frac{\lambda}{32\pi^2\epsilon}\eta^2. \tag{3.7}$$

In practice, the renormalization at higher orders can be done as above. That is, $\mathcal{O}(\delta^n)$ diagrams belonging to the first class will be renormalized exactly as in the original theory at $\mathcal{O}(\lambda^n)$. This is obvious from the fact that all the diagrams in this class are of order-$\delta^n\lambda^n$. It is easy to check that for those diagrams the most divergent terms will display ϵ^{-n} poles. On the other hand, $\mathcal{O}(\delta^n)$ diagrams belonging to the second class will make use of the counterterm $\delta B^{\delta^{n-1}}\eta^2$, where $B^{\delta^{1-n}}$ has been evaluated in a previous order. One can also easily check that for these diagrams the most divergent terms will have ϵ^{n-1} poles. Moreover, power counting reveals that those $\delta\eta^2$ insertions make the loops more convergent. For example, *all* one loop diagrams of order $\mathcal{O}(\delta^n)$, with $n \geq 3$ are finite.

Finally, one should note that the renormalization prescription adopted here is in accordance with the one suggested in Ref. [11], where the order by order renormalization was shown to hold at any higher orders in δ.

IV. NUMERICAL RESULTS

One can now set $\delta = 1$ and apply the PMS to the finite thermal mass. First let us set $m = 0$ so that our results for the thermal mass can be compared directly with the resummed perturbative expansion (RPE) results of Ref. [8]. At order-δ one gets

$$\bar{\eta} = 2\pi T\left[\ln\left(\frac{4\pi T^2}{\mu^2}\right) - \gamma_E\right]^{-1}, \tag{4.1}$$

TABLE I. Results for M_T^2/μ^2 $(\times 10^{-2})$.

T/μ	$\overline{\eta}/\mu$	$O(\delta^2)$	RPE
0.5	0.033	0.098	0.099
1.0	0.065	0.393	0.396
1.5	0.098	0.884	0.892
2.0	0.130	1.572	1.587
2.5	0.163	2.457	2.481
3.0	0.195	3.538	3.574
3.5	0.228	4.815	4.867
4.0	0.260	6.289	6.358
4.5	0.293	7.960	8.049
5.0	0.325	9.827	9.939
5.5	0.358	11.890	12.029
6.0	0.390	14.151	14.317
6.5	0.423	16.607	16.805
7.0	0.455	19.260	19.493

which, clearly, does not depend on the the coupling and cannot generate nonperturbative information. However, nonperturbative results appear already at second order in δ. Table I shows our results and the results furnished by the RPE for M_T^2 in units of μ for $\lambda = 0.1$. We also show, in units of μ, the optimal values of η.

Let us now obtain the critical temperature for the phase transition at $\mathcal{O}(\delta^2)$. Taking $\lambda = 0.1$, we reset $m^2 = -\mu^2$ in M_T^2 observing a second order phase transition at the critical temperature $T_c = 15.57\ \mu$ whereas the modified perturbation scheme (MPS) of Banerjee and Mallik [7] predicts $T_c = 15.63\ \mu$. Choosing $\lambda = 0.01$ we find $T_c = 49.03\ \mu$ whereas the value $T_c = 49.05\ \mu$ is predicted by the MPS. Note that in the calculation of the critical temperature performed in Ref. [7] the propagator has been effectively dressed up to the leading order correction in the temperature, which is set by the tadpole term in Eq. (2.12) [see their Eq. (4.5)]. Here, on the other hand, we are definitely working with all higher order corrections up to the two-loop contribution. The fact that our results for the critical temperature are slightly different than those obtained in Ref. [7] is an indication of the importance of these higher order corrections and is in accordance with well known results concerning the study of phase transitions in the context of the electroweak effective potential beyond 1-loop [13]. The results are also in accordance with recent results for the finite temperature effective potential of the $\lambda\phi^4$ theory, obtained with the superdaisy approximation [35].

V. CONCLUSIONS

Using the $\lambda\phi^4$ model we have shown how the optimized δ expansion can be useful in extracting nonperturbative information through an essentially perturbative evaluation of Feynman graphs. Our $\mathcal{O}(\delta^2)$ results for the thermal mass and for the phase transition critical temperature are in excellent agreement with the ones given by other methods [7,8]. However, although providing very similar results, these methods differ, from the δ expansion, in some aspects which may become important if one tries to consider higher orders. For example, within the latter methods the effective mass used in the modified propagator changes order by order turning the propagator into a coupling dependent quantity from the start. One can then expect the selection and evaluation of higher contributions to become complicated quickly. The δ-expansion method avoids these potential problems by using Ω in the modified propagator which is used at any order calculation. Therefore, after drawing the relevant graphs which contribute to a given order, one does not have to worry about bookkeeping inconsistencies nor renormalization problems since this is done as in perturbation theory. The extension to higher order in δ is immediate and, as discussed above, leads to a consistent resummation procedure in finite temperature field theory.

Although we have not attempted to prove the possibility of convergence of our results we have explicitly shown that the procedure interpolation, renormalization, and optimization in the finite temperature domain can be consistently handled to furnish encouraging results.

We also note that the linear δ expansion can be extended to the case of gauge theories, where it has already been used as a tool to study the electroweak phase transition on the lattice [24]. Recently, it has been shown [36] that the method does not spoil gauge invariance. In this context, the linear δ expansion may be a useful technique to analytically study the nonperturbative aspects and difficulties associated, for example, with the electroweak phase transition as well as other problems in high temperature gauge theories.

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