QED effective action at finite temperature

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The QED effective Lagrangian in the presence of an arbitrary constant electromagnetic background field at finite temperature is derived in the imaginary-time formalism to one-loop order. The boundary conditions in imaginary time reduce the set of gauge transformations of the background field, which allows for a further gauge invariant and puts restrictions on the choice of gauge. The additional invariant enters the effective action by a topological mechanism and can be identified with a chemical potential; it is furthermore related to Debye screening. In concordance with the real-time formalism, we do not find a thermal correction to Schwinger's pair-production formula. The calculation is performed on a maximally Lorentz covariant and gauge invariant stage. [S0556-2821(99)07018-6]

I. INTRODUCTION

The construction of an effective action for quantum electrodynamics (QED) in the presence of various external conditions has been a challenge since the early days of the theory. The study of generalizations of the Heisenberg-Euler Lagrangian that include finite-temperature effects has been initiated by Dittrich [1], who considered the case of a constant external magnetic field at finite temperature using the imaginary time formalism. An extension of this work to the case of arbitrary constant electromagnetic fields turned out to be qualitatively more substantial than naively expected. Employing the real-time formalism, this situation was investigated by Cox, Hellman and Yildiz [2] and Loewe and Rojas [3]. A more comprehensive study of the problem has been performed by Elmfors and Skagerstam [4], who corrected the preceding findings and additionally introduced a chemical potential. An attempt employing the imaginary-time formalism was made by Ganguly, Kaw and Parikh [5] for the case of an external electric field. Recently, the finite-temperature effective action for electromagnetic fields was studied by Shovkovy [6] in the world line approach, where finite temperature is also introduced via an imaginary-time formalism.

This paper is devoted to the derivation of the effective action of arbitrary constant electromagnetic fields at finite temperature in the imaginary-time formalism. Similarly to the above-mentioned papers, our approach is based on Schwinger's proper-time formalism [7] and refers to the one-loop level. By assigning a 4-velocity vector to the motion of the observer with respect to the heat bath, a manifest covariant notation is obtained [8] which enables us to formulate the problem in terms of gauge invariant and covariant quantities.

However, gauge transformations of the finite-temperature generating functional are *a priori* restricted to periodic gauge functions Λ_p in order to leave the boundary conditions of the functional integral over the fluctuating field invariant. This requires a more careful choice of gauge for the background field than at zero temperature, since the reduced class of

gauge transformations allows for more physical information to be carried in the explicit form of the gauge potential. The additional information can be associated with a chemical potential.

Apart from subtleties with the correct choice of gauge, we largely agree with the findings of the "real-time" investigations [4]. We finally comment on the apparent controversy in the literature concerning the (non-)vanishing of the imaginary part of the thermal effective action that is related to pair production [2–5,9,10]. In concordance with the findings of the real-time calculations, we do not find a thermal contribution to the pair-production rate to this order of calculation.

It is, of course, obligatory to point out that the implications of the present calculation may not be immediately interpretable, since the presence of an electric field violates the thermal equilibrium assumption of the imaginary-time formalism. In particular, a constant electric field transfers energy to thermally fluctuating charged particles. On a formal level, it is not clear whether the periodicity in imaginary time can be identified with the physical temperature of the heat bath. However, there are field configurations allowing for thermal equilibrium, e.g., a shallow potential well as suggested in [4], for which the constant field approximation can be applicable.

Moreover, the knowledge of the effective action given below depending on the complete set of invariants of an electromagnetic field including an additional Lorentz vector (temperature times heat-bath velocity) might be useful even in the limit of vanishing electric fields.

II. IMAGINARY-TIME FORMALISM

The one-loop effective action of QED is characterized by the fact that the fluctuating charged fermions which couple to the external field to all orders have been integrated out. In this way, finite temperature is introduced via the imaginary-time formalism by postulating anti-periodic boundary conditions for these fluctuating fermions in the direction of imaginary time with period $\beta = 1/T$. Regarding the complete generating functional of QED, the external field is treated as a background field [11]. To maintain invariance of the fermionic integral under gauge transformations of the back-

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ground field, it is important to restrict the gauge functions $\Lambda(x)$ to be β-periodic in imaginary time[1]:

$$\{\Lambda_p\}: \quad \Lambda_p(x^\mu + i\beta u^\mu) = \Lambda_p(x^\mu), \tag{1}$$

where u^μ denotes the 4-velocity vector of the heat bath. Although the QED action as well as the integration measure is invariant under arbitrary gauge transformations $\Lambda(x)$ of the background field, the anti-periodic boundary conditions will be modified if $\Lambda(x) \notin \{\Lambda_p\}$; in particular $\psi(0) = -\psi(\beta) \rightarrow \psi(0) = -e^{ie(\Lambda(\beta)-\Lambda(0))}\psi(\beta)$. At zero temperature, the fermion determinant can only depend on the field strength $F^{\mu\nu}$ that arises from the background field; the explicit form of A_μ is subject to arbitrary gauge transformations. In contrast, the restricted class of gauge transformations Λ_p at finite temperature allows for further gauge invariant quantities of the type

$$\bar{A}_u(\mathbf{x}) = \frac{1}{\beta}\int_0^\beta d\tau\, A_u(x^\mu + i\tau u^\mu), \qquad A_u := A^\mu u_\mu, \tag{2}$$

where $\mathbf{x}$ denotes the components of x^μ orthogonal to u^μ. Already at this stage, one might suspect that the physical meaning of $\bar{A}_u$ is related to a chemical potential μ which would enter the QED action by adding $\mu\gamma^\mu u_\mu$ to the Dirac operator: $\slashed{\Pi} = (-i\slashed{\partial} - e\slashed{A}) \rightarrow (-i\slashed{\partial} - e\slashed{A} + \mu\slashed{u})$. In the following, we will further establish this relation between $\bar{A}_u$ and μ and especially demonstrate that the appearance of $\bar{A}_u$ in the effective action is of topological origin. Instead of employing the functional integral formalism, we will closely follow Schwinger's proper-time formalism, which provides for a detailed study of gauge invariance.

We therefore begin with the fermionic Green's function in an external electromagnetic field at zero temperature satisfying the differential equation

$$[(\gamma^\mu \Pi_\mu) + m]\, G(x,x'|A) = \delta(x-x'), \tag{3}$$

with $\Pi_\mu = -i\partial_\mu - eA_\mu$. Following Schwinger [7], we can solve Eq. (3) formally on an operator level $[G(x,x'|A) = \langle x|G[A]|x'\rangle]$:

$$G[A] = (m - \gamma\Pi)\, i\int_0^\infty ds\, e^{-im^2 s}\, e^{i(\gamma\Pi)^2 s}. \tag{4}$$

Convergence of this proper-time integral and the following is ensured by the implicit prescription $m^2 \rightarrow m^2 - i\epsilon$. The proper-time transition amplitude

$$K(x,x';s|A) := \langle x|\, e^{is(\gamma\Pi)^2}\, |x'\rangle \tag{5}$$

in the integrand of Eq. (4) also enters the proper-time formula for the effective (unrenormalized) one-loop Lagrangian:

$$\mathcal{L}^1 = \lim_{x'\to x} \frac{i}{2}\, \text{tr}_\gamma \int_0^\infty \frac{ds}{s}\, e^{-ism^2}\langle x|\, e^{is(\gamma\Pi)^2}\, |x'\rangle. \tag{6}$$

Introducing the scalar propagator

$$\Delta(x,x'|A) = i\int_0^\infty ds\, e^{-im^2 s}\, K(x,x';s|A), \tag{7}$$

which is related to the fermion's Green's function via $G[A] = (m - \gamma\Pi)\,\Delta[A]$, we implicitly find an equation for $K(x,x';s|A)$ which is the Green's function equation for $\Delta(x,x'|A)$:

$$D[A]\,\Delta(x,x'|A) := [m^2 - (\gamma\Pi)^2]\,\Delta(x,x'|A) = \delta(x-x'), \tag{8}$$

where $D[A]$ abbreviates the differential operator. Obviously, $K(x,x';s|A)$ as well as the Green's functions $G(x,x'|A)$ and $\Delta(x,x'|A)$ is gauge dependent. For constant electromagnetic fields, the solution for the transition amplitude $K(x,x';s|A)$ can most conveniently be found in the Schwinger-Fock gauge that eliminates the gauge potential in favor of the field strength:

$$A_{\text{SF}}^\mu := -\frac{1}{2}F^{\mu\nu}(x-x')_\nu. \tag{9}$$

The solution reads [12]

$$K(x,x';s|A_{\text{SF}}) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-x')} e^{i(e/2)\sigma F s} e^{-Y(is)} e^{-pX(is)p}, \tag{10}$$

where $\sigma F := \sigma_{\mu\nu}F^{\mu\nu}$, $\sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu]$, and the quantities Y and X additionally depend on the field strength,

$$Y(s) = \frac{1}{2}\,\text{tr}\ln[\cos(e\mathsf{F}s)], \quad \mathsf{X}(s) = \frac{\tan(e\mathsf{F}s)}{e\mathsf{F}}, \tag{11}$$

and we used matrix notation, e.g., $F_\mu{}^\nu \equiv (\mathsf{F})_\mu{}^\nu$. By insertion of Eq. (10) into Eqs. (7), (3) and (6), we obtain the explicit representation for the scalar propagator, the fermion's Green's function and the effective Lagrangian, respectively, for constant external fields at zero temperature (in the Schwinger-Fock gauge).

To introduce finite temperature via the imaginary-time formalism, one is tempted to replace the p_0 integration in Eq. (10) by a sum over Matsubara frequencies.[2] However, this would lead to an incorrect or, at least, incomplete result, since the gauge dependence of the Green's functions has to be taken into account.

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[1]In principle, one could additionally allow for an integer multiple of $2\pi/e$ on the right-hand side of Eq. (1). But since such a term does not contribute to the present situation, we will simply omit it in the following.

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[2]For theories without gauge symmetries, of course, this procedure has been applied successfully in [13].
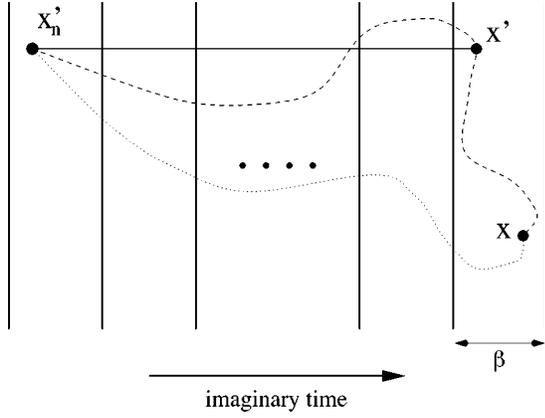
FIG. 1. The positions of the different points x, x' and x'_n are exhibited. The dotted line represents an arbitrary path from $x'_n = x' - i\beta nu$ to x. As a first step, this path is continuously deformed in such a way that x' becomes an element of the path (dashed line). Second, the path from x'_n to x' can be deformed to a straight line (solid line), which gives rise to Eq. (19).

As can be shown, the complete gauge dependence can be treated multiplicatively by a *holonomy* factor. In particular, the transition amplitude in an arbitrary gauge is related to the one in the Schwinger-Fock gauge by

$$K(x,x';s|A) = \Phi(x,x'|A) K(x,x';s|A_{\text{SF}}), \qquad (12)$$

where the holonomy factor reads

$$\Phi(x,x'|A) = \exp\left[ie \int_{x'}^{x} d\xi_\mu \left(A^\mu(\xi) + \frac{1}{2} F^{\mu\nu}(\xi - x')_\nu \right) \right]. \qquad (13)$$

Identical relations hold for the Green's functions. Note that the integrand is curl-free and hence the integral in Eq. (13) is path-independent as long as the configuration space is simply connected. Concerning the effective Lagrangian (6) at zero temperature, the holonomy factor plays no role, since $\Phi(x,x'|A) \to 1$ in the coincidence limit $x \to x'$. Consequently, the effective action is gauge invariant.

The situation changes substantially at finite temperature: since the imaginary time becomes compactified according to the anti-periodic boundary conditions, the configuration space is no longer simply connected. As a consequence, the holonomy factor is only invariant under continuous deformations of the integration path but can pick up a winding number by closing the path via the anti-periodic boundary.

The simplest way to establish anti-periodicity in imaginary time is to apply the method of image sources to the Green's function equation. Therefore, let x and x' belong to the same topological sector; i.e., there is a straight path from x to x' which does not cross the imaginary-time boundaries. Then we define the reflection points of x' along the imaginary-time axis by (Fig. 1)

$$x'_n = x' - i\beta nu. \qquad (14)$$

Applying the image-source construction, e.g., to Eq. (8), we obtain

$$\sum_{n=-\infty}^{\infty} (-1)^n \delta(x,x'_n) = \sum_{n=-\infty}^{\infty} (-1)^n D[A] \Delta(x,x'_n|A)$$
$$= D[A] \Delta^T(x,x'|A), \qquad (15)$$

where u^μ denotes the 4-velocity of the heat bath, the periodicity scale is set by the inverse temperature β, the factor $(-1)^n$ stems from the *anti*-periodic boundary conditions, and we have defined the thermal Green's function

$$\Delta^T(x,x'|A) = \sum_{n=-\infty}^{\infty} (-1)^n \Delta(x,x'_n|A). \qquad (16)$$

Transition to Fourier space and separation of the temperature-dependent parts leads us to

$$\Delta^T(x,x'|A) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-x')} \Delta(p) \Phi(x,x'|A)$$
$$\times \sum_{n=-\infty}^{\infty} (-1)^n e^{-ip(i\beta nu)} \Phi(x',x'_n|A)$$
$$\times e^{-in(e/2)\beta(iuF(x-x'))}. \qquad (17)$$

The separation

$$\Phi(x,x'_n|A) = \Phi(x,x'|A) \Phi(x',x'_n|A) e^{-in(e/2)\beta(iuF(x-x'))} \qquad (18)$$

was achieved by a continuous deformation of the integration path in such a way that, on the one hand, x' becomes an element of the path and, on the other hand, the path of $\Phi(x,x'|A)$ lies entirely in the topological trivial sector.

Concerning $\Phi(x',x'_n|A)$, we can deform the integration path to a straight line (SL) along the imaginary u^μ direction:

$$\Phi(x',x'_n|A) = \exp\left[ie \int_{\substack{x'-i\beta nu \\ \text{SL}}}^{x'} d\xi_\mu A^\mu(\xi) \right]. \qquad (19)$$

As mentioned above, the exponent in Eq. (19) is an invariant quantity under periodic gauge transformations Λ_p but will depend on the explicit form of A^μ in a certain manner. At this stage, it is important to point out that the background field potential is not necessarily subject to periodic boundary conditions, since it does not correspond to thermalized particles; it is not an integration variable even in the complete theory. To specify the form of A^μ, more physical input is required: in the present paper, we assume that the system under consideration is homogeneous. Since the effective Lagrangian for a homogeneous system such as the constant field configuration has to be independent of x, the coincidence limit $x \to x'$ of the thermal transition amplitude $K^T(x,x';s|A)$ must also be independent of x; the same requirement holds for $\Delta^T(x,x'|A)$. With regard to Eq. (17), this is only satisfied if $\Phi(x',x'_n|A)$ is independent of x'. Thereby, we obtain the gauge condition

$$0 = \int_0^1 d\tau\, \partial_{x'}^\mu A_u[x'^\mu - \mathrm{i}\beta n u^\mu + \tau(\mathrm{i}\beta n u^\mu)]. \qquad (20)$$

Condition (20) is satisfied if

$$A_u \equiv \bar{A}_u = \text{const}, \qquad (21)$$

which is the generic choice. Any other solution is gauge equivalent to Eq. (21), $A_u \rightarrow A_u + \partial_u \Lambda_\mathrm{p}$. Equation (20) also fixes the choice for the spatial components **A**: since A^μ should produce a constant electric field via

$$\text{const} = \mathbf{E} = \nabla A_u - \overset{(21)}{\partial_u \mathbf{A}} = -\partial_u \mathbf{A},$$

the generic choice for **A** in the heat bath rest frame ($\partial_u \overset{\hat{}}{=} \partial/\partial t$) reads

$$\mathbf{A} = -\mathbf{E}t + \mathbf{a}(\mathbf{x}), \qquad (22)$$

whereby the function $\mathbf{a}(\mathbf{x})$ is defined by $\mathbf{B} =: \nabla \times \mathbf{a}$. Again, other choices for **A** are given by its gauge transforms with respect to Λ_p. Note that these gauge conditions are different from those found in Ref. [4] employing the real-time formalism.[3]

Taking these considerations into account, the holonomy factor (19) eventually yields

$$\Phi(x', x'_n|A) = \exp[\mathrm{i}e(\mathrm{i}\beta n)\bar{A}_u]. \qquad (23)$$

With the aid of a Poisson resummation, we obtain, for the sum in Eq. (17),

$$\sum_{n=-\infty}^{\infty} (-1)^n\, e^{-\mathrm{i}p(\mathrm{i}\beta n u)} \Phi(x', x'_n|A)\, e^{-\mathrm{i}n(e/2)\beta(\mathrm{i}u F(x-x'))}$$

$$= 2\pi \mathrm{i}T \sum_{n=-\infty}^{\infty} \delta(p_u - e(\bar{A} - A_\mathrm{SF})_u + \mathrm{i}\pi T(2n+1)), \qquad (24)$$

with $p_u = u^\mu p_\mu$, and $A_{\mathrm{SF}\,u} = -\frac{1}{2} u_\mu F^{\mu\nu}(x-x')_\nu$. Inserting Eq. (24) into Eq. (17) leads us to the final expression for $\Delta^T(x,x'|A)$. Similarly, the thermal transition amplitude $K^T(x,x';s|A)$ as well as the thermal fermion's Green's function can be derived. Note that these objects contain

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[3]In the real-time formalism, the A^0 component of the gauge field enters the propagators as well as the effective action via the thermal distribution function that is given as an external condition. Hence, there is no intrinsic criterion for an appropriate choice of A^μ, and one has to rely on other arguments. E.g., in Ref. [4], it was argued that a gauge condition of the form $(d/dt)A^\mu = 0$ is required for obtaining a clear separation of fermionic and electromagnetic energies. This implies that the constant electric background field is produced by a spatially non-constant A^0, $\mathbf{E} = -\nabla A^0$, and has to be interpreted as a spatially non-constant chemical potential (cf. later). This demonstrates that different gauge choices which are not gauge-equivalent with respect to $\{\Lambda_\mathrm{p}\}$ correspond to different physical settings.

temperature-dependent contributions as well as the zero-temperature part. The question of the gauge dependence of the thermal fermionic Green's function in a purely magnetic background has also been addressed in [14].

We observe that the Matsubara prescription finally reads

$$\int \frac{dp_u}{2\pi} f(p_u^2)$$

$$\rightarrow \mathrm{i}T \sum_{n=-\infty}^{\infty} f\left\{ -\left[\pi T\left(2n+1 + \frac{\mathrm{i}}{\pi} \frac{e(\bar{A} - A_\mathrm{SF})_u}{T} \right) \right]^2 \right\}. \qquad (25)$$

The explicit appearance of $A_{\mathrm{SF}\,u}$ hints at the fact that this modified Matsubara prescription will be applied to an object which has been calculated in the Schwinger-Fock gauge. Equation (25) finally states that it is a gauge field-shifted momentum in the u^μ direction, $(p - e(A - A_\mathrm{SF}))_u$, which is replaced by Matsubara frequencies instead of the canonical momentum. This implies a dependence of the Green's functions and the transition amplitude on the gauge field invariant $\bar{A}_u$ even in the coincidence limit $x' \rightarrow x$ (note that $A_\mathrm{SF} \rightarrow 0$ for $x' \rightarrow x$). As a consequence, the effective Lagrangian will be invariant under periodic gauge transformations Λ_p but not under arbitrary gauge transformations Λ. Of course, this was expected from our initial considerations. The physical role of $\bar{A}_u$ will be elucidated at the end of Sec. IV.

III. COVARIANT FORMULATION

The imaginary-time formalism has often been criticized because it exhibits the explicit non-covariant feature of leading to discrete energies but continuous momenta for the quantized fields. In the present work, we want to demonstrate that it is nevertheless possible to establish covariance at any stage of this calculation, since the above-mentioned disproportion between energy and momentum only appears in internal propagators, all of which are integrated out. Manifest covariance is achieved by constructing a reference frame that completely relies on the covariant and gauge invariant building blocks of the problem.

These building blocks in the present problem of constant electromagnetic (EM) fields at finite temperature are the field strength tensors, $F^{\mu\nu}$ and $^\star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$; furthermore, we encounter the heat-bath vector n^μ [15] that is on the one hand characterized by the value of its invariant scalar product,[4] $n^\mu n_\mu = -T^2$, where T denotes the heat-bath temperature, and on the other hand related to the heat-bath 4-velocity via the invariant parameter T: $n^\mu = T u^\mu$. There are 10 independent components in $F^{\mu\nu}$, $^\star F^{\mu\nu}$, and n^μ. The number of generators of the Lorentz group is 6; hence we can transform 6 components to zero, since there is no little group that leaves $F^{\mu\nu}$, $^\star F^{\mu\nu}$, and n^μ invariant.[5] Therefore,

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[4]We employ the metric $g = \text{diag}(-1,1,1,1)$.

[5]For pure EM fields, the dimension of the little group would be 2, since boosts along and rotation around the field direction in a system where **E** and **B** fields are parallel leave the fields invariant.

we are left with four Lorentz and gauge invariant scalars (or pseudo-scalars). For reasons of convenience, we choose the following set:

$$a := (\sqrt{\mathcal{F}^2 + \mathcal{G}^2} + \mathcal{F})^{1/2},$$

$$b := (\sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F})^{1/2},$$

$$c := \frac{1}{T^2}(n_\alpha F^{\alpha\mu})(n_\beta F^\beta{}_\mu) \equiv (u_\alpha F^{\alpha\mu})(u_\beta F^\beta{}_\mu),$$

$$T = \sqrt{-n^\mu n_\mu}. \tag{26}$$

The *secular* invariants a, b are related to the solutions to the secular equation of $F^{\mu\nu}$; e.g., the standard invariants can be expressed in terms of a and b according to

$$\mathcal{F} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2) \equiv \frac{1}{2}(a^2 - b^2),$$

$$|\mathcal{G}| = \left| \frac{1}{4} {}^\star F^{\mu\nu} F_{\mu\nu} \right| = |-\mathbf{E} \cdot \mathbf{B}| \equiv ab. \tag{27}$$

Without loss of generality, we confine ourselves to the case of $\mathcal{G} > 0$ (or $\mathbf{E} \cdot \mathbf{B} < 0$) and drop the absolute value notation. E.g., in a system where $\mathbf{B}$ is anti-parallel to $\mathbf{E}$, we find $a = |\mathbf{B}|$ and $b = |\mathbf{E}|$. Note that c is positive-definite since n^μ is a time-like vector; e.g., in the rest frame of the heat bath, we find $c = \mathbf{E}^2$. It is obvious that any gauge invariant Lorentz scalar appearing in the problem is expressible in terms of this set of invariants (26). In the following, we are going to introduce a convenient coordinate system in which even the components of any Lorentz vector or tensor of the problem can be expressed in terms of these invariants. We define the *vierbein* $e^{A\mu}$ which mediates between the given system labeled by $\mu, \nu, \ldots = 0,1,2,3$ and the desired system labeled by the (Lorentz) indices $A, B, \ldots = 0, 1, 2, 3$ by

$$e_0{}^\mu := u^\mu,$$

$$e_1{}^\mu := \frac{u_\alpha F^{\alpha\mu}}{\sqrt{c}},$$

$$e_2{}^\mu := \frac{1}{\sqrt{d}}(u^\alpha F_{\alpha\beta} F^{\beta\mu} - c\, e_0{}^\mu),$$

$$e_3{}^\mu := \epsilon^{\alpha\beta\gamma\mu} e_{0\alpha} e_{1\beta} e_{2\gamma}, \tag{28}$$

where the quantity d abbreviates the combination of invariants

$$d := 2\mathcal{F}c - \mathcal{G}^2 + c^2. \tag{29}$$

The vierbein satisfies the identity

$$e_{A\mu} e_B{}^\mu = g_{AB} \equiv \mathrm{diag}(-1,1,1,1), \tag{30}$$

where $g_{AB} \sim g^{AB}$ denotes the metric which raises and lowers capital indices. By a direct computation, we can transform the field strength tensors and the heat-bath vector:

$$n^A := g^{AB} e_B{}^\mu n_\mu = (T,0,0,0),$$

$$F_{AB} := e_{A\mu} F^{\mu\nu} e_{B\nu} = \begin{vmatrix} 0 & \sqrt{c} & 0 & 0 \\ -\sqrt{c} & 0 & \sqrt{d/c} & 0 \\ 0 & -\sqrt{d/c} & 0 & -\mathcal{G}/\sqrt{c} \\ 0 & 0 & \mathcal{G}/\sqrt{c} & 0 \end{vmatrix},$$

$${}^\star F_{AB} := e_{A\mu} {}^\star F^{\mu\nu} e_{B\nu} = \begin{vmatrix} 0 & -\mathcal{G}/\sqrt{c} & 0 & \sqrt{d/c} \\ \mathcal{G}/\sqrt{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{c} \\ -\sqrt{d/c} & 0 & \sqrt{c} & 0 \end{vmatrix}. \tag{31}$$

So indeed, the components of these tensors are completely expressed in terms of invariants. Hence, any tensor algebraic manipulation involving the objects from Eq. (31) can immediately be performed on the level of gauge and Lorentz invariants.

It is worthwhile to point out at this stage that a duality transformation of the type $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$ does not only imply an interchange of a and b (and a sign flip for $\mathcal{G}$) but also demands for $c \to c + 2\mathcal{F} = c + a^2 - b^2$. Hence, it is not sufficient in the finite-temperature case to perform a calculation for magnetic fields and then draw an analogy for electric fields by replacing $\mathbf{B} \to -i\mathbf{E}$–in contrast to a zero-temperature calculation.

IV. EFFECTIVE ACTION

From Eq. (6), we can read off the definition of the effective Lagrangian at finite temperature:

$$\mathcal{L}^{1+1T} = \frac{i}{2} \mathrm{tr}_\gamma \int_0^\infty \frac{ds}{s} e^{-ism^2} K^T(s|A), \tag{32}$$

where the superscript implies that $\mathcal{L}^{1+1T}$ consists of the zero-temperature as well as the finite-temperature one-loop part: $\mathcal{L}^{1+1T} = \mathcal{L}^1 + \mathcal{L}^{1T}$. The thermal transition amplitude in the integrand is simply obtained by applying the modified Matsubara prescription (25) to the zero-temperature transition amplitude, Eqs. (10)–(12), in the coincidence limit:

$$K^T(s;A) = iT \sum_{n=-\infty}^\infty \int \frac{d^3p}{V(2\pi)^3} e^{i(e/2)\sigma Fs} e^{-Y(is)}$$

$$\times e^{-pX(is)p} \Big|_{p_u = e\bar{A}_u - i\pi T(2n+1)}, \tag{33}$$

where V denotes the 3-space volume orthogonal to the u^μ direction. We now perform the computation of Eqs. (32) and (33) within the system that we established in the previous section. With respect to the capital labels, this volume is

related to the components $A, B, \ldots = 1, 2, 3$, whereas the components along the u^μ direction correspond to the label $A, B, \ldots = 0$. The X matrix in the exponent of Eq.(33) can now be written as

$$X_{AB}^{(11)} = i \left[\frac{\tan eas}{ea} \frac{(b^2 g_{AB} - F_{AB}^2)}{a^2 + b^2} \right.$$

$$\left. + \frac{\tanh ebs}{eb} \frac{(a^2 g_{AB} + F_{AB}^2)}{a^2 + b^2} \right], \qquad (34)$$

where $F_{AB}^2 = F_A{}^C F_{CB}$. Incidentally, the identical equation also holds, of course, with the labels A, B replaced by μ, ν, but then the components are not related to gauge and Lorentz invariants. The only non-vanishing components of the symmetric tensor X_{AB} are the diagonal elements as well as X_{02} and X_{13}. The Gaussian momentum integration in Eq. (33) therefore results in

$$\int_V \frac{d^3 p}{(2\pi)^3} e^{-p^A X_{AB} p^B}$$

$$= \frac{e^{(X_{11} X_{33} - X_{13}^2) p_0^2 / X_{22}}}{(4\pi)^{3/2}} [(X_{11} X_{33} - X_{13}^2) X_{22}]^{-1/2}, \qquad (35)$$

where we made use of the fact that $-(X_{00} X_{22} - X_{02}^2) = (X_{11} X_{33} - X_{13}^2)$. Substituting the modified Matsubara frequencies $p_0 \equiv p_u = e\bar{A}_u - i\pi T(2n+1)$ into the exponent of Eq. (35), the summation over n in Eq. (33) can be reorganized according to the Poisson formula:

$$\sum_{n=-\infty}^{\infty} e^{-\sigma(n-z)^2} = \sum_{n=-\infty}^{\infty} \left(\frac{\pi}{\sigma}\right)^{1/2} e^{(-\pi^2/\sigma)n^2 - 2\pi i zn}. \qquad (36)$$

In this case, we set $z = -\frac{1}{2} - ie\bar{A}_u/2\pi T$ and $\sigma = (4\pi^2 T^2/X_{22})(X_{11} X_{33} - X_{13}^2)$. At this point, we have to mention that formula (36) is not valid for Re $\sigma < 0$, which would lead to a divergent behavior of the sum. This will be checked later on.

The Poisson resummation serves the purpose of separating the zero-temperature from the finite-temperature part, since the complete loop-momentum integration/summation in Eq. (33) now yield

$$iT \sum_{n=-\infty}^{\infty} \int_V \frac{d^3 p}{(2\pi)^3} e^{-p X_p}$$

$$= \frac{i}{16\pi^2} (X_{11} X_{33} - X_{13}^2)^{-1} \left[1 + 2 \sum_{n=1}^{\infty} (-1)^n \right.$$

$$\left. \times \exp\left(-\frac{X_{22}}{X_{11} X_{33} - X_{13}^2} \frac{n^2}{4T^2}\right) \cosh\frac{e\bar{A}_u n}{T} \right]. \qquad (37)$$

Keeping only the ''1'' in the second line leads to the standard proper-time expression for the zero-temperature effec-

tive action, while the sum represents the thermal correction. Employing the standard results for the remaining terms in Eqs. (32) and (33),

$$e^{-Y(is)}_{(11)} = (\cos eas \cosh ebs)^{-1},$$

$$\mathrm{tr}_\gamma e^{(ie/2)\,\sigma Fs} = 4\cos eas \cosh ebs,$$

and inserting the explicit representations of the X_{AB} into Eq. (37), we end up with the desired expression for the one-loop contribution to the effective Lagrangian for constant electromagnetic fields at finite temperature:

$$\mathcal{L}^{1+1T} = \mathcal{L}^1 + \mathcal{L}^{1T},$$

$$\mathcal{L}^1 = \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-im^2 s} \left[eas \cot eas \, ebs \coth ebs \right.$$

$$\left. - \frac{e^2(a^2 - b^2)s^2}{3} - 1 \right], \qquad (38)$$

$$\mathcal{L}^{1T} = \frac{1}{4\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-im^2 s} \, eas \cot eas \, ebs \coth ebs$$

$$\times \sum_{n=1}^{\infty} (-1)^n e^{ih(s)n^2/4T^2} \cosh\frac{e\bar{A}_u n}{T}. \qquad (39)$$

At the zero-temperature part $\mathcal{L}^1$, we subtracted the divergent terms, which corresponds to a field strength and charge renormalization. The function $h(s)$ in the exponent of $\mathcal{L}^{1T}$ is obtained from

$$h(s) := \frac{iX_{22}}{X_{11} X_{33} - X_{13}^2} = \frac{b^2 - c}{a^2 + b^2} ea \cot eas$$

$$+ \frac{a^2 + c}{a^2 + b^2} eb \coth ebs. \qquad (40)$$

In the rest frame of the heat bath where $c = \mathbf{E}^2$, we recover the findings of Ref. [4] for $h(s)$. Note that $h(s)$ is strictly real, so there are no apparent convergence problems in employing the Poisson resummation (36). However, it is not a straightforward exercise to obtain numerical estimates for Eq. (39), due to the wildly oscillatory behavior of the whole integrand, especially in the sum. Let us for the moment remark that $h(s)$ reduces to $1/s$ in the limit of vanishing electric fields [the limit is most conveniently taken for $\mathbf{E}$ and $\mathbf{B}$ (anti-)parallel]. And assuming $\bar{A}_u = 0$, we recover the findings of Ref. [1] for a purely magnetic field at finite temperature. Furthermore, the general form of Eq. (39) coincides with the representation found in the world line approach [6] (in the heat-bath rest frame), except for the dependence on the gauge potential; the importance of the holonomy factor has been overlooked in [6].

The physical interpretation of $\bar{A}_u$ can most easily be illuminated in the limiting case of vanishing field invariants, $a, b, c = 0$; under these circumstances we are able to rotate

the contour $s \rightarrow -is$ and an interchange of integration and summation is permitted in Eq. (39):

$$\mathcal{L}^{1T}(\bar{A}_u) = -\frac{1}{4\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-m^2 s} \sum_{n=1}^\infty (-1)^n e^{-n^2/4T^2 s} \cosh\frac{e\bar{A}_u n}{T}$$

$$= -\frac{2}{\pi^2} m^2 T^2 \sum_{n=1}^\infty \frac{(-1)^n}{n^2} \cosh\frac{e\bar{A}_u n}{T} K_2\left(\frac{m}{T} n\right)$$

$$= -\frac{2T}{\pi^2} \sum_{n=1}^\infty \frac{(-1)^n}{n} \cosh\frac{e\bar{A}_u n}{T}$$

$$\times \int \frac{d^3 k}{4\pi} e^{[-(\sqrt{k^2+m^2})/T]n}$$

$$= T 2 \int \frac{d^3 k}{(2\pi)^3} [\ln(1+e^{-\beta(E+e\bar{A}_u)})$$

$$+ \ln(1+e^{-\beta(E-e\bar{A}_u)})], \qquad (41)$$

where we introduced the particle energy for the $e^+ e^-$ gas, $E := \sqrt{k^2 + m^2} \equiv |k_u|$, and two different representations of the modified Bessel function K_2 were taken advantage of [16]. According to the relation

$$\mathcal{L}^{1T}(\bar{A}_u) = T \frac{\ln Z(T, \bar{A}_u)}{V},$$

we indeed find the general expression for the partition function Z of an ideal $e^+ e^-$-gas in which the $\bar{A}_u$ field plays the role of a chemical potential. If we had started the computation including a chemical potential, we would always have encountered the combination $-e\bar{A}_u - \mu \hat{=} e\bar{A}^0 - \mu$, which is therefore the only physical quantity. In other words, we can identify $e\bar{A}_u$ with a chemical potential during the complete calculation; hence, the additional information (compared to the zero temperature case) which is required to define the correct choice of the background gauge potential A^μ is obtained from the value of the chemical potential of the system under consideration. If one wants to perform a gauge transformation beyond the class of periodic Λ_p, one has to redefine the chemical potential to obtain the same physical system.

The case of the effective Lagrangian of a constant magnetic field at non-zero chemical potential has been discussed in [17]. Based on the real-time formalism, a comprehensive study of this situation including finite temperature can be found in [18] where also astrophysical implications are discussed. The same physical situation was investigated employing the imaginary-time formalism in [19] where high- and low-temperature expansions were approached in a more direct way. As is demonstrated in these references, the zero-temperature limit of Eq. (39) at a chemical potential obeying $\mu > m$ requires careful study.

A detailed weak-field expansion of the effective Lagrangian at finite temperature and chemical potential was performed in [20], relying on the "real-time" representation of the effective action as given in [4].

V. DISCUSSION

Going beyond the constant field approximation, the effective Lagrangian in Eqs. (38) and (39) can be viewed as the zeroth order of a gradient expansion of the one-loop effective action which governs the dynamics of the background gauge field $A^\mu(x)$. An immediate physical consequence of the fact that the $\bar{A}_u$ field appears explicitly in the Lagrangian is the well-known Debye screening of electric fields. A weak-field expansion will take the form $\mathcal{L}^{1T} = -\frac{1}{2} \partial_\mu \bar{A}_u \partial^\mu \bar{A}_u + (m_{\text{eff}}^2/2)(\bar{A}_u)^2 + \mathcal{O}(\bar{A}_u^4)$, where the effective photon mass (inverse Debye screening length) is given by

$$m_{\text{eff}}^2(T) = \frac{\partial^2 \mathcal{L}^{1T}}{\partial \bar{A}_u^2}\bigg|_{\bar{A}_u = 0}. \qquad (42)$$

Considering the zero-field limit for simplicity, we find

$$\frac{\partial^2 \mathcal{L}^{1T}}{\partial \bar{A}_u^2}\bigg|_{\bar{A}_u = 0} = -\frac{2e^2}{\pi^2} m^2 \sum_{n=1}^\infty (-1)^n K_2\left(\frac{m}{T} n\right), \qquad (43)$$

where K_2 denotes a modified Bessel function. In the high-temperature limit, $T \gg m$, the sum can be expanded, e.g., employing the techniques described in the Appendix B of [21], leading to $[\Sigma_{n=1}^\infty (-1)^n K_2(m/T n) \simeq \pi^2 T^2/6m^2 + \mathcal{O}(1)]$. We finally arrive at

$$m_{\text{eff}}^2(T) = \frac{(eT)^2}{3}, \qquad (44)$$

which is the well-known result found in the literature. The leading corrections to the Debye mass in the high-density and high-temperature limit can be looked up in the erratum of Ref. [4].

At last we turn to the question of whether Schwinger's famous formula for the pair-production probability renders finite-temperature corrections at one-loop order. While no thermal contributions have been found to the imaginary part of $\mathcal{L}^{1T}$ in [2] or [4] within the real-time formalism or in [10] employing the functional Schrödinger representation, an imaginary part seems to appear in the imaginary-time formalism [5]. Besides, the latter result had also been computed in the real-time formalism in [3].

Although our findings for the effective thermal Lagrangian, Eq. (39), in the heat-bath rest frame formally coincide with those found in [3] (up to numerical pre-factors and an interchange of proper-time integration and summation), we do not agree with their computation of the imaginary part, which follows the line of the zero-temperature calculation. Various obstacles are encountered when proceeding in this way for the finite-temperature case: since the function $h(s)$ in the exponential of Eq. (39) reduces to

$$h(s) = eE \coth eEs \qquad (45)$$

for a purely electric field, $E = |\mathbf{E}|$, (i) a rotation of the contour, $s \rightarrow -is$, becomes useless due to the $\coth$ term in the exponent of Eq. (39); (ii) each term in the sum of Eq. (39)

exhibits an essential singularity at the poles of the coth term on the imaginary axis; the use of the rule $\cot z \to \mathrm{P}\cot z + i\pi\Sigma_{z_0}\delta(z-z_0)$ is therefore senseless (cf. [3]); (iii) proper-time integration and summation must not be interchanged. If they are, the imaginary parts of the successive terms in the sum diverge exponentially, as can be shown by evaluating the residues of the singularities on the imaginary s axis. Incidentally, we do not agree with the imaginary part computed in [5], simply because the expressions for the effective Lagrangians do not coincide.

However, we can give an indirect argument for the vanishing of the imaginary part following Ref. [4]: because of the formal resemblance between our result Eq. (39) and the findings of Loewe and Rojas [3] for the effective Lagrangian (not for its imaginary part), we can follow their steps backwards and end up with the starting point of the real-time formalism,

$$\frac{\partial\Gamma^{1T}}{\partial m} = -i\mathrm{Tr}\left\{f_{\mathrm{F}}(k_u,\bar{A}_u)\left(\frac{1}{\not{V}-m+i\epsilon}-\frac{1}{\not{V}-m-i\epsilon}\right)\right\},$$
(46)

where $f_{\mathrm{F}}(k_u,\bar{A}_u)$ denotes a (real) thermal distribution function for the fermions and $\Gamma = \int d^4x\,\mathcal{L}^{1T}$. Obviously, since the right-hand side is purely real, there is no imaginary part in the thermal contribution to the effective one-loop action and hence no thermal correction to the Schwinger pair-production formula to this order of calculation.

VI. CONCLUSION

In the present work, we studied the derivation of the effective QED action to one-loop order in presence of arbitrary constant electromagnetic fields at finite temperature in the imaginary-time formalism. Although the final expression for the effective action is well known and has been studied extensively, especially in the real-time formalism [4], the problem as treated in the imaginary-time formalism reveals some delicate features.

Gauge invariance of the classical action turns out to be restricted to periodic gauge transformations Λ_{p} on the quantum level in order to leave the boundary conditions of the functional integral unaltered. This implies the existence of further gauge invariant quantities beside the field strength which are constructed from the background field A^μ. Additional information about the system under consideration has to be employed to fix the form of the gauge potential A^μ. In the present case, the demand for homogeneity (constant fields and constant chemical potential) gives rise to the additional gauge invariant quantity $\bar{A}_u$.

The way in which $\bar{A}_u$ enters the effective action can be viewed as a topological effect that arises from the compactification of the finite-temperature configuration space in imaginary time; the configuration space, namely, loses its property of being simply connected and allows for infinitely many topologically inequivalent paths to connect two different points in space-time. Each path can be classified by its winding number around the space-time cylinder. The holonomy factor that carries the gauge dependence of the Green's function is sensitive to these inequivalent paths, since it represents a mapping of the paths in configuration space into the gauge group. A Poisson resummation of the sum over the winding number leads to a sum over Matsubara frequencies shifted by the $\bar{A}_u$ field. The quantity $e\bar{A}_u$ indeed acts like a chemical potential in the partition function and therefore can be identified with μ.

The gauge non-invariance of the effective action under non-periodic gauge transformations furthermore manifests itself in giving a mass to the (integrated) time-like gauge field component $\bar{A}_u$, which is, of course, nothing but the screening mass of the Debye mechanism.

As a second focus of this work, we introduced a certain coordinate frame related to the given Lorenz vectors and tensors of the problem that allowed for a manifest covariant computation in the sense of relativistic thermodynamics. This procedure helped us to present the result in terms of a complete set of Lorenz and gauge invariants.

As an immediate application of the final formula for the effective Lagrangian, we discussed a possible thermal contribution to the Schwinger pair-production formula. In agreement with the results of the real-time formalism, we do not find an imaginary part in the thermal contribution to the effective Lagrangian to this order of calculation. On a heuristic level, it appears plausible that a thermal contribution to pair production can arise from higher loop graphs. E.g., the two-loop process contains the mass operator (in the presence of an external field), which can be associated with collective excitations at finite temperature. These can be approximately taken into account by replacing the fermion mass by an effective T or μ dependent mass [22]. However, since these effective masses generally exceed the fermion mass, such thermal contributions are expected to be of subdominant importance.

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