Smearing of propagators by gravitational fluctuations on the Planck scale

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We examine the effects of quantum fluctuations in the gravitational field on the propagator of a spin-zero matter field. At short range, the propagator is a function of the square of the geodesic distance, and it can be expressed as an explicit functional of the gravitational field. Calculation of the path integral over the gravitational field is then possible, and this yields a gravitationally modified propagator. A renormalization of the gravitational constant is required, such that the bare gravitational constant approaches zero. To order G, the modified propagator is then $-mK_1(im\sqrt{x^2-L^2-i\epsilon})/4\pi^2\sqrt{x^2-L^2-i\epsilon}$, where $L\simeq$Planck length. This result is consistent with the smeared propagator proposed in a previous paper, and it makes the radiative corrections in QED and in other field theories finite. [S0556-2821(99)00222-2]

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I. INTRODUCTION

In Ref. [1], we proposed a modification of the Feynman propagator on the basis of some qualitative considerations of a smearing of the light cone by gravitational quantum fluctuations on the Planck scale. For a spin-zero particle of mass m, the proposed smeared propagator was

$$\bar{G}_F(x^2)=\int d\lambda\, f(\lambda)G_F(x^2-\lambda)$$
$$=-\frac{m}{4\pi^2}\int d\lambda\, f(\lambda)\,\frac{K_1(im\sqrt{x^2-\lambda-i\epsilon})}{\sqrt{x^2-\lambda-i\epsilon}}. \quad (1)$$

Here K_1 is the modified Bessel function of order 1, and $f(\lambda)$ is a weight function, or spectral function, whose exact form was undetermined. However, on physical grounds the spectral function was expected to have a peak near the square of the Planck length, that is, near $\lambda\simeq L_*^2=\hbar G/c^3=(1.6\times10^{-33}\,\mathrm{cm})^2$. Causality arguments and the global commutativity theorem suggest that only positive values of λ should be included in the spectral integral; that is, the light cone should be smeared toward the interior of the Minkowski light cone $x^2=0$, which implies a reduction of the maximum attainable speed of propagation.

For $|x^2|\gg L_*^2$, the smeared propagator approaches the conventional Feynman propagator, but for $|x^2|\ll m^2$, it approaches

$$\bar{G}_F(x^2)\rightarrow\frac{i}{4\pi^2}\int_0^\infty d\lambda\, f(\lambda)\,\frac{1}{x^2-\lambda-i\epsilon}. \quad (2)$$

This propagator has no singularity at $x^2=0$, and its Fourier transform has an effective cutoff at the Planck mass, that is, at $|p^2|\simeq1/L_*^2=M_*^2=(1.2\times10^{19}\,\mathrm{GeV})^2$. As shown in [1], this leads to finite results for the charge, mass, and wave function renormalizations in QED.

In this paper, we will show from the quantum theory of gravitation that the quantum fluctuations of the gravitational field do indeed generate a modification of the short-range behavior of the Feynman propagator of just the form proposed in Eq. (1). To establish this result, we follow a scheme sketched by Deser a long time ago [2]. Deser conjectured that a gravitational modification and regularization of the propagator would emerge from the evaluation of the path integral

$$\bar{G}_F(x_1,x_2)=N^{-1}\int Dg\int D\phi\;\phi(x_1)\phi(x_2)$$
$$\times\exp\left[i\int d^4x\,\mathcal{L}(\phi)+i\int d^4x\,\mathcal{L}(g)\right], \quad (3)$$

with

$$N=\int Dg\int D\phi\,\exp\left[i\int d^4x\,\mathcal{L}(\phi)+i\int d^4x\,\mathcal{L}(g)\right]. \quad (4)$$

Here $\int D\phi$ and $\int Dg$ indicate path integrations over the scalar field $\phi(x)$ and the gravitational field $g_{\mu\nu}(x)$, and $\int d^4x\,\mathcal{L}(\phi)$ and $\int d^4x\,\mathcal{L}(g)$ are, respectively, the actions for the scalar and gravitational fields.

The obstacle to the implementation of this scheme is that the evaluation of the path integrals in Eq. (3) is not feasible. As a first step in such an evaluation, we would have to perform the path integral over $\phi(x)$ and, thereby, obtain the Feynman propagator $G_F(x_1,x_2;g)$ in the given geometry $g_{\mu\nu}(x)$. This propagator is a Green's function in this given background geometry, and it obeys the differential equation

$$(-g)^{-1/2}\partial_\mu[(-g)^{1/2}g^{\mu\nu}(x_2)\partial_\nu G_F(x_1,x_2;g)]$$
$$+[m^2+\xi R(x_2)]G_F(x_1,x_2;g)$$
$$=-(-g)^{-1/2}\delta^4(x_1-x_2), \quad (5)$$

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where $\partial_\mu = \partial/\partial x_2^\mu$ and where the term $\xi R(x_2)$ represents a nonminimal coupling to the curvature scalar ($\xi=\frac{1}{6}$ for "conformal" coupling). As a second step, we would then perform the path integral over $g_{\mu\nu}(x)$, to obtain the gravitationally modified propagator. However, to perform this second path integral, we need an explicit expression for G_F as a functional of $g_{\mu\nu}(x)$. In principle, the differential equation (5) and its boundary conditions determine the dependence of G_F on $g_{\mu\nu}(x)$, but in the absence of an explicit expression, we cannot proceed with Deser's scheme.

We can bypass this obstacle if instead of trying to find the full propagator, we focus on its short-range behavior. In that case it is possible to write down an explicit expression for G_F as a functional of $g_{\mu\nu}$ by taking advantage of the known general properties of the differential equation (5). This is a linear partial differential equation of the second order, and when x_2 is near x_1, the solution of such a differential equation has the Hadamard elementary form

$$G_F(x_1,x_2;g)=\frac{U}{\sigma}+V\ln\sigma+W, \tag{6}$$

where $\sigma=\sigma(x_1,x_2)$ is one-half of the square of the geodesic distance between the points x_1 and x_2, and U, V, and W are regular functions of x_1 and x_2, that is, functions without singularities as $x_2\to x_1$. The functions U, V, and W can be developed in a Hadamard expansion [3,4] or a DeWitt-Schwinger expansion [5,6]. In either case, for $x_2\to x_1$, the dominant, singular terms in Eq. (6) are

$$G_F(x_1,x_2;g)\to\frac{i}{8\pi^2(\sigma-i\epsilon)}-\frac{i}{16\pi^2}\left[m^2-\left(\frac{1}{6}-\xi\right)\times R(x_1)\right]\ln\frac{1}{2}m^2\sigma. \tag{7}$$

The function $\sigma(x_1,x_2)$ is a functional of the metric tensor $g_{\mu\nu}$, and as we will see in Sec. II, this functional can be expressed in an explicit form as a perturbation series in ascending powers of the gravitational field. Thus the substitution of Eq. (7) into Eq. (3) leads to a calculable path integral.

The technique for expressing G_F in terms of σ, and σ in terms of $g_{\mu\nu}$ was introduced by Ford [7] in an investigation of the modifications of the photon propagator caused by a fluctuating background of gravitational waves. Ford considered only fluctuations arising from "squeezed" gravitational-wave states. He did not investigate the case of the vacuum state, which contains divergences which require renormalization. Ford used operator methods rather than path integrals in his calculation, and he adopted a transverse, noncovariant gauge. This is unsuitable for the treatment of vacuum fluctuations, since the final result must necessarily be Lorentz invariant. By the use of path integrals instead of operator methods and by taking advantage of the Faddeev-Popov gauge-fixing procedure, we can maintain Lorentz invariance throughout the calculation. Furthermore, Ford dealt in detail only with the linear terms ($\propto h_{\mu\nu}$) in the functional σ; he mentioned the quadratic terms ($\propto h_{\mu\nu}h_{\alpha\beta}$) briefly and then decided to ignore them. As we will see below, for the vacuum fluctuations, the quadratic terms are dominant—their contribution to the propagator is much larger than that of the linear terms.

In Sec. II we begin with an evaluation of σ up to terms of second order in the gravitational field. In Sec. III we calculate the path integral $\int Dg$ for the propagator (7). A renormalization of the gravitational constant is required to make the effects of the fluctuations of the spacetime geometry finite. In Sec. IV we examine the modifications of the graviton propagator caused by quantum fluctuations of the matter fields, and we deal with the consequent renormalization of the gravitational constant and with the short-range behavior of the graviton propagator. The result is a mutual regularization of the propagators of matter fields and of the gravitational field: The quantum fluctuations of the gravitational field regularize the propagator of the matter fields, while the quantum fluctuations of the matter fields regularize the propagator of the gravitational field. In Sec. V we discuss applications to QED. And in Sec. VI we briefly discuss the difficulties arising from higher-order contributions to the path integral.

II. SQUARE OF THE GEODESIC DISTANCE

To express the square of the geodesic distance between two points x_1 and x_2 as a functional of the metric tensor, we begin with the formula given by Synge [8]:

$$2\sigma(x_1,x_2)=g_{\mu\nu}(x_1)U^\mu(x_1)U^\nu(x_1), \tag{8}$$

where $U^\mu(x_1)$ is the tangent vector dx^μ/du at x_1 for the geodesic connecting the points, and u is an affine parameter for this geodesic, normalized in such a way that x_1 corresponds to $u=0$ and x_2 corresponds to $u=1$. We can determine U^μ by integrating the geodesic equation. This second-order differential equation can be written as a pair of first-order differential equations,

$$\frac{dU^\mu}{du}+\Gamma^\mu_{\alpha\beta}U^\alpha U^\beta=0 \tag{9}$$

and

$$\frac{dx^\mu}{du}=U^\mu \tag{10}$$

or, equivalently, as a pair of integral equations,

$$U^\mu(u)=n^\mu-\int_0^u du'\,\Gamma^\mu_{\alpha\beta}U^\alpha U^\beta \tag{11}$$

and

$$x^\mu(u)=x_1^u+\int_0^u du'\,U^\mu, \tag{12}$$

where $n^\mu = U^\mu(0)$ is a constant. These integral equations can be solved by successive approximations, starting with the zeroth approximation $U^\mu = \Delta x^\mu = x_2^\mu - x_1^\mu$. Details are given in the Appendix. Adopting the usual definition for the gravitational field $h_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{16\pi G} h_{\mu\nu}, \tag{13}$$

we obtain the result

$$\sigma = \frac{1}{2}\,\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu + \sqrt{16\pi G} A_1 + 16\pi G (A_2 + A_2'), \tag{14}$$

where

$$A_1 = \frac{1}{2}\,\Delta x^\mu \Delta x^\nu \int_0^1 du\, h_{\mu\nu}, \tag{15}$$

$$\begin{aligned} A_2 = & -\frac{1}{2}\,\Delta x^\mu \Delta x^\alpha \Delta x^\beta \int_0^1 du \int_0^u du'\, h_\mu{}^\rho h_{\alpha\beta,\rho} - \Delta x^\mu \Delta x^\alpha \Delta x^\beta \int_0^1 du \int_0^u du'\, h_\mu{}^\rho h_{\rho\alpha,\beta} \\ & + \frac{1}{2}\,\Delta x^\nu \Delta x^\alpha \Delta x^\kappa \int_0^1 du \int_0^u du'\, h_{\alpha\nu,\beta} \int_0^1 dv\, h_\kappa{}^\beta - \frac{1}{2}\,\Delta x^\mu \Delta x^\nu \Delta x^\alpha \int_0^1 du\, h_{\mu\nu,\rho} \int_0^u du'\, h_\alpha{}^\rho \\ & + \frac{1}{2}\,\Delta x^\mu \Delta x^\nu \Delta x^\alpha \int_0^1 du\, u h_{\mu\nu,\rho} \int_0^1 du'\, h_\alpha{}^\rho + \frac{1}{2}\,\Delta x^\alpha \Delta x^\kappa \int_0^1 du\, h_{\nu\alpha} \int_0^1 dv\, h_\kappa{}^\nu - \frac{1}{2}\,\Delta x^\mu \Delta x^\nu h_{\alpha\mu}(0) h_\nu{}^\alpha(0), \end{aligned} \tag{16}$$

and

$$\begin{aligned} A_2' = & \frac{1}{2}\,\Delta x^\nu \Delta x^\alpha \Delta x^\kappa \Delta x^\rho \int_0^1 du \int_0^u du'\, h_{\nu\alpha,\beta} \int_0^{u'} du''\, h_{\kappa\rho}{}^{,\beta} - \frac{3}{8}\,\Delta x^\alpha \Delta x^\beta \Delta x^\kappa \Delta x^\rho \int_0^1 du \int_0^u du'\, h_{\alpha\beta,\nu} \int_0^1 dv \int_0^v dv'\, h_{\kappa\rho}{}^{,\nu} \\ & - \frac{1}{4}\,\Delta x^\mu \Delta x^\nu \Delta x^\alpha \Delta x^\beta \int_0^1 du\, u h_{\mu\nu,\rho} \int_0^1 dv \int_0^v dv'\, h_{\alpha\beta}{}^{,\rho} + \frac{1}{4}\,\Delta x^\mu \Delta x^\nu \Delta x^\alpha \Delta x^\beta \int_0^1 du\, h_{\mu\nu,\rho} \int_0^u du' \int_0^{u'} du''\, h_{\alpha\beta}{}^{,\rho}. \end{aligned} \tag{17}$$

All of the integrals in Eqs. (15)–(17) are to be understood as line integrals along the straight line from x_1 to x_2; for instance, in an integration over u, the integrand is to be evaluated at the point $u x_2 + (1-u) x_1$. Equation (14) is valid for arbitrarily large separations of the points x_1 and x_2, but it assumes that the gravitational fields are weak, so perturbation theory is applicable.

The quantities A_1, A_2, and A_2' differ in their behavior upon path integration: A_1 gives logarithmically divergent contributions to the path integral, A_2 gives quadratically divergent contributions, and A_2' gives even worse divergent contributions. However, as we will see, the contributions from A_2' cancel. This leaves the contributions from A_2 as the dominant terms, and these can be reduced to a finite value by assuming that the bare gravitational constant approaches zero. With this renormalization of the gravitational constant, the logarithmic contributions from A_1 disappear.

III. PATH INTEGRAL FOR THE SMEARED PROPAGATOR

For the sake of simplicity, in the following calculation we provisionally ignore the linear term A_1 in Eq. (14), and we consider the contribution from this term later, at the end of the calculation. Likewise, we provisionally ignore the term $\propto R(x_1)$ in Eq. (7) and we consider its contribution later.

For the calculation of the path integral, it is convenient to write Eq. (7) as an exponential

$$G_F(x_1, x_2; g) = \frac{i}{8\pi^2(\sigma - i\epsilon)} - \frac{i}{16\pi^2}\, m^2 \ln \frac{1}{2}\, m^2 \sigma \tag{18}$$

$$\simeq -\frac{1}{16\pi^2} \int_0^\infty \frac{ds}{s^2} \times \exp[-i m^2 s - i(\sigma - i\epsilon)/2s]. \tag{19}$$

Equations (18) and (19) differ by a fractional amount of the order of $m^2\sigma$, which is negligible if σ is small. The path integral (3) then becomes

$$\begin{aligned} \overline{G}_F(x_1, x_2) &= -\frac{1}{16\pi^2 N'} \int_0^\infty \frac{ds}{s^2} \int Dh \exp\Big[-i m^2 s - i(\sigma - i\epsilon)/2s + i \int d^4x\, \mathcal{L}(g)\Big] \\ &= -\frac{1}{16\pi^2 N'} \int_0^\infty \frac{ds}{s^2} \int Dh \exp\Big[-i m^2 s \end{aligned}$$

$$-i\frac{(x_2-x_1)^2}{4s}-\sqrt{16\pi G}i\frac{A_1}{2s}$$
$$-16\pi iG\frac{A_2+A_2'}{2s}+i\int d^4x\,\mathcal{L}(g)\Bigg],\tag{20}$$

with

$$N'=\int Dh\exp\left[i\int d^4x\,\mathcal{L}(g)\right].\tag{21}$$

To lowest order, the gravitational action is quadratic in the field variables,

$$\int d^4x\,\mathcal{L}(g)=-\frac{1}{2}\int d^4x\int d^4x'\,\mathcal{D}^{\mu\nu\alpha\beta}(x,x')\times h_{\mu\nu}(x)h_{\alpha\beta}(x'),\tag{22}$$

where $\mathcal{D}^{\mu\nu\alpha\beta}$ is a differential operator that consists of derivatives of delta functions. In an abbreviated symbolic notation, in which each subscript stands for the discrete tensor indices and also for the continuous variable x (see, e.g., Weinberg [9]), this action can be written as

$$\int d^4x\,\mathcal{L}(g)=-\frac{1}{2}\mathcal{D}_{rs}h_rh_s\tag{23}$$

and the path integral of this action is

$$\int Dh\exp\left(-\frac{1}{2}i\mathcal{D}_{rs}h_rh_s\right)=[\mathrm{Det}(i\mathcal{D}/2\pi)]^{-1/2}.\tag{24}$$

In Eq. (20) the quadratic terms $16\pi iG(A_2+A_2')/2s$ in the exponent can be regarded as an increment to $\mathcal{D}_{rs}$:

$$\delta\mathcal{D}_{rs}h_rh_s=16\pi G(A_2+A_2')/s.\tag{25}$$

Accordingly, if we ignore the term A_1, the propagator (20) can be evaluated from the corresponding fractional increment in the right-hand side of Eq. (24):

$$\overline{G}_F(x_1,x_2)=-\frac{1}{16\pi^2}\int_0^\infty\frac{ds}{s^2}\exp[-im^2s-i(x_2-x_1)^2/4s]\times\{\mathrm{Det}[i(\mathcal{D}+\delta\mathcal{D})/2\pi]\}^{-1/2}[\mathrm{Det}(i\mathcal{D}/2\pi)]^{1/2}\tag{26}$$

$$\simeq-\frac{1}{16\pi^2}\int_0^\infty\frac{ds}{s^2}\exp\left[-im^2s-i(x_2-x_1)^2/4s-\frac{1}{2}\mathrm{Tr}\,\mathcal{D}^{-1}\delta\mathcal{D}\right].\tag{27}$$

We first consider the contributions to $-\frac{1}{2}\mathrm{Tr}\mathcal{D}^{-1}\delta\mathcal{D}$ from A_2'. The inverse of the operator $\mathcal{D}$ is the conventional graviton propagator,

$$\mathcal{D}^{-1}_{\mu\nu\alpha\beta}(x,x')=-\frac{i}{4\pi^2(x-x')^2}(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta}).\tag{28}$$

Here it is assumed that the gauge problem that arises in the inversion of $\mathcal{D}$ has been handled by the Faddeev-Popov procedure [10]. Each of the terms in A_2' contains contracted derivatives $h_{\mu\nu}{}^{,\rho}(u)h_{\alpha\beta,\rho}(u')$ of the gravitational field; corresponding to this, $-\frac{1}{2}\mathrm{Tr}\,\mathcal{D}^{-1}\delta\mathcal{D}$ receives a contribution

$$\propto\partial'_\rho\partial^\rho\frac{1}{(x-x')^2}\propto\delta^4(x-x')\propto\delta^4(x_2-x_1).\tag{29}$$

But according to Eq. (17), these terms are multiplied by $(x_2-x_1)^4$, which effectively makes them zero.

Next we consider the contributions arising from A_2. The first two terms in A_2 involve the product of a field and its derivative at the same point; the contribution to $-\frac{1}{2}\mathrm{Tr}\,\mathcal{D}^{-1}\delta\mathcal{D}$ from these terms is proportional to the first derivative of the propagator (28) evaluated at $x=x'$. This vanishes if we perform the limit $x\to x'$ symmetrically.

The next four terms yield divergent contributions to $-\frac{1}{2}\mathrm{Tr}\mathcal{D}^{-1}\delta\mathcal{D}$, among which the dominant part is

$$\frac{2iG}{s}\int_0^1du\int_0^u du'\frac{1}{(u-u')^3}.\tag{30}$$

To render this divergent expression finite, we need to set the gravitational constant equal to zero. Specifically, we assume that G in Eq. (30) is actually the bare gravitational constant G_B, and we introduce a (provisional) regularization in the u' integral by replacing $u-u'$ by $u-u'+\epsilon'$, where it is understood that the limit $\epsilon'\to0$ will be taken. Then Eq. (30) becomes

$$\frac{iG_B}{s\epsilon'^2}+\mathcal{O}\left(\frac{G_B}{\epsilon'}\right).\tag{31}$$

To keep this finite, we must take $G_B\propto\epsilon'^2$. In Sec. IV we will see how this conforms with the renormalized gravitational constant observed in gravitational interactions of particles. Anticipating the result that G_B/ϵ'^2 is of the order of magnitude of the renormalized gravitational constant G, we see that Eq. (31) can be written as $ibG/4s$ or $ibL_*^2/4s$, where b is a number of the order of magnitude of 1.

The last term in A_2 gives another divergent contribution to $-\frac{1}{2}\mathrm{Tr}\,\mathcal{D}^{-1}\delta\mathcal{D}$ involving a propagator evaluated at $x=x'$. To determine the limit $x\to x'=0$, we take $x-x'=\epsilon'\sqrt{G}$ and we let $\epsilon'\to0$. The contribution to $-\frac{1}{2}\mathrm{Tr}\,\mathcal{D}^{-1}\delta\mathcal{D}$ is then $\propto G_B/\epsilon^2G\simeq1$. Therefore the last term in A_2 gives a contribution $\simeq\eta_{\mu\nu}\Delta x^\mu\Delta x^\nu$, with a numerical factor of the order of 1. This contribution simply has the effect of multiplying the geodesic distance of flat spacetime by a numerical factor of the order of 1. Physically, this means that, on the average, the gravitational quantum fluctuations modify the measured distances in spacetime. To restore the desirable correspondence between coordinates and measured distances, we have to rescale our coordinates and thereby reset the numerical factor to 1. This rescaling effectively eliminates, or compensates, the last term in A_2.

Finally, we consider the contributions arising from the terms A_1 and $R(x_1)$, which we have ignored so far. The linear terms A_1 in Eq. (20) can be eliminated by a change of variable in the path integral (completion of the square). This generates a quadratic term and gives a logarithmically divergent contribution $\propto G_B \ln \epsilon'$ in the exponent of Eq. (27). But this contribution disappears when we perform the renormalization $G_B = \epsilon'^2 G \to 0$ of the gravitational constant.

The term $(\frac{1}{6} - \xi) R(x_1)$ in Eq. (7) causes troubles in the path integral, because the vacuum expectation value of $R(x_1)$ is highly divergent, and this divergence is not suppressed by our renormalization of the gravitational constant. However, we can deal with this problem by a mass renormalization. We rewrite the bracket in Eq. (7) as

$$\left[m_{\mathrm{ren}}^2 - \left(\frac{1}{6} - \xi \right) R_{\mathrm{ren}}(x_1) \right], \tag{32}$$

where

$$m_{\mathrm{ren}}^2 = m^2 - \left(\frac{1}{6} - \xi \right) \langle R(x_1) \rangle \tag{33}$$

and

$$R_{\mathrm{ren}}(x_1) = R(x_1) - \langle R(x_1) \rangle. \tag{34}$$

This renormalized curvature scalar has a vacuum expectation value of zero, and it is easy to verify that the vacuum expectation value of $R_{\mathrm{ren}}(x_1) h_{\mu\nu}(x)$ is also zero, except for terms of order G^2 and higher. Consequently, in the path integral for the propagator, the terms arising from the renormalized curvature scalar involve only contributions of order G^2 and higher. These contributions are similar to the contributions arising from other higher-order corrections to σ (see Sec. VI).

To order G, the net result is then merely the term (31), which gives a smeared propagator

$$\bar{G}_F(x_1, x_2) = -\frac{1}{16\pi^2} \int_0^\infty \frac{ds}{s^2} \exp[-im^2 s - i(x_2 - x_1)^2/4s + i(bL_*^2 + i\epsilon)/4s] \tag{35}$$

$$= -\frac{m}{4\pi^2} \frac{K_1[im\sqrt{(x_2 - x_1)^2 - bL_*^2 - i\epsilon}]}{\sqrt{(x_2 - x_1)^2 - bL_*^2 - i\epsilon}}. \tag{36}$$

Evidently, this is of the form proposed in Eq. (1), with a spectral function $f(\lambda) = \delta(\lambda - bL_*^2)$.

The result established here for a single propagator can be easily generalized to the product of several propagators, such as occur in a typical Feyman diagram. At short range, each of the several propagator will contribute a σ term in an exponent as in Eq. (19), and the path integration $\int Dh$ will yield a product of propagators of the form (36), with the same value of b for each propagator.

In all of the above we have concentrated on the short-range behavior of the propagator. We expect that the smeared propagator (36) will be a good approximation for $|x^2| \ll 1/m^2$. At longer ranges, the propagator will differ from Eq. (36) by terms proportional to m^2, m^4, etc. These terms are hard or impossible to calculate explicitly, but we expect that the conventional Feynman propagator will be a good approximation for $|x^2| \gg 1/M_*^2$. If m is a typical particle mass (say, no more than a few hundred GeV/c^2), then there is a wide overlap between these ranges of $|x^2|$, and this means we have a good approximation for the propagator for *all* values of $|x^2|$. For instance, we can adopt the smeared propagator (36) for the range $|x^2| < 1/mM_*$ and the conventional Feynman propagator $-mK_1(im\sqrt{x^2 - i\epsilon})/4\pi^2\sqrt{x^2 - i\epsilon}$ for the range $|x^2| > 1/mM_*$. At $|x^2| = 1/mM_*$, the fractional deviation between these two expressions is of the order of only $m/M_* \simeq 10^{-19}$. In practice, we can simply adopt the smeared propagator for all values of x^2, since it smoothly merges into the conventional propagator when $|x^2| \gg 1/M_*^2$. The advantage of using the smeared propagator for all values of x^2 is that this results in a simple formula for the Fourier transform, valid for all values of the momentum:

$$\bar{G}_F(p) = -i\sqrt{bL_*^2} \frac{K_1[-\sqrt{bL_*^2(p^2 - m^2 + i\epsilon)}]}{\sqrt{p^2 - m^2 + i\epsilon}}. \tag{37}$$

This expression permits the calculation of Feynman diagrams by familiar techniques [1].

IV. GRAVITON PROPAGATOR; RENORMALIZATION OF *G*

The regularization of the propagator by quantum fluctuations in the geometry applies to all matter propagators. But it does not apply to the graviton propagator. The trouble is that the differential equation for the gravitational field is the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{38}$$

This is not a linear differential equation [11], it yields no simple equation for the graviton propagator, and we cannot assume that this propagator has the Hadamard form (6). Hence we must find a different mechanism to regularize the graviton propagator.

The obvious mechanism arises from quantum fluctuations of the matter fields. It is known that in a curved spacetime background, the quantum fluctuations of the matter fields generate a vacuum energy momentum that alters the matter action by a constant term (a cosmological constant), a term proportional to the curvature scalar R, and a term proportional to the square of the curvature. When these terms are added to the usual action for the Einstein equation (with its bare gravitational constant G_B), we obtain a net action for the gravitational field [12],

$$\int d^4x\sqrt{-g}\left[A+\left(B+\frac{1}{16\pi G_B}\right)R+C\left(\frac{3}{2}R^2+3R_{\mu\nu}R^{\mu\nu}\right)\right],\tag{39}$$

where A, B, and C are divergent coefficients. If the divergences are regularized by point splitting, with a point separation $\tau\rightarrow 0$, then the coefficients take the form

$$A\propto\frac{1}{\tau^4},\quad B\propto\frac{1}{\tau^2},\quad C\propto\ln\tau.\tag{40}$$

The coefficient of R in Eq. (39) gives us the renormalized, observed gravitational constant [13]

$$\frac{1}{16\pi G}=B+\frac{1}{16\pi G_B}.\tag{41}$$

To attain a finite value of G, we need $B\rightarrow -1/16\pi G_B$, and therefore $G_B\propto\tau^2\rightarrow 0$. In the preceding section we regularized the divergences by means of a dimensionless parameter ϵ', which characterized the point splitting. On dimensional grounds, the dimensional and dimensionless point splitting parameters must be related by the Planck length, that is, $\tau^2\propto G\epsilon'^2$. With this, $G_B/\epsilon'^2\propto\tau^2/\epsilon'^2\propto G$, as anticipated in the preceding section [see Eq. (31)].

The other divergent terms in Eq. (39) must be canceled by counterterms. It is desirable not to cancel the C term completely, but to retain a finite remainder, so the renormalized value of C is of the order of 1. This has the advantage that the fourth-order derivatives contained in this term modify the conventional graviton propagator $\propto 1/k^2$ in momentum space to

$$\propto\frac{1}{k^2-C'k^4/M_*^2}.\tag{42}$$

where C' is a constant of the same order of magnitude as C. As shown by Stelle [14], the extra two powers of k in the denominator of Eq. (42) are sufficient to render all gravitational Feynman diagrams renormalizable (although not finite). In this way the quantum fluctuations of matter, which contribute the C term, regularize the propagator for the gravitational field.

Furthermore, if the graviton of the fourth-derivative gravitational theory is supplemented by supersymmetric partners, then the resulting fourth-derivative supersymmetric gravitational theory is finite [15]. The contributions from high-order loops, which in supersymmetric gravity are divergent (and nonrenormalizable), are reduced to finite values by the fourth derivatives, and the low-order loops, which in fourth-derivative theory are divergent, are cancelled by supersymmetry conditions.

We can rewrite the modified graviton propagator as

$$\frac{1}{k^2-C'k^4/M_*^2}=\frac{1}{k^2}-\frac{1}{k^2-M_*^2/C'}.\tag{43}$$

This shows that the modification of the propagator is actually a Pauli-Villars regularization. However, this regularization is the result of a physically motivated alteration of the Lagrangian, in contrast to the conventional Pauli-Villars regularization, which is a purely formal trick for manipulating divergences. The negative sign of the second term on the right-hand side of Eq. (43) indicates that this term is the propagator of a state with negative norm in Hilbert space (a ghost). This renders the physical interpretation of these states problematic, if they occur in external lines of Feynman diagrams. But since their mass is $\simeq M_*$, we do not have to worry about this unless the c.m. energy reaches 10^{19} GeV. Such energies are attained only at a very early stage in the big bang, before the Planck time, 10^{-43} s. At that stage, the physics of the universe is likely to be dominated by nonlinear quantum gravity effects (densely packed black holes of mass $\simeq M_*$?), and the S-matrix and scattering processes of elementary particles in asymptotically free states are probably irrelevant, so the issue of probability conservation for the ghost states never needs to be faced [16]. At ordinary energies, the ghost states occur only in internal lines of Feynman diagrams, where they are no more troublesome than the Faddeev-Popov ghosts needed in internal lines to achieve gauge invariance.

V. APPLICATION TO QED; GAUGE INVARIANCE

In Ref. [1] we used the smeared propagator to calculate the radiative corrections in QED for photons interacting with a charged spin-$\frac{1}{2}$ field, such as electrons. The second-order radiative corrections are all finite, with numerical values of a few percent (roughly the same as those that would be obtained in a calculation with the conventional Feynman propagator modified by an invariant cutoff at M_*^2). In Ref. [1] we had no definite expression for the spectral function $f(\lambda)$, but we conjectured that the spectral function is concentrated at $x^2\simeq L_*^2$. Equation (36) confirms this conjecture and confirms the numerical estimate for the radiative corrections. By keeping the radiative corrections finite and small, we avoid a well-known inconsistency of conventional QED: the charge renormalization factor Z_3 is large and negative, $Z_3\rightarrow -\infty$, and consequently the square of the bare charge is negative, $e_B^2=e^2/Z_3\rightarrow -0$. Landau and Pomeranchuk [17] considered this a fatal flaw of perturbative QED, but modern treatments of QED usually prefer to ignore this issue.

The calculations in Ref. [1] suffer from a deficiency in that the vacuum polarization has a gauge-noninvariant part of large magnitude, $\simeq M_*^2$. According to conventional wisdom, a modification of the electron propagator requires a modification of the photon-electron vertex, to preserve gauge invariance. In Ref. [18] we explored this option, and we constructed a modification of the vertex. Physically, the vertex modification looks like a contribution from charged entities of mass M_* and size L_*, possibly charged black holes.

Here we want to propose a different and more elegant solution of the gauge problem. It has been known for some time [19] that the problem of gauge invariance of the vacuum polarization can be solved by a careful definition of the current operator. For electrons, the current is defined as an operator product $j^\mu=\overline{\psi}\gamma^\mu\psi$, and the two operators on the right-hand side must be defined by point splitting; concomitantly, an extra term $\propto A^{\mu}$ must be added to the current, to preserve gauge invariance. With this refinement in the definition of the current, the Schwinger terms, which break the gauge invariance of the vacuum polarization, are cancelled exactly by seagull terms arising from the extra term in the current. Thus, with a careful, gauge-invariant definition of the current operator, gauge invariance is automatically attained in calculations with the conventional Feynman propagator (and special tricks, such as Pauli-Villars regularization or dimensional regularization, become superfluous for attaining gauge invariance).

Since we want to use the smeared propagator in our calculation, and not the conventional Feynman propagator, how does this help? The answer is that before we perform the path integration $\int Dg$ in Eq. (3)—or in a corresponding expression for the vacuum polarization—we must perform the path integration $\int D\phi$, and we can also elect to perform the integrations $\int d^4x$ over all the spacetime variables that appear in the construction of the vacuum polarization [20]. The path integration $\int D\phi$ leads to conventional Feynman propagators (in the given geometry $g_{\mu\nu}$,), and if the current has been defined in the appropriate gauge-invariant way, the integration over spacetime variables will then result in an automatic cancellation of the gauge-noninvariant part of the vacuum polarization. The subsequent path integration $\int Dg$ will not resurrect the gauge-noninvariant part, and the final result is necessarily gauge invariant.

VI. HIGHER-ORDER TERMS

In the preceding sections we dealt only with the contributions to the smeared propapator arising from the quadratic terms in the gravitational field, and we neglected all the higher-order terms. Such higher-order terms arise from taking more steps in the successive approximation for σ, and they also arise from taking into account the nonlinear interaction terms in the gravitational Lagrangian $\mathcal{L}(g)$.

The quadratic terms are exceptional in that the path integral reduces to a Gaussian integral which can be evaluated analytically. With the inclusion of terms beyond the quadratic terms $\propto hh$, the path integral (3) becomes a non-Gaussian integral, and its evaluation becomes impossible, except by means of an expansion of the exponential in a series in powers of h. But for our purposes such an expansion is useless, since the path integral then leads to a series in powers of $L_*^2/(x_2-x_1)^2$, which does not converge if $|(x_2-x_1)^2|\ll L_*^2$ and, therefore, cannot tell us anything about the short-range behavior of the propagator. The only obvious alternative to a power series is a numerical calculation of the path integral. Whether such a numerical calculation is feasible and how the smeared propagator is modified by higher-order corrections remain open questions.

Although the effect of higher-order corrections is unknown, our second-order results indicate a mutual regularization of matter and gravitation: the quantum fluctuations of the geometry regularize the propagator for matter fields, and the quantum fluctuations of the matter fields regularize the propagator for gravitons.

One odd aspect of this regularization generated by the quantum fluctuations of the geometry is that the smeared propagator $\overline{G_F}$ does not satisfy any simple wave equation and it is not a Green's function. Of course, for any given geometry the Feynman propagator G_F does satisfy the wave equation (5), and it is a Green's function. But when we average these propagators over a large variety of different geometries, there will be no single background geometry and no definite wave equation, and no Green's function. However, if the initial conditions for a propagating wave involve only wavelengths much longer than the Planck length, then the smeared propagator can be effectively replaced by the conventional Feynman propagator, and this resurrects the Green's function. The absence of a solution for the initial value problem on the Planck scale merely reflects the absence of a well-defined background geometry on this scale—if there is no well-defined geometry, then the initial value problem is meaningless. The absence of a well-defined (Riemannian) geometry manifests itself in the shift of the effective light cone in the smeared propagator: instead of a light cone $(x_2-x_1)^2=0$, the smeared propagator has a light cone $(x_2-x_1)^2-bL_*^2=0$. We can also recognize such a modification of the geometry from a direct calculation of the vacuum expectation value of the square of the geodesic distance by means of the path integral $\langle\sigma\rangle \propto\int Dh\sigma\exp[i\int d^4x\,\mathcal{L}(g)]$. The (normalized) value of this path integral has the same form as the expression in the exponent in Eq. (27), that is, $\langle\sigma\rangle=(x_2-x_1)^2-(i/2)\,\mathrm{Tr}\,\mathcal{D}^{-1}\delta\mathcal{D}$, except that $\delta\mathcal{D}$ now does not include the factor $1/s$ appearing in Eq. (25). Upon renormalization of the divergent terms, the result for the expectation value is then $\langle\sigma\rangle=(x_2-x_1)^2-bL_*^2$, and the light cone deduced from this is the same as that deduced from the smeared propagator. For $x_2\to x_1$, the expectation value of the square of the geodesic distance does not approach zero, but $-bL_*^2$. This indicates that even for very small separations between the points, there are fluctuations of the order of L_*^2 in the square of the geodesic distance.

APPENDIX

Substitution of the zeroth approximation $U^{\mu}=x_2^{\mu}-x_1^{\mu}$ into the right-hand side of Eq. (11) gives a first approximation for U^{μ}, which when resubstituted gives the second approximation:

$$U^{\mu}(u)=n^{\mu}-n^{\alpha}n^{\beta}\int_0^u\Gamma^{\mu}_{\alpha\beta}du' +2n^{\alpha}n^{\kappa}n^{\rho}\int_0^u\Gamma^{\mu}_{\alpha\beta}du'\int_0^{u'}\Gamma^{\beta}_{\kappa\rho}du''. \tag{A1}$$

Equation (12) then gives

$$x_2^{\mu}=x_1^{\mu}+n^{\mu}-n^{\alpha}n^{\beta}\int_0^1du\int_0^u\Gamma^{\mu}_{\alpha\beta}du' +2n^{\alpha}n^{\kappa}n^{\rho}\int_0^1du\int_0^u\Gamma^{\mu}_{\alpha\beta}du'\int_0^{u'}\Gamma^{\beta}_{\kappa\rho}du''. \tag{A2}$$

This equation can, again, be solved for n^μ by successive approximations, starting with the zeroth approximation $n^\mu = \Delta x^\mu = x_2^\mu - x_1^\mu$. The second approximation then yields

$$\begin{aligned}
\sigma &= \frac{1}{2} g_{\mu\nu}(0) U^\mu(0) U^\nu(0) = \frac{1}{2} g_{\mu\nu}(0) n^\mu n^\nu \\
&= \frac{1}{2} \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu + \frac{1}{2} h_{\mu\nu}(0) \Delta x^\mu \Delta x^\nu + \eta_{\mu\nu} \Delta x^\mu \Delta x^\alpha \Delta x^\beta \int_0^1 du \int_0^u du' \, \Gamma^\nu_{\alpha\beta} + h_{\mu\nu}(0) \Delta x^\mu \Delta x^\alpha \Delta x^\beta \int_0^1 du \int_0^u du' \, \Gamma^\nu_{\alpha\beta} \\
&\quad - 2 \eta_{\mu\nu} \Delta x^\nu \Delta x^\alpha \Delta x^\kappa \Delta x^\rho \int_0^1 du \int_0^u du' \, \Gamma^\mu_{\alpha\beta} \int_0^{u'} du'' \Gamma^\beta_{\kappa\rho} + 2 \eta_{\mu\nu} \Delta x^\nu \Delta x^\alpha \Delta x^\kappa \Delta x^\rho \int_0^1 du \int_0^u du' \, \Gamma^\mu_{\alpha\beta} \int_0^1 dv \int_0^v dv' \, \Gamma^\beta_{\kappa\rho} \\
&\quad + \frac{1}{2} \eta_{\mu\nu} \Delta x^\alpha \Delta^\beta \Delta x^\kappa \Delta x^\rho \int_0^1 du \int_0^u du' \, \Gamma^\mu_{\alpha\beta} \int_0^1 dv \int_0^v dv' \, \Gamma^\nu_{\kappa\rho} + \frac{1}{2} \Delta x^\mu \Delta x^\alpha \Delta x^\beta \int_0^1 du \int_0^u du' \, \partial_\mu (h_{\alpha\beta,\rho} \delta x^\rho), \qquad \text{(A3)}
\end{aligned}$$

where

$$\delta x^\rho(u') = \Delta x^\kappa \Delta x^\nu \left[u' \int_0^1 du'' \int_0^{u''} du''' \Gamma^\rho_{\kappa\nu} - \int_0^{u'} du'' \int_0^{u''} du''' \Gamma^\rho_{\kappa\nu} \right]. \qquad \text{(A4)}$$

Here all of the integrals are to be understood as line integrals along the straight line from x_1 to x_2; for instance, in an integration over u, the integrand is to be evaluated at the point $u x_2 + (1-u) x_1$. The deviation of the actual geodesic from the straight line has been taken into account in these equations by appropriate Taylor series expansions relative to the straight line. To second order, the only integral that needs this correction is that in the third term on the first line of Eq. (A3), and this correction is written in the last term on the last line.

With

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} (\eta^{\mu\nu} - \sqrt{16\pi G} h^{\mu\nu})(h_{\alpha\nu,\beta} + h_{\beta\nu,\alpha} - h_{\alpha\beta,\nu}) + \cdots, \qquad \text{(A5)}$$

σ becomes a functional of $h_{\mu\nu}$ and its derivatives $h_{\mu\nu,\alpha}$. Some of these derivatives can be eliminated by means of the identity

$$\Delta x^\alpha \partial_\alpha F(x) = dF/du, \qquad \text{(A6)}$$

where $x = u x_2 + (1-u) x_1$. This permits integration of the total derivative appearing on the right-hand side. The final result is as stated in Eq. (14).

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[11] If we treat a (weak) gravitational-wave field as a perturbation of a fixed classical background, we obtain a linear differential equation in that background. But this does not help us here, since any attempt at rearranging the path integral $\int Dg$ into a double path integral $\int\int Dg_0 Dh$ over the background and the perturbation would mean that we are adopting two independent gravitational fields, with a doubling of the degrees of freedom.

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[13] A negative value of B is required for the cancellation of the divergences in B and $1/G_B$. The sign of B depends on the signs of τ^2 and of $(\frac{1}{6} - \xi)$, since $B \propto (\frac{1}{6} - \xi)/\tau^2$. Hence τ^2 must be taken spacelike if $(\frac{1}{6} - \xi) > 0$ and timelike if $(\frac{1}{6} - \xi) < 0$. The conformal case $\xi = \frac{1}{6}$ does not lend itself to the proposed renormalization.

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