Naked singularities and Seifert's conjecture

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It is shown that for a general nonstatic spherically symmetric metric of the Kerr-Schild class several energy-momentum complexes give the same energy distribution as in the Penrose prescription, obtained by Tod. This result is useful for investigating the Seifert conjecture for naked singularities. The naked singularity forming in the Vaidya null dust collapse supports the Seifert conjecture. Further, an example and a counter-example to this conjecture are presented in the Einstein massless scalar theory. [S0556-2821(99)00520-2]

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In a series of seminal papers it was established that under a wide variety of circumstances a spacetime singularity is inevitable in a physically realistic complete gravitational collapse [1]. However, these studies were silent on a very important question: whether or not a spacetime singularity arising therefrom will be visible to observers. To this end about three decades ago, Penrose [2] in his review on gravitational collapse asked ''Does there exist a cosmic censor who forbids the occurrence of naked singularities, clothing each one in an absolute event horizon?'' Though there is no precise statement of a *cosmic censorship hypothesis* (CCH), roughly speaking it states that naked singularities do not occur in generic realistic gravitational collapse. There are two versions of the CCH: the hypothesis that, generically, singularities forming due to gravitational collapse are hidden inside black holes is known as the *weak cosmic censorship hypothesis*, whereas the *strong cosmic censorship hypothesis* states that, generically, timelike singularities never occur (see [3] and references therein). No proof for any version of the CCH is known and it is considered the most important unsolved problem in classical general relativity [4–6]. The subject of singularities fascinated many researchers (see [1–16] and references therein).

Since the CCH was proposed, numerous counterexamples to this hypothesis have been found in the literature (see [8–11] and references therein). Iguchi *et al.* [12] examined the stability of nakedness of singularity in the Lemaitre-Tolman-Bondi collapse against the odd-parity modes of nonspherical linear perturbations for the metric and reported that the perturbations do not diverge but are well defined even in the neighborhood of a central naked singularity. Much work has been done on black holes as well as on naked singularities, but it was not known how black holes could be differentiated observationally from naked singularities (if these indeed exist). Recently it has been shown that the gravitational lensing could be possibly helpful for this purpose [13]. As the known counterexamples to CCH are of special geometric nature, it is still believed by most that the conjecture correctly characterizes realistic gravitational collapse. Recently, Wald [3] reviewed the status of the weak cosmic censorship and expressed his view that naked singularities cannot arise generically.

The idea of naked singularities has been disliked by many physicists as their existence is thought to give serious problems. For instance, there can be production of matter and/or radiation out of extremely high gravitational fields and the mechanism for that is not known and there can be other causal influences from the infinitely compressed material. When the first few counterexamples to the CCH were obtained it was clear that the CCH could not be proved in generality. Penrose [14] imposed a condition that a naked singularity must be proved to be stable against perturbations of initial conditions as well as equations of state. Seifert [15] conjectured that any singularity that occurs if a finite nonzero amount of matter tends to collapse into one point is hidden and naked singularities occur only if one has singularities along lines or surfaces or if the central singularities are carefully arranged that they contain only zero mass.

There have been some discussions on ''mass'' of naked singularities [5,16], but not many studies have been done. The investigation of this subject is difficult as there is no agreed and precise definition of local or quasilocal mass in general relativity and it has been a ''recalcitrant'' problem since the outset of this theory. Bergqvist [17] performed calculations with several different definitions of mass (their uses are not restricted to ''Cartesian coordinates'') and reported that not any two of them give the same result for the Reissner-Nordström and Kerr spacetimes. On the other hand it is also known that several *energy-momentum complexes* (their uses are restricted to ''Cartesian coordinates'') give the same and reasonable results for some well known spacetimes [18–20]. These are encouraging results and we will use some of them to investigate the Seifert conjecture. We use geometrized units and follow the convention that Latin (Greek) indices run from $0\ldots3$ $(1\ldots3)$. The comma and semicolon, respectively, stand for the partial and covariant derivatives.

The Einstein energy-momentum complex is [21]

$$\Theta_i{}^k = \frac{1}{16\pi} H_i{}^{kl}{}_{,l}, \tag{1}$$

where

$$H_i{}^{kl} = -H_i{}^{lk} = \frac{g_{in}}{\sqrt{-g}}[-g(g^{kn}g^{lm} - g^{ln}g^{km})]_{,m}. \tag{2}$$

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$\Theta_0{}^0$ and $\Theta_\alpha{}^0$ stand for the energy and momentum density components, respectively.[1] The energy and momentum components are given by

$$P_i = \int\int\int \Theta_i{}^0 dx^1 dx^2 dx^3. \tag{3}$$

Applying Gauss's theorem one obtains

$$P_i = \frac{1}{16\pi}\int\int H_i{}^{0\alpha}\, n_\alpha\, dS, \tag{4}$$

where n_α is the outward unit normal vector over the infinitesimal surface element dS and P_0 and P_α stand for the energy and momentum components, respectively.

The symmetric energy-momentum complex of Landau and Lifshitz is [23]

$$L^{ik} = \frac{1}{16\pi}\lambda^{iklm}{}_{,lm}, \tag{5}$$

where

$$\lambda^{iklm} = -g(g^{ik}g^{lm} - g^{il}g^{km}). \tag{6}$$

L^{00} and $L^{\alpha 0}$ are the energy and energy current (momentum) density components. The energy and momentum are given by

$$P^i = \int\int\int L^{i0} dx^1 dx^2 dx^3. \tag{7}$$

Using Gauss's theorem, the energy and momentum components are

$$P^i = \frac{1}{16\pi}\int\int \lambda^{i0\alpha m}{}_{,m}\, n_\alpha\, dS. \tag{8}$$

The symmetric energy-momentum complex of Papapetrou is [24]

$$\Sigma^{ik} = \frac{1}{16\pi}N^{iklm}{}_{,lm}, \tag{9}$$

where

$$N^{iklm} = \sqrt{-g}(g^{ik}\eta^{lm} - g^{il}\eta^{km} + g^{lm}\eta^{ik} - g^{lk}\eta^{im}), \tag{10}$$

with

$$\eta^{ik} = \mathrm{diag}(1,-1,-1,-1). \tag{11}$$

Σ^{00} and $\Sigma^{\alpha 0}$ are the energy and energy current (momentum) density components. The energy and momentum components are given by

$$P^i = \int\int\int \Sigma^{i0} dx^1 dx^2 dx^3. \tag{12}$$

For the time-independent metrics, one has

$$P^i = \frac{1}{16\pi}\int\int N^{i0\alpha\beta}{}_{,\beta}\, n_\alpha\, dS. \tag{13}$$

The symmetric energy-momentum complex of Weinberg is [25]

$$W^{ik} = \frac{1}{16\pi}D^{lik}{}_{,l}, \tag{14}$$

where

$$D^{lik} = \frac{\partial h^a{}_a}{\partial x_l}\eta^{ik} - \frac{\partial h^a{}_a}{\partial x_i}\eta^{lk} - \frac{\partial h^{al}}{\partial x^a}\eta^{ik} + \frac{\partial h^{ai}}{\partial x^a}\eta^{lk} + \frac{\partial h^{lk}}{\partial x_i} - \frac{\partial h^{ik}}{\partial x_l} \tag{15}$$

and

$$h_{ik} = g_{ik} - \eta_{ik}. \tag{16}$$

η_{ik} is the Minkowski metric. Indices on h_{ik} or $\partial/\partial x_i$ are raised or lowered with help of η's. It is clear that

$$D^{lik} = -D^{ilk}. \tag{17}$$

W^{00} and $W^{\alpha 0}$ are the energy and energy current (momentum) density components. The energy and momentum components are given by

$$P^i = \int\int\int W^{i0} dx^1 dx^2 dx^3. \tag{18}$$

Using Gauss's theorem, one has

$$P^i = \frac{1}{16\pi}\int\int D^{\alpha 0 i}\, n_\alpha\, dS. \tag{19}$$

Though the uses of the energy-momentum complexes are restricted to ''Cartesian coordinates'' (i.e., these give meaningful results in these coordinates), these satisfy the *local conservation laws* ($\Theta_i{}^k{}_{,k}=0$, $L_{,k}{}^{ik}=0$, $\Sigma_{,k}{}^{ik}=0$, $W_{,k}{}^{ik}=0$) in all systems of coordinates.

We first discuss some of our earlier results in brief. Then we use them to show that the energy-momentum complexes of Einstein, Landau and Lifshitz, Papapetrou and Weinberg, and the Penrose quasilocal definition give the same result for a general nonstatic spherically symmetric metric of the Kerr-Schild class. The Kerr-Schild class spacetimes are given by metrics g_{ik} of the form

$$g_{ik} = \eta_{ik} - H l_i l_k, \tag{20}$$

[1] Though the energy-momentum complex obtained by Tolman differs in form from the Einstein energy-momentum complex, both are equivalent in import [22]. The present author was earlier unaware of this [18–20].

where $\eta_{ik}=\mathrm{diag}(1,-1,-1,-1)$ is the Minkowski metric. H is the scalar field and l_i is a null, geodesic and shear free vector field in the Minkowski spacetime, which are respectively expressed as

$$\begin{aligned} \eta^{ab}l_a l_b&=0,\\ \eta^{ab}l_{i,a}l_b&=0,\\ (l_{a,b}+l_{b,a})l^a{}_{,c}\eta^{bc}-(l^a{}_{,a})^2&=0. \end{aligned} \tag{21}$$

An interesting feature of the Kerr-Schild class metric g_{ik} in Eq. (20) is that the vector field l_i remains null, geodesic and shear-free with the metric g_{ik}. Equations (21) lead to

$$\begin{aligned} g^{ab}l_a l_b&=0,\\ g^{ab}l_{i;a}l_b&=0,\\ (l_{a;b}+l_{b;a})l^a{}_{;c}g^{bc}-(l^a{}_{;a})^2&=0. \end{aligned} \tag{22}$$

There are several well-known spacetimes of the Kerr-Schild class, for instance, Schwarzschild, Reissner-Nordström, Kerr, Kerr-Newman, Vaidya, Dybney *et al.*, Kinnersley, Bonnor-Vaidya, and Vaidya-Patel (for references see in [26]).

It is known that the energy-momentum complexes of Einstein $\Theta_i{}^k$, Landau and Lifshitz L^{ik}, Papapetrou Σ^{ik} and Weinberg W^{ik} "coincide" for any Kerr-Schild class metric [20].

These energy-momentum complexes for any Kerr-Schild class metric are given by [20]

$$\Theta_i{}^k=\eta_{ij}L^{jk}, \tag{23}$$

$$L^{ik}=\Sigma^{ik}=W^{ik}=\frac{1}{16\pi}\Lambda^{iklm}{}_{,lm}, \tag{24}$$

where

$$\Lambda^{ikpq}\equiv H(\eta^{ik}l^p l^q+\eta^{pq}l^i l^k-\eta^{ip}l^k l^q-\eta^{kq}l^i l^p). \tag{25}$$

To obtain the above result for the Landau and Lifshitz, Papapetrou and Weinberg complexes in terms of the scalar function H and the vector l_i, only null condition of Eqs. (21) was used while for the Einstein complex the null as well as geodesic conditions were used. The shear-free condition was not required to obtain Eqs. (23)–(25). Thus, these energy-momentum complexes "coincide" for a class of solutions more general than the Kerr-Schild class. The energy and momentum components are

$$P^i=\frac{1}{16\pi}\int\int \Lambda^{i0\alpha m}{}_{,m}\, n_\alpha\, dS. \tag{26}$$

The energy-momentum complexes of Landau and Lifshitz, Papapetrou and Weinberg are symmetric in their indices and therefore have been used to define angular momentum; the spatial components of angular momentum are (though we do not use this in this paper we give here for completeness)

$$J^{\alpha\beta}=\frac{1}{16\pi}\int\int (x^\alpha\Lambda^{\beta0\sigma m}{}_{,m}-x^\beta\Lambda^{\alpha0\sigma m}{}_{,m}+\Lambda^{\alpha0\sigma\beta})n_\sigma\, dS. \tag{27}$$

Now we consider a general nonstatic spherically symmetric spacetime of the Kerr-Schild class given by the line element

$$ds^2=B(u,r)du^2-2du dr-r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{28}$$

Transforming the above line element in t,x,y,z coordinates according to

$$\begin{aligned} u&=t+r,\\ x&=r\sin\theta\cos\phi,\\ y&=r\sin\theta\sin\phi,\\ z&=r\cos\theta, \end{aligned} \tag{29}$$

one gets

$$ds^2=dt^2-dx^2-dy^2-dz^2-(1-B(t,x,y,z))\times\left[dt+\frac{xdx+ydy+zdz}{r}\right]^2. \tag{30}$$

This is obviously a Kerr-Schild class metric with $H=1-B$ and $l_i=(1,\, x/r,y/r,z/r)$. Using these in Eqs. (25) and (26) one gets the expression for energy

$$E=\frac{r}{2}(1-B(u)). \tag{31}$$

Tod [27], using the Penrose quasilocal mass definition, got the same result. This is indeed an encouraging result. The Penrose mass is called quasilocal because it is obtained over two-surface and is not found by an integral over a spanning three-surface of a local density. The above expression can be used to test the Seifert conjecture for the naked singularities arising due to a general spherical collapse described by Eq. (28) and satisfying the *weak energy condition*. These investigations could give conditions on $B(u)$ for the Seifert conjecture to be true and to be false. It could be possible that the Seifert conjecture is true in all these cases. The Vaidya null dust collapse is extensively studied (see [9] and references therein). $B=1-2M(u)/r$ in Eq. (28) gives the Vaidya null dust collapse solution. For this, Eq. (31) gives $E=M(u)$ (see also [19]). It is known that for $M=\lambda u$, a naked singularity occurs at $r=0$, $u=0$ for $\lambda\leqslant 1/16$. Physically, dM/du represents the power (energy flowing per unit time) imploding through a two-sphere of radius r and it must be non-negative for the *weak energy condition* to be satisfied. Using Eq. (31) one finds that the naked singularity in the Vaidya null dust collapse is massless. This supports the Seifert conjecture. For the Bonnor-Vaidya spacetime $B=1-2M(u)/r+Q(u)^2/r^2$, where $Q(u)$ is the charge parameter. This represents charged null dust collapse. The result in Eq. (31) can be useful for investigations of the Seifert conjecture in the Bonnor-Vaidya collapse.

Now we wish to investigate the Seifert conjecture in the Einstein massless scalar (EMS) theory, given by equations

$$R_{ij}-\frac{1}{2}Rg_{ij}=8\pi T_{ij} \tag{32}$$

and

$$\Phi_{,i}{}^{;i}=0. \tag{33}$$

R_{ij} is the Ricci tensor and R is the Ricci scalar. Φ stands for the massless scalar field. T_{ij}, the energy-momentum tensor of the massless scalar field, is given by

$$T_{ij}=\Phi_{,i}\Phi_{,j}-\frac{1}{2}g_{ij}g^{ab}\Phi_{,a}\Phi_{,b}. \tag{34}$$

Equation (32) with Eq. (34) can be expressed as

$$R_{ij}=8\pi\Phi_{,i}\Phi_{,j}. \tag{35}$$

It is known that the most general static spherically symmetric solution to the EMS equations (with the cosmological constant $\Lambda=0$) is asymptotically flat and until recently it was known that this is the well-known Wyman solution [28]. Recently, it has been shown that the Janis-Newman-Winicour (JNW) solution (which was obtained about thirteen years before the Wyman solution) is the same as the Wyman solution [29]. This solution is characterized by two constant parameters, the mass M and the ''scalar charge'' q, and is given by the line element (see in [29])

$$ds^2=\left(1-\frac{b}{r}\right)^{\gamma}dt^2-\left(1-\frac{b}{r}\right)^{-\gamma}dr^2 -\left(1-\frac{b}{r}\right)^{1-\gamma}r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{36}$$

and the scalar field

$$\Phi=\frac{q}{b\sqrt{4\pi}}\ln\left(1-\frac{b}{r}\right), \tag{37}$$

where

$$\gamma=\frac{2M}{b}, \qquad b=2\sqrt{M^2+q^2}. \tag{38}$$

For the ''scalar charge'' zero this solution reduces to the Schwarzschild solution. $r=b$ is a globally naked strong curvature singularity (see [30] with [29]). It is of interest to investigate whether or not this naked singularity is massless. Obviously Eq. (31) cannot be used for this spacetime. In the following we obtain the energy expression for the most general nonstatic spherically symmetric metric described by the line element

$$ds^2=B(r,t)dt^2-A(r,t)dr^2-2F(r,t)dtdr -D(r,t)r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{39}$$

We transform the line element (39) to ''Cartesian coordinates'' t,x,y,z (according to $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$) and then calculate energy distribution associated with this using definitions of Einstein, Landau and Lifshitz, and Weinberg; these are respectively given by

$$E_E=\frac{r[B(A-D-D'r)-F(r\dot{D}-F)]}{2\sqrt{AB+F^2}}, \tag{40}$$

$$E_{LL}=\frac{rD(A-D-rD')}{2}, \tag{41}$$

$$E_W=\frac{r(A-D-rD')}{2}. \tag{42}$$

The prime and dot denote for the partial derivative with respect to the coordinates r and t, respectively. These energy-momentum complexes have an advantage that one can easily apply Gauss's theorem to get energy contained inside a two-surface for any spacetime. The same is possible with the Papapetrou energy-momentum complex for the Kerr-Schild class metrics, but not for arbitrary spacetimes. However, while using the Papapetrou energy-momentum complex one can apply Gauss's theorem for any time-independent metric [see Eq. (13)]. Thus, for static and spherically symmetric spacetimes ($F=0$ and A,B and D depending only on radial coordinate) one gets

$$E_P=\frac{r}{8(AB)^{3/2}}[4AB^2(A-D)+r(A^2B'D-2A^2BD' -AA'BD-2AB^2D'-ABB'D+A'B^2D)]. \tag{43}$$

In a series of papers there are encouraging results (see in [18–20]). Several energy-momentum complexes give the same results for the Einstein-Rosen as well as any metric of the Kerr-Schild class (the Kerr-Schild class has several well-known solutions). As mentioned earlier these energy-momentum complexes ''coincide'' for a more general class than the Kerr-Schild class. However, it is obvious from Eqs. (40)–(43) that the energy-momentum complexes disagree for the most general nonstatic spherically symmetric metric. While obtaining the energy distribution [given by Eq. (31)] for the spacetime described by the line element (28) we used Kerr-Schild Cartesian coordinates; however, for the case of the most general nonstatic spherically symmetric metric we used what we call ''Schwarzschild Cartesian coordinates.'' The line element (28) is a special case of the line element (39). The Einstein energy-momentum complex gives the same result in both the cases [consider Eq. (31) as a special case of Eq. (40)] which is the same as Tod found using the Penrose definition. However, other definitions disagree with their own results obtained in Kerr-Schild Cartesian and Schwarzschild Cartesian coordinates [as Eqs. (41), (42) and (43) do not yield Eq. (31) as a special case]. For a simple case of the Schwarzschild metric, $E_{LL}=E_P=E_W=M$ when calculations are performed in the Kerr-Schild Cartesian coordinates; however, in Schwarzschild Cartesian coordinates one has $E_{LL}=E_W=M(1-2M/r)^{-1}, E_P=M(r^2-2Mr+2M^2)/(r-2M)^2$. It is not clear why different energy-momentum complexes "coincide" in the Kerr-Schild Cartesian coordinates, but not in the Schwarzschild Cartesian coordinates. The Einstein-Rosen metric is not of the Kerr-Schild class and therefore calculations were not done in Kerr-Schild Cartesian coordinates; however, different energy-momentum complexes gave the same and reasonable result. Therefore, the fact that different energy-momentum complexes give the same result for some spacetimes is not restricted to the use of the Kerr-Schild Cartesian coordinates. It is known that the quasilocal mass definitions also have some problems (see in [17,31]); for instance, it has not been possible to obtain the Penrose quasilocal mass for the Kerr metric.

The present investigations indicate that the Einstein energy-momentum complex is better than other energy-momentum complexes. Therefore, the result in Eq. (40) may be useful for investigating the Seifert conjecture in the context of naked singularities arising due to spherical collapses, which are being investigated (see [10,11] and references therein). $F=0$ in Eq. (40) gives one obtained earlier (see in [29]). For the JNW spacetime this equation immediately gives

$$E=M. \tag{44}$$

The energy expression is independent of the radial distance and therefore the entire energy is confined to the singularity. This is a counterexample to the Seifert conjecture. However, for the purely scalar field ($M=0$ in the JNW solution), the globally naked strong curvature singularity, $r=2q$, is massless. This supports the Seifert conjecture. The "scalar charge" in the JNW spacetime does not contribute to the energy, because contributions from the matter and the field energy cancel. The anonymous referee mentioned that there are some indications that the JNW solution does not occur generically. Therefore, the present counterexample to the Seifert conjecture may not be taken seriously. However, if Eq. (40) can be considered to be the correct expression for the energy distribution, then Eq. (44) demonstrates that one cannot prove the Seifert conjecture in generality.

Now we summarize these in the following. The possibility of the energy localization in general relativity has been debated and there are mutually contradicting viewpoints on this issue [32]. According to Bondi, a nonlocalizable form of energy is inadmissible in relativity and its location can in principle be found [32]. In fact, a unanimously agreed precise definition of local or even quasilocal mass would have been very much useful to understand some important issues in relativity. For instance, for investigating the Seifert conjecture (we have discussed in this paper) and the *hoop conjecture* [33], which states that horizons form when and only when a mass M gets compacted into a region whose circumference in *every direction* $\leqslant 2\pi M$, these concepts are useful. However, no adequate prescription is known and it is not clear if it is possible at all. The results obtained using the energy-momentum complexes are usually not taken seriously, because their uses are restricted to "Cartesian coordinates" and the quasilocal mass definitions are also not satisfactory. Bergqvist [17] showed that not any two of seven quasilocal mass definitions he considered give the same result for the Reissner-Nordström as well as Kerr spacetimes. The well-known Penrose quasilocal mass definition could not deal with the Kerr metric [31]. On the other hand, several energy-momentum complexes are known to give the same and "reasonable" result for many well known solutions. We showed that different energy momentum complexes disagree when they are evaluated in Schwarzschild Cartesian coordinates and give the same result in Kerr-Schild Cartesian coordinates; however, the Einstein energy-momentum complex still gives consistent results in both cases. It is not clear why different definitions "coincide" when calculations are carried out in Kerr-Schild Cartesian coordinates, but disagree in Schwarzschild Cartesian coordinates. At this stage it is not known if this is accidental or points out something interesting. Any example or counterexample to the Seifert conjecture may not be taken seriously unless an adequate prescription for localization or quasilocalization of mass is known and is applied.

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