Scattering by a Reissner-Nordström black hole: The dipole radiation

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The coupled gravitational and electromagnetic dipole perturbations of the Reissner-Nordström solution are studied and the relation between the energy of the incident, reflected, and absorbed electromagnetic waves is obtained. The effect on the polarization of the dipole electromagnetic waves is also discussed and it is shown that the radial functions appearing in the solution of the perturbation equations satisfy differential relations analogous to the Teukolsky-Starobinsky identities. [S0556-2821(99)04620-2]

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I. INTRODUCTION

The linear perturbations of the Reissner-Nordström (RN) solution have been the subject of many investigations (see, e.g., Refs. [1–9]). In spite of the coupling of the gravitational and electromagnetic perturbations produced by the background electromagnetic field, the symmetry of the RN solution allows us to decompose the perturbations into independent multipoles, characterized by the integers j and m, and to reduce the perturbation equations for each multipole to one-dimensional Schrödinger-type equations.

A very convenient way of studying the perturbations of the RN solution is based on the use of expressions for the complete metric and vector potential perturbations in terms of a set of complex potentials that obey a system of four first-order partial differential equations [10,11,7]. As shown in Refs. [9,12], by means of this approach one can easily derive many results associated with this problem; in particular, for each multipole with $j\geqslant 2$, one can explicitly demonstrate the conservation of the energy of the combined gravitational and electromagnetic radiation, find the expression for the energy reflection and transmission coefficients and the correct conversion factors of the energy of gravitational radiation into energy of electromagnetic radiation, and vice versa, as well as the effect on the polarization of the waves and differential identities of Teukolsky-Starobinsky type.

An interesting fact is that the differential equations arising in the treatment of the perturbations of the RN solution by means of the complex potentials mentioned above coincide with those obtained using other approaches, such as the one based on the perturbations of the Newman-Penrose quantities (see Ref. [3] and Sec. IV) and the one based on the direct integration of the linearized Einstein-Maxwell equations [4].

As pointed out in Refs. [7,13], the multipoles with $j=1$, corresponding to the dipole perturbations, require a separate treatment, which is related to the fact that there is no dipole gravitational radiation (though the metric and the curvature are perturbed). In this paper we extend the results of Refs. [3,9,12] to the dipole perturbations. Specifically, we find expressions for the energy of the dipole electromagnetic radiation that is reflected and absorbed by a RN black hole, showing that the energy is conserved. We show that, as in the case of the multipoles with $j\geqslant 2$, the electric charge of the black hole makes that a circularly polarized incident wave be reflected as an elliptically polarized wave and we obtain differential identities satisfied by the radial functions appearing in the solution of the perturbation equations. We follow the notation employed in Ref. [9], which differs only slightly from that of Refs. [6,7] (mainly in those symbols used to denote the spin coefficients or the tetrad vectors).

II. REDUCTION OF THE PERTURBATION EQUATIONS

As shown in Refs. [10,11], the metric and vector potential perturbations of a solution of the Einstein-Maxwell equations, such that the background electromagnetic field is algebraically general and one of its principal null directions is geodetic and shear-free (as the RN solution), in a frame such that φ_1 is the only nonvanishing component of the background electromagnetic field and $\kappa=\sigma=0$, are given by

$$h_{\mu\nu}=2\{l_\mu l_\nu[(\delta+3\beta+\bar{\alpha}-\tau)M_{1'}-\bar{\lambda}M_{0'}]+m_\mu m_\nu(D+3\varepsilon-\bar{\varepsilon}-\rho)M_{0'}-l_{(\mu}m_{\nu)}[(D+3\varepsilon+\bar{\varepsilon}-\rho+\bar{\rho})M_{1'}+(\delta+3\beta-\bar{\alpha}-\tau-\bar{\pi})M_{0'}]\}+\text{c.c.},\tag{1}$$

$$b_\mu=\tfrac{1}{2}[l_\mu(\delta+2\beta+\tau)\psi_{\rm E}-m_\mu(D+2\varepsilon+\rho)\psi_{\rm E}]+\text{c.c.},\tag{2}$$

up to gauge transformations, with the complex scalar potentials $M_{0'},M_{1'},\psi_{\rm E}$ governed by the equations

$$2\varphi_1[(\Delta+2\gamma+\mu)\psi_{\rm E}-(\delta+4\beta+2\tau)\psi_{\rm G}]=(3\Psi_2-2\Phi_{11})M_{1'},$$

$$2\varphi_1[(\bar{\delta}+2\alpha+\pi)\psi_{\rm E}-(D+4\varepsilon+2\rho)\psi_{\rm G}]=(3\Psi_2+2\Phi_{11})M_{0'},$$

$$(\bar{\delta}+3\alpha+\bar{\beta}-\bar{\tau})M_{1'}-(\Delta+3\gamma-\bar{\gamma}+\bar{\mu})M_{0'}=2\varphi_1\psi_{\rm G},$$

$$(D+3\varepsilon+\bar{\varepsilon}+3\rho-\bar{\rho})M_{1'}-(\delta+3\beta-\bar{\alpha}+3\tau+\bar{\pi})M_{0'}$$
$$=4\varphi_1\psi_{\rm E}, \tag{3}$$

where $\psi_{\rm G}$ is an auxiliary complex potential that does not enter explicitly in Eqs. (1) and (2).

Using the null tetrad employed in Refs. [3,6,7], Eqs. (3) admit separable solutions of the form [7]

$$\psi_{\rm G}=(N(r)/r)\,{}_{-2}Y_{jm}(\theta,\varphi)e^{-i\omega t},$$
$$\psi_{\rm E}=\sqrt{2}G(r)\,{}_{-1}Y_{jm}(\theta,\varphi)e^{-i\omega t},$$
$$M_{0'}=2Qg(r)\,{}_{-2}Y_{jm}(\theta,\varphi)e^{-i\omega t},$$
$$M_{1'}=\sqrt{2}Q(n(r)/r)\,{}_{-1}Y_{jm}(\theta,\varphi)e^{-i\omega t}, \tag{4}$$

where Q is the electric charge of the black hole, ${}_sY_{jm}$ are spin-weighted spherical harmonics [14], j and m are integers with $j\geqslant 1$, and $|m|\leqslant j$. Since ${}_sY_{jm}=0$ for $j<|s|$, in the case where $j=1$ only $\psi_{\rm E}$ and $M_{1'}$ do not vanish. Then, from Eqs. (3) one obtains the radial equations

$$r^3\left(\partial_r-i\omega\frac{r^2}{\chi}\right)\frac{n}{r^3}=\frac{2G}{r},$$
$$\frac{\chi^2}{r^3}\left(\partial_r+i\omega\frac{r^2}{\chi}\right)\frac{r^3G}{\chi}=\left(3M-\frac{2Q^2}{r}\right)2n, \tag{5}$$

where $\chi\equiv r^2-2Mr+Q^2$ and M is the mass of the black hole described by the RN solution. Making use of the definitions

$$q_1\equiv 6M,\quad q_2\equiv-\frac{2Q^2}{3M}, \tag{6}$$

and

$$Y_{(-)}\equiv q_1\frac{n}{r^3},\quad X_{(-)}\equiv\frac{r^3G}{\chi}, \tag{7}$$

in terms of the variable r_* defined by $dr_*/dr=r^2/\chi$, Eqs. (5) can be rewritten as

$$\left(\frac{d}{dr_*}-i\omega\right)Y_{(-)}=\frac{\chi^2}{r^8}\frac{2q_1}{r}X_{(-)},$$
$$\left(\frac{d}{dr_*}+i\omega\right)X_{(-)}=\frac{r^4}{\chi}\left(1+\frac{q_2}{r}\right)Y_{(-)}. \tag{8}$$

(The functions $Y_{(-)}$ and $X_{(-)}$ are the analogues of the functions X_{-2} and Y_{-1} introduced by Chandrasekhar [3,6].)

The solution of Eqs. (8) can be expressed in terms of a single radial function $Z^{(+)}$ or $Z^{(-)}$ which obey the Schrödinger-type equations [7,15]

$$\left(-\frac{d^2}{dr_*^2}+V^{(\pm)}\right)Z^{(\pm)}=\omega^2Z^{(\pm)}, \tag{9}$$

where

$$V^{(\pm)}\equiv\pm q_2\frac{df}{dr_*}+q_2^2f^2+2f, \tag{10}$$

$$f\equiv\frac{\chi}{r^3(r+q_2)}. \tag{11}$$

It can be shown that

$$Y_{(-)}=V^{(\pm)}Z^{(\pm)}+(W^{(\pm)}+2i\omega)\left(\frac{d}{dr_*}+i\omega\right)Z^{(\pm)}, \tag{12}$$

$$X_{(-)}=\mp q_2Z^{(\pm)}+f^{-1}\left(\frac{d}{dr_*}+i\omega\right)Z^{(\pm)}, \tag{13}$$

where

$$W^{(\pm)}\equiv-\frac{d}{dr_*}\ln f\mp q_2f, \tag{14}$$

satisfy Eqs. (8) provided that $Z^{(\pm)}$ obeys Eq. (9). Equations (10) and (11) yield the expressions

$$V^{(+)}=\frac{\chi(2r^4+6Mq_2r^2+8Mq_2^2r+3Mq_2^3)}{r^6(r+q_2)^2},$$
$$V^{(-)}=\frac{2\chi}{r^6}(r^2+2Q^2). \tag{15}$$

Since the potentials $V^{(\pm)}$ are of short range, the auxiliary functions $Z^{(\pm)}$ have the asymptotic behaviors $e^{\pm i\omega r_*}$ for $r_*\to\infty$ (i.e., $r\to\infty$) and $r_*\to-\infty$ (the horizon of the black hole). Assuming that there are no waves emerging from the horizon, the functions $Z^{(\pm)}$ must have the asymptotic forms

$$Z^{(\pm)}\to A^{(\pm)}e^{-i\omega r_*}+B^{(\pm)}e^{i\omega r_*}\quad(r_*\to+\infty),$$
$$\to C^{(\pm)}e^{-i\omega r_*}\quad(r_*\to-\infty). \tag{16}$$

Since the potentials $V^{(\pm)}$ are real, it follows from Eq. (9) that $|A^{(\pm)}|^2-|B^{(\pm)}|^2=|C^{(\pm)}|^2$. Hence, the amplitudes

$$R^{(\pm)}\equiv\frac{B^{(\pm)}}{A^{(\pm)}},\quad T^{(\pm)}\equiv\frac{C^{(\pm)}}{A^{(\pm)}} \tag{17}$$

satisfy the relations

$$|R^{(\pm)}|^2+|T^{(\pm)}|^2=1. \tag{18}$$

We shall show below that, as in the cases considered in Refs. [3,6], the two potentials $V^{(+)}$ and $V^{(-)}$ lead to the same reflection and transmission coefficients [see Eq. (29) below].

III. ENERGY AND POLARIZATION OF THE DIPOLE RADIATION

Making use of Eqs. (2), (4), (6), and (7), we find that the components φ_0 and φ_2 of the dipole electromagnetic perturbations with respect to the original tetrad (distinguished by a superscript B) are given by [7]

$$\overline{\varphi_0^{\rm B}} = \frac{1}{\sqrt{2}}\left[\left(\partial_r - i\omega\frac{r^2}{\chi}\right)^2 \frac{\chi}{r^3} X_{(-)}\right]{}_{-1}Y_{1m}e^{-i\omega t}, \quad (19)$$

$$\overline{\varphi_2^{\rm B}} = \frac{1}{\sqrt{2}}\frac{\chi}{r^5}X_{(-)}\,{}_1Y_{1m}e^{-i\omega t} - \frac{q_2}{4\sqrt{2}r^2}\overline{\left[\left(\partial_r - i\omega\frac{r^2}{\chi}\right)r^2 Y_{(-)}\right]{}_{-1}Y_{1m}e^{i\omega t}}, \quad (20)$$

with $m=0,\pm 1$.

On the other hand, from Eqs. (9)–(14) one finds that if $r_*\to\infty$ and $Z^{(\pm)}\to e^{-i\omega r_*}$, then

$$Y_{(-)}\to -\frac{q_1 K^{(\mp)}}{2\omega^2 r^5}e^{-i\omega r_*}, \quad X_{(-)}\to \frac{iK^{(\mp)}}{2\omega}e^{-i\omega r_*}, \quad (21)$$

where

$$K^{(\pm)} \equiv 2 \mp 2i\omega q_2, \quad (22)$$

while, if $r_*\to\infty$ and $Z^{(\pm)}\to e^{i\omega r_*}$,

$$Y_{(-)}\to -4\omega^2 e^{i\omega r_*}, \quad X_{(-)}\to 2i\omega r^2 e^{i\omega r_*}. \quad (23)$$

Finally, if $r_*\to -\infty$ and $Z^{(\pm)}\to e^{-i\omega r_*}$,

$$Y_{(-)}\to \frac{\chi^2}{r_+^8} \times \frac{K^{(\mp)}(2q_1/r)e^{-i\omega r_*}}{4[-i\omega + (Mr_+ - Q^2)/r_+^3][-i\omega + 2(r_+ - M)/r_+^2]},$$

$$X_{(-)}\to \frac{K^{(\mp)}e^{-i\omega r_*}}{2[-i\omega + (Mr_+ - Q^2)/r_+^3]}, \quad (24)$$

where $r_+\equiv M + (M^2 - Q^2)^{1/2}$ [cf. Ref. [6], Chap. 5, Eqs. (285) and (286)]. Note that, since $r_+^2 - 2Mr_+ + Q^2 = 0$, $(Mr_+ - Q^2)/r_+^3 = (r_+ - M)/r_+^2$.

Thus, according to Eqs. (16), (19)–(21), (23), and (24) we find that, when $r_*\to\infty$,

$$r\overline{\varphi_0^{\rm B}} \to -i\sqrt{2}\omega K^{(\mp)}A^{(\pm)}e^{-i\omega(t+r_*)}{}_{-1}Y_{1m}, \quad (25)$$

$$2r\overline{\varphi_2^{\rm B}} \to i\sqrt{2}\omega[2B^{(\pm)}e^{-i\omega(t-r_*)}{}_1Y_{1m} - 2i\omega q_2\overline{B^{(\pm)}}e^{i\omega(t-r_*)}\overline{{}_{-1}Y_{1m}}]. \quad (26)$$

Similarly, when $r_*\to -\infty$

$$\chi\overline{\varphi_0^{\rm B}} \to -i\sqrt{2}\omega r_+ K^{(\mp)}C^{(\pm)}e^{-i\omega(t+r_*)}{}_{-1}Y_{1m}. \quad (27)$$

These equations imply that if a given dipole perturbation is expressed in terms of functions $Z^{(+)}$ and $Z^{(-)}$, which have the asymptotic forms (16), then the coefficients $A^{(\pm)}, B^{(\pm)}$, and $C^{(\pm)}$ must be related by

$$A^{(+)}K^{(-)} = A^{(-)}K^{(+)}, \quad B^{(+)} = B^{(-)},$$
$$C^{(+)}K^{(-)} = C^{(-)}K^{(+)}, \quad (28)$$

hence, using Eqs. (17) and (22),

$$|R^{(+)}| = \left|R^{(-)}\frac{K^{(-)}}{K^{(+)}}\right| = |R^{(-)}|, \quad T^{(+)} = T^{(-)} \quad (29)$$

(assuming ω real) (see Refs. [3,6]).

Making use of Eqs. (25)–(27) one finds that the flux of energy per unit time of the dipole electromagnetic radiation coming from infinity is given by

$$\frac{dE_{\rm in}}{dt} = \frac{1}{8\pi}\int \lim_{r\to\infty}|r\varphi_0^{\rm B}|^2 d\Omega = \frac{\omega^2}{4\pi}|K^{(\mp)}A^{(\pm)}|^2 \quad (30)$$

and the *time-averaged* flux of energy per unit time of the outgoing dipole electromagnetic radiation is

$$\left\langle\frac{dE_{\rm out}}{dt}\right\rangle = \left\langle\frac{1}{2\pi}\int \lim_{r\to\infty}|r\varphi_2^{\rm B}|^2 d\Omega\right\rangle = \frac{\omega^2}{4\pi}|K^{(\mp)}B^{(\pm)}|^2. \quad (31)$$

The flux of energy per unit time across the event horizon of the dipole electromagnetic radiation is [Ref. [6], Chap. 8, Eq. (258)]

$$\frac{dE_{\rm hole}}{dt} = \frac{1}{8\pi}\int \lim_{r\to r_+ + 0}\frac{|\chi\varphi_0^{\rm B}|^2}{2Mr_+ - Q^2}d\Omega = \frac{\omega^2}{4\pi}|K^{(\mp)}C^{(\pm)}|^2. \quad (32)$$

Thus, by virtue of Eqs. (17), (18), and (30)–(32) we conclude that

$$\frac{dE_{\rm in}}{dt} = \left\langle\frac{dE_{\rm out}}{dt}\right\rangle + \frac{dE_{\rm hole}}{dt}. \quad (33)$$

Furthermore, the energy reflection and transmission coefficients are given by

$$\frac{\langle dE_{\rm out}/dt\rangle}{dE_{\rm in}/dt} = \left|\frac{B^{(\pm)}}{A^{(\pm)}}\right|^2 = |R^{(\pm)}|^2,$$
$$\frac{dE_{\rm hole}/dt}{dE_{\rm in}/dt} = \left|\frac{C^{(\pm)}}{A^{(\pm)}}\right|^2 = |T^{(\pm)}|^2, \quad (34)$$

respectively, i.e., these coefficients coincide with the reflection and transmission coefficients of either of the Schrödinger equations (9).

By contrast with the multipole electromagnetic radiation with $j\geqslant 2$, which gives rise to multipole gravitational radiation of the same order j and conversely, the dipole electromagnetic radiation cannot be converted into gravitational radiation, since there is no dipole gravitational radiation, and the energy of the electromagnetic radiation alone is conserved.

As discussed in Ref. [9] (see also Ref. [16]), Eq. (25) corresponds to an incident circularly polarized wave, while the presence of both factors $e^{-i\omega t}$ and $e^{i\omega t}$ in Eq. (26) implies that the reflected wave is elliptically polarized. This change in the polarization of the electromagnetic waves is a consequence of the existence of a nonvanishing background electromagnetic field; in the case of the Schwarzschild solution, a circularly polarized electromagnetic wave gives rise to a circularly polarized reflected wave [16].

IV. DIFFERENTIAL IDENTITIES

As in the case of the radial functions appearing in the study of the perturbations of the RN solution with $j \geqslant 2$, there exist differential identities involving the radial functions $X_{(-)}$ and $Y_{(-)}$ defined above. Making use of the Einstein-Maxwell equations and the Ricci and Bianchi identities, in the case of a solution of the Einstein-Maxwell equations such that the background electromagnetic field is algebraically general and one of its principal null directions is geodetic and shear-free, in a frame such that φ_1 is the only nonvanishing component of the background electromagnetic field and $\kappa = \sigma = 0$ (hence, $\Psi_0 = \Psi_1 = 0$), one obtains the decoupled system of equations [11]

$$\begin{aligned}
&(\overline{\delta} - 4\alpha + \pi)(\Psi_0)^{\mathrm{B}} - (D - 2\varepsilon - 4\rho)(\tilde{\Psi}_1)^{\mathrm{B}} \\
&\quad = (3\Psi_2 - 2\Phi_{11})(\tilde{\kappa})^{\mathrm{B}}, \\
&(\Delta - 4\gamma + \mu)(\Psi_0)^{\mathrm{B}} - (\delta - 2\beta - 4\tau)(\tilde{\Psi}_1)^{\mathrm{B}} \\
&\quad = (3\Psi_2 + 2\Phi_{11})(\tilde{\sigma})^{\mathrm{B}}, \\
&(D - 3\varepsilon + \overline{\varepsilon} - 3\rho - \overline{\rho})[\varphi_1(\tilde{\sigma})^{\mathrm{B}}] - (\delta - 3\beta - \overline{\alpha} - 3\tau + \overline{\pi}) \\
&\quad \times [\varphi_1(\tilde{\kappa})^{\mathrm{B}}] = \varphi_1(\Psi_0)^{\mathrm{B}}, \\
&(\overline{\delta} - 3\alpha + \overline{\beta} - \overline{\tau})[\varphi_1(\tilde{\sigma})^{\mathrm{B}}] - (\Delta - 3\gamma - \overline{\gamma} + \overline{\mu})[\varphi_1(\tilde{\kappa})^{\mathrm{B}}] \\
&\quad = 2\varphi_1(\tilde{\Psi}_1)^{\mathrm{B}},
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
(\tilde{\Psi}_1)^{\mathrm{B}} &\equiv [2\varphi_1(\Psi_1)^{\mathrm{B}} - 3\Psi_2(\varphi_0)^{\mathrm{B}}]/2\varphi_1, \\
(\tilde{\kappa})^{\mathrm{B}} &\equiv (\kappa)^{\mathrm{B}} + (D - 2\varepsilon - \rho)[(\varphi_0)^{\mathrm{B}}/2\varphi_1], \\
(\tilde{\sigma})^{\mathrm{B}} &\equiv (\sigma)^{\mathrm{B}} + (\delta - 2\beta - \tau)[(\varphi_0)^{\mathrm{B}}/2\varphi_1],
\end{aligned} \tag{36}$$

and a symbol similar to $(\varphi_0)^{\mathrm{B}}$ denotes the first-order variation of the corresponding null tetrad component, which may not coincide with the component with respect to the background null tetrad of the first-order variation of the electromagnetic field, denoted above by φ_0^{B}. The difference between, e.g., $(\varphi_0)^{\mathrm{B}}$ and φ_0^{B} comes from the variations of the metric, which lead to variations of the null tetrad.

In the case of the RN solution, Eqs. (35) admit separable solutions of the form [12]

$$\begin{aligned}
(\Psi_0)^{\mathrm{B}} &= \sqrt{2} F(r)\, {}_2Y_{jm}(\theta,\varphi) e^{-i\omega t}, \\
(\tilde{\Psi}_1)^{\mathrm{B}} &= H(r)\, {}_1Y_{jm}(\theta,\varphi) e^{-i\omega t}, \\
(\tilde{\kappa})^{\mathrm{B}} &= 2 f(r)\, {}_1Y_{jm}(\theta,\varphi) e^{-i\omega t}, \\
(\tilde{\sigma})^{\mathrm{B}} &= \sqrt{2} h(r)\, {}_2Y_{jm}(\theta,\varphi) e^{-i\omega t}.
\end{aligned} \tag{37}$$

When $j = 1$, only $(\tilde{\Psi}_1)^{\mathrm{B}}$ and $(\tilde{\kappa})^{\mathrm{B}}$ can be different from zero and substituting Eqs. (37) into Eqs. (35) we obtain

$$\begin{aligned}
\left(\partial_r - i\omega \frac{r^2}{\chi}\right)(r^4 H) &= \left(3M - \frac{2Q^2}{r}\right) 2rf, \\
\left(\partial_r + i\omega \frac{r^2}{\chi}\right)\left(\frac{\chi^2}{r^5} f\right) &= \frac{2\chi H}{r^3},
\end{aligned} \tag{38}$$

or, making use of the definitions

$$Y_{(+)} \equiv q_1 \frac{\chi^2}{r^5} f, \qquad X_{(+)} \equiv r^4 H, \tag{39}$$

we have

$$\begin{aligned}
\left(\frac{d}{dr_*} + i\omega\right) Y_{(+)} &= \frac{\chi^2}{r^8} \frac{2q_1}{r} X_{(+)}, \\
\left(\frac{d}{dr_*} - i\omega\right) X_{(+)} &= \frac{r^4}{\chi}\left(1 + \frac{q_2}{r}\right) Y_{(+)}.
\end{aligned} \tag{40}$$

(The functions $Y_{(+)}$ and $X_{(+)}$ are the analogues of the functions Y_{+1} and X_{+2} introduced by Chandrasekhar [6].) Thus, for ω real, the pair of functions $(Y_{(+)}, X_{(+)})$ satisfies the same equations as $(\overline{Y_{(-)}}, \overline{X_{(-)}})$ [see Eqs. (8)].

Since the potentials (4) generate the complete perturbations by means of Eqs. (1) and (2), we can write $(\tilde{\Psi}_1)^{\mathrm{B}}$ and $(\tilde{\kappa})^{\mathrm{B}}$ in terms of ψ_{E} and $M_{1'}$ and obtain expressions equivalent to those given by Eqs. (37). In fact, making use of the formulas

$$\begin{aligned}
(\Psi_{ACDE})^{\mathrm{B}} = &-\tfrac{1}{4} h_\mu{}^\mu \Psi_{ACDE} + \tfrac{1}{2} h_{(AC}{}^{R'S'} \Phi_{DE)R'S'} \\
&+ \tfrac{1}{2} \nabla_{(A}{}^{R'} \nabla_C{}^{S'} h_{DE)R'S'},
\end{aligned} \tag{41}$$

where $h_{ABC'D'}$ is the spinor equivalent of the metric perturbation, $\Phi_{ABC'D'}$ and Ψ_{ABCD} are the Ricci and Weyl spinors of the background solution, and

$$(\Gamma_{ACDE'})^{\mathrm{B}} = -\tfrac{1}{2} \nabla_{(A}{}^{R'} h_{C)DR'E'} - \tfrac{1}{2} \Gamma_{AC}{}^{RS'} h_{RDS'E'}, \tag{42}$$

which give the first-order variations of the components of the Weyl curvature and of the spin coefficients, respectively, and Eqs. (1), (2), (4), (7), and (36) we find that

$$(\tilde{\Psi}_1)^{\rm B}=\frac{1}{4}(D-\varepsilon+\bar{\varepsilon}+\rho-\bar{\rho})(D+2\bar{\varepsilon}+\rho-\bar{\rho})(D+\varepsilon+3\bar{\varepsilon}+\rho-\bar{\rho})\overline{M_{1'}}-\frac{3\Psi_2}{4\varphi_1}(D-\varepsilon+\bar{\varepsilon}-\bar{\rho})(D+2\bar{\varepsilon}+\bar{\rho})\overline{\psi_{\rm E}}$$

$$=\frac{Q}{2\sqrt{2}q_1}(\mathcal{D}^*\mathcal{D}^*\mathcal{D}^*r^2\overline{Y_{(-)}})\overline{{}_{-1}Y_{1m}}e^{i\omega t}+\frac{3}{\sqrt{2}Qr^2}(Mr-Q^2)\left(\mathcal{D}^*\mathcal{D}^*\frac{\chi}{r^3}\overline{X_{(-)}}\right)\overline{{}_{-1}Y_{1m}}e^{i\omega t}, \tag{43}$$

where

$$\mathcal{D}\equiv\partial_r-i\omega\frac{r^2}{\chi},\quad \mathcal{D}^*\equiv\partial_r+i\omega\frac{r^2}{\chi}. \tag{44}$$

Making use of the relation

$$r^2\mathcal{D}^*\mathcal{D}^*\mathcal{D}^*r^2\overline{Y_{(-)}}=2q_1\mathcal{D}^*\mathcal{D}^*\left(\frac{\chi}{r^3}\overline{X_{(-)}}\right)-4q_1\mathcal{D}^*\left(\frac{\chi}{r^4}\overline{X_{(-)}}\right), \tag{45}$$

which follows from Eq. (8), Eq. (43) can be rewritten as

$$(\tilde{\Psi}_1)^{\rm B}=\frac{q_1}{2\sqrt{2}Qr^4}\left\{r^2(r+q_2)\mathcal{D}^*\mathcal{D}^*\left(\frac{\chi}{r^3}\overline{X_{(-)}}\right)+q_2r^2\mathcal{D}^*\left(\frac{\chi}{r^4}\overline{X_{(-)}}\right)\right\}\overline{{}_{-1}Y_{1m}}e^{i\omega t}. \tag{46}$$

Similarly, from Eqs. (1), (2), (4), (7), (36), and (42), we obtain

$$(\tilde{\kappa})^{\rm B}=\frac{1}{4}(D-2\varepsilon-\rho)\frac{1}{\varphi_1}(D-\varepsilon+\bar{\varepsilon}-\bar{\rho})(D+2\bar{\varepsilon}+\bar{\rho})\overline{\psi_{\rm E}}=\frac{r}{\sqrt{2}Q}\left(\mathcal{D}^*\mathcal{D}^*\mathcal{D}^*\frac{\chi}{r^2}\overline{X_{(-)}}\right)\overline{{}_{-1}Y_{1m}}e^{i\omega t}. \tag{47}$$

[It should be noticed that Eqs. (41) and (42) involve a specific choice of the perturbed null tetrad; however, this fact has no consequence on expressions (46) and (47), since the combinations (36) are invariant under the rotations of the perturbed null tetrad [11].] On the other hand, from Eqs. (37) and (39) we have, for the $j=1$ perturbations

$$(\tilde{\Psi}_1)^{\rm B}=\frac{1}{r^4}X_{(+)}\,{}_1Y_{1m}e^{-i\omega t},\quad (\tilde{\kappa})^{\rm B}=\frac{2r^5}{q_1\chi^2}Y_{(+)}\,{}_1Y_{1m}e^{-i\omega t}. \tag{48}$$

Thus, by comparing Eqs. (47) and (48), taking into account the fact that $(Y_{(+)},X_{(+)})$ and $(Y_{(-)},X_{(-)})$ satisfy complex-conjugate equations, it follows that

$$\frac{\chi^2}{r^4}\mathcal{D}^*\mathcal{D}^*\mathcal{D}^*\frac{\chi}{r^2}X_{(+)}=CY_{(-)}, \tag{49}$$

for some constant C and, by suitably normalizing the functions $Y_{(\pm)}$, $X_{(\pm)}$,

$$\frac{\chi^2}{r^4}\mathcal{D}\mathcal{D}\mathcal{D}\frac{\chi}{r^2}X_{(-)}=\tilde{C}Y_{(+)}, \tag{50}$$

where $\tilde{C}$ is another constant such that $\tilde{C}=\bar{C}$ for ω real.

In a similar manner, from Eqs. (46) and (48) one finds that

$$r^2(r+q_2)\mathcal{D}^*\mathcal{D}^*\frac{\chi}{r^3}X_{(+)}+q_2r^2\mathcal{D}^*\frac{\chi}{r^4}X_{(+)}=CX_{(-)} \tag{51}$$

and

$$r^2(r+q_2)\mathcal{D}\mathcal{D}\frac{\chi}{r^3}X_{(-)}+q_2r^2\mathcal{D}\frac{\chi}{r^4}X_{(-)}=\tilde{C}X_{(+)}, \tag{52}$$

with the same constants appearing in Eqs. (49) and (50). [Alternatively, one could demonstrate the validity of Eqs. (49)–(52) starting from Eqs. (8) and (40), by a direct procedure analogous to that followed in Ref. [6] to prove the Teukolsky-Starobinsky identities, which involves many computations and requires the previous knowledge of the appropriate operators.] Substituting Eq. (52) into Eq. (51), making use of Eqs. (8), one obtains the relation

$$C\tilde{C}=4+4\omega^2q_2^2, \tag{53}$$

i.e., $C\tilde{C}=K^{(+)}K^{(-)}$.

V. CONCLUDING REMARKS

The results of this paper, together with those of Refs. [9,12], show many interesting relations between the quantities characterizing the perturbations of the RN solution. By contrast with other approaches, the use of the complex potentials to express the complete perturbations allows us to obtain these relations in a relatively simple way. (As a matter of fact, a detailed analysis of the perturbations, similar to that presented in this paper and in Refs. [9,12], has not been obtained by the alternative approaches.)

As in the case of the multipoles with $j\geqslant 2$, the equations for the potentials (3) can be reduced to the same radial equations obtained from the linearized Newman-Penrose equations (35), which is useful in finding the differential identities satisfied by the radial functions. For all the multipole orders, the differential identities found in connection with the perturbations of the RN solution are more involved than the Teukolsky-Starobinsky identities obtained in the study of the perturbations by massless fields of the type D vacuum spacetimes.

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