Extreme Kerr throat geometry: A vacuum analog of $AdS_2 \times S^2$

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We study the near-horizon limit of a four-dimensional extreme rotating black hole. The limiting metric is a completely nonsingular vacuum solution, with an enhanced symmetry group $SL(2,R)\times U(1)$. We show that many of the properties of this solution are similar to the $AdS_2\times S^2$ geometry arising in the near-horizon limit of extreme charged black holes. In particular, the boundary at infinity is a timelike surface. This suggests the possibility of a dual quantum mechanical description. A five-dimensional generalization is also discussed. [S0556-2821(99)03320-2]

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I. INTRODUCTION

There is growing evidence in support of the conjecture that string theory with asymptotically anti–de Sitter (AdS) boundary conditions is completely described by a dual conformal field theory (CFT) [1]. One of the features of AdS spacetime that makes this correspondence possible is that the (conformal) boundary at infinity is a timelike surface. In many applications, one can imagine the dual CFT residing on this timelike boundary. For this reason, the AdS-CFT correspondence is often called holographic [2]. In contrast, asymptotically flat boundary conditions lead to a boundary at infinity consisting of two null surfaces (together with a point at spatial infinity). It is far from clear what form a holographic description will take in this case.[1]

We describe below some vacuum spacetimes with asymptotic structure similar to AdS spacetime. They are obtained by taking the near-horizon geometry of extreme rotating black hole solutions. In four dimensions, there is a one-parameter family of solutions labeled by the total angular momentum J. These vacuum spacetimes are completely nonsingular and can be used to construct classical solutions to all string theories. One simply takes the product with a Ricci flat internal space to obtain a 10- (or 11-) dimensional solution. Since the curvature is bounded everywhere and small for large J, the α' (or M theory) corrections should be small. One advantage over the AdS solutions is that there are no background Ramond-Ramond fields, so string propagation in these backgrounds is straightforward. However, an obvious disadvantage is that these solutions do not admit covariantly constant spinors and, hence, are not supersymmetric. The effect of quantum corrections remains to be investigated.

It is tempting to speculate that there is some type of dual CFT description of string theory on spacetimes which asymptotically approach these vacuum solutions. Starting with the four-dimensional Kerr solution, one obtains a vacuum solution which resembles $AdS_2\times S^2$. It has a symmetry group $SL(2,R)\times U(1)$ and a timelike boundary at infinity. Since the boundary of AdS_2 is one dimensional, one expects the dual theory to have a conformal quantum mechanical description. Unfortunately, despite much effort [4,5] the AdS_2-CFT_1 correspondence is still poorly understood. We will not be able to describe the dual theory in our case, although we will make some comments in Sec. V.

Since the AdS-CFT duality is better understood in higher dimensions, it is natural to ask whether spacetimes analogous to higher-dimensional AdS arise in the near-horizon limits of rotating black holes in higher dimensions. Several scientists have studied the near-horizon geometry of rotating charged black holes and p-branes [6,7,8,9]. However, in all these cases, the charge plays an essential role, and most solutions are asymptotically AdS. What about higher-dimensional analogs of the Kerr solution [10]? If the spacetime has only one component of angular momentum nonzero, then there is no extremal black hole in more than five dimensions. A given mass black hole can have arbitrarily large angular momentum. If all components of the angular momentum are nonzero, then there is an extremal limit, but we expect that the near-horizon geometry will still resemble AdS_2. This is because the effective cosmological constant in these vacuum solutions can be thought of as arising from the off-diagonal terms in a Kaluza-Klein reduction of the metric. This always produces a two-form Maxwell field which can act like a cosmological constant in two dimensions only. To obtain AdS_n for $n>2$ one would need a higher-rank form, which does not arise naturally in a vacuum solution.

In the next section we derive the near-horizon limit of the extreme Kerr solution and show that it has enhanced symmetry. We also discuss geodesics and find that timelike geodesics with sufficiently large angular momentum can escape to infinity. However, just like radial null geodesics in AdS spacetime, they do so in finite coordinate time, but with infinite affine parameter. This shows that the throat solution is geodesically complete and has a timelike boundary.

The possibility of viewing the vicinity of the extreme Kerr horizon as a complete vacuum spacetime in its own right was suggested by the results of Bardeen and Wagoner [11] (see also [12]). They studied the exterior metric of a uniformly rotating disk in the extreme relativistic limit. If the relativistic limit (infinite redshift from the center of the disk) is taken before taking the limit of infinite affine distance from the disk, the asymptotic geometry is the throat of an

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[1] For a discussion of obtaining this as a limit of the AdS-CFT correspondence, see [3].

extreme Kerr black hole rather than an asymptotically flat spacetime.

In Sec. III, we discuss the propagation of a massless scalar test field in this background. Most modes have discrete frequencies and are confined. However, a few modes with large azimuthal angular momentum can propagate to infinity. We will argue that this can lead to an analog of superradiance where a wave can scatter and return with more energy than it started with. When back reaction is included, the throat solution is definitely unstable. We give a general argument (independent of superradiance) that nearby solutions will be singular. This argument also shows that $AdS_2\times S^2$ is unstable, but not higher-dimensional AdS spacetimes. In [5] another argument for the instability of $AdS_2\times S^2$ is given, and it is suggested that the dual theory may describe only ''ground states'' of string theory, and not finite energy excitations. The same may be true in our case as well. Alternatively, the instability may simply indicate that the addition of any amount of energy produces a black hole. The dual theory could perhaps describe these black states as well.

Our results can be extended to the general Kerr-Newman solution describing a charged rotating black hole. The near-horizon limit of this solution is briefly discussed in Sec. IV. In the following section we make some comments about the dual quantum mechanical theory. Perhaps the key observation is that the area of the event horizon is related to the effective cosmological constant of AdS_2 in a universal way that is independent of whether the extreme black hole has only angular momentum, charge, or both. This suggests that the quantum mechanical theories which are dual to these backgrounds are closely related.

Finally, in Sec. VI, we discuss the near-horizon limit of the five-dimensional extreme Kerr solution. The resulting geometry is more complicated, but qualitatively similar to the four-dimensional case. The limiting solution resembles $AdS_2\times S^3$.

II. KERR THROAT SOLUTION AND ITS PROPERTIES

We begin with the (four-dimensional) Kerr metric in Boyer-Linquist coordinates:

$$ds^2=-e^{2\nu}d\tilde{t}^2+e^{2\psi}(d\tilde{\phi}-\omega d\tilde{t})^2+\rho^2(\Delta^{-1}d\tilde{r}^2+d\theta^2), \tag{2.1}$$

where

$$\rho^2\equiv\tilde{r}^2+a^2\cos^2\theta, \quad \Delta\equiv\tilde{r}^2-2M\tilde{r}+a^2 \tag{2.2}$$

and

$$e^{2\nu}=\frac{\Delta\rho^2}{(\tilde{r}^2+a^2)^2-\Delta a^2\sin^2\theta}, \quad e^{2\psi}=\Delta\sin^2\theta e^{-2\nu}, \tag{2.3}$$

$$\omega=\frac{2M\tilde{r}a}{\Delta\rho^2}e^{2\nu}.$$

The total mass is M, and the angular momentum is $J=Ma$, which we will assume is positive.[2] The extremal limit corresponds to $a^2=M^2$, so $\Delta=(\tilde{r}-M)^2$ and the event horizon is at $\tilde{r}=M$. The area of the extremal horizon is

$$A=8\pi M^2=8\pi J. \tag{2.4}$$

The value of ω at the horizon is called the angular velocity of the horizon and, in the extremal case, is simply $\omega=1/2M$. Since $g_{\tilde{r}\tilde{r}}=\rho^2/\Delta$, it is clear that the spatial distance to the extremal horizon in a constant $\tilde{t}$ surface is infinite. In analogy to the extreme charged black holes, we wish to extract the limiting geometry as one moves down this throat.

To describe this near-horizon geometry, we set

$$\tilde{r}=M+\lambda r, \quad \tilde{t}=\frac{t}{\lambda}, \quad \tilde{\phi}=\phi+\frac{t}{2M\lambda} \tag{2.5}$$

and take the limit $\lambda\to 0$. The shift from $\tilde{\phi}$ to ϕ makes $\partial/\partial t$ tangent to the horizon. In other words, the coordinates corotate with the horizon. The result is

$$ds^2=\left(\frac{1+\cos^2\theta}{2}\right)\left[-\frac{r^2}{r_0^2}dt^2+\frac{r_0^2}{r^2}dr^2+r_0^2d\theta^2\right] + \frac{2r_0^2\sin^2\theta}{1+\cos^2\theta}\left(d\phi+\frac{r}{r_0^2}dt\right)^2, \tag{2.6}$$

where we have defined $r_0^2\equiv 2M^2$. This spacetime is no longer asymptotically flat. We will show that it is similar to $AdS_2\times S^2$ in many respects. For example, if one sets $\theta=0$ (or π), one sees that the spacetime along the axis is *precisely* AdS_2. It is clear that by rescaling t, one can ensure that r_0 only appears as an overall factor in front of the metric.

It is easy to see that Eq. (2.6) has enhanced symmetry. In addition to the $\partial/\partial t$ and $\partial/\partial\phi$ symmetries present in Kerr spacetime, Eq. (2.6) is clearly invariant under $r\to cr$, $t\to t/c$ for any constant c. So Eq. (2.6) has the dilation symmetry of AdS_2. It is less obvious, but still true, that Eq. (2.6) is also invariant under (an analog of) the global time translation in AdS_2. To see this, note that the (r,t) coordinates are analogous to Poincaré coordinates on AdS_2. To find the extra global time translation symmetry, we introduce new coordinates which are related to (r,t) in the same way that the global coordinates of AdS_2 are related to the Poincaré coordinates. For simplicity, we set $r_0=1$. Let

$$r=(1+y^2)^{1/2}\cos\tau+y, \quad t=\frac{(1+y^2)^{1/2}\sin\tau}{r}. \tag{2.7}$$

The new axial angle coordinate φ is chosen so that $g_{\varphi y}=0$, with the result

$$\phi=\varphi+\log\left|\frac{\cos\tau+y\sin\tau}{1+(1+y^2)^{1/2}\sin\tau}\right|. \tag{2.8}$$

[2]We have set $G=1$ where G is the four-dimensional Newton's constant.

In these new coordinates, the throat metric (2.6) takes the form

$$ds^2=\left(\frac{1+\cos^2\theta}{2}\right)\left[-(1+y^2)d\tau^2+\frac{dy^2}{1+y^2}+d\theta^2\right]$$
$$+\frac{2\sin^2\theta}{1+\cos^2\theta}(d\varphi+y\,d\tau)^2. \tag{2.9}$$

Note that the $\tau=0$ hypersurface coincides with a $t=0$ hypersurface and that $\varphi=\phi$ on this hypersurface (and also at infinity, for all time).

The throat solution (2.9) thus has all of the symmetries of AdS_2 plus translations in φ: Its isometry group is $SL(2,R)\times U(1)$. All geometric quantities depend only on θ. We will show below that the coordinates in Eq. (2.9) (with $-\infty<\tau<\infty$, $-\infty<y<\infty$) cover the entire spacetime. The surfaces of constant τ are always spacelike, so τ is a global time function and this spacetime has no closed timelike curves. However, the Killing field $\partial/\partial\tau$ is not timelike everywhere. It is timelike for all θ when $y^2<1/3$, but asymptotically is spacelike for $\sin\theta>(1+\cos^2\theta)/2$ or $\sin\theta>0.536$, within 32.4° of the equatorial plane. This is a consequence of the rotation and is analogous to the ergosphere in the extreme Kerr solution.[3] It will be convenient to define the vector field

$$\chi=\frac{\partial}{\partial\tau}-y\frac{\partial}{\partial\varphi}. \tag{2.10}$$

χ is a future-directed timelike vector everywhere, which is orthogonal to the surfaces of constant τ and tangent to the surfaces of constant y.

There are two boundaries at infinity corresponding to $y=\pm\infty$. Since a surface of constant y is always timelike, the limiting surfaces $y=\pm\infty$ must be timelike or null. The boundary will be timelike if geodesics can reach it in a finite value of τ. We now show that this is the case. In the process, we will also show that Eq. (2.9) is geodesically complete. Hence the solution is nonsingular, and the coordinates (τ,y,θ,φ) cover the entire spacetime.

Since we are mostly interested in the asymptotic properties of the geodesics, it suffices to consider geodesics with constant θ. This corresponds to $\theta=0$, $\pi/2$. It is clear that test particles moving along the axis $\theta=0$ will behave exactly as in AdS_2. In particular, timelike geodesics never reach infinity, and null geodesics reach infinity in a finite time. The equatorial plane $\theta=\pi/2$ is a three-dimensional homogeneous space with symmetry group $SL(2,R)\times U(1)$. It is a twisted product of AdS_2 and a circle of constant radius. Consider a geodesic with momentum $P=\dot\tau(\partial/\partial\tau)+\dot y(\partial/\partial y)+\dot\varphi(\partial/\partial\varphi)$, where an overdot denotes a derivative with respect to an affine parameter. The conserved quantities are

$$L=P\cdot\partial/\partial\varphi=2(\dot\varphi+y\dot\tau),\quad E=-P\cdot\partial/\partial\tau=\frac{1+y^2}{2}\dot\tau-Ly. \tag{2.11}$$

Setting $g_{\mu\nu}P^\mu P^\nu=-\mu^2$ yields

$$\dot y^2-4(E+Ly)^2+(2\mu^2+L^2)(1+y^2)=0. \tag{2.12}$$

Geodesics with zero angular momentum again behave exactly like geodesics in AdS_2. That is, massive particles feel an infinite potential barrier and stay confined to finite $|y|$. Massless particles can reach infinity, but since y is proportional to the affine parameter, these geodesics are obviously complete. One can easily verify that these geodesics reach infinity in a finite time τ. Timelike geodesics with $L^2<2\mu^2/3$ are also confined. However, geodesics with $L^2>2\mu^2/3$ can escape to infinity. This is a qualitatively new feature of Eq. (2.9) which is not present in $AdS_2\times S^2$. Since χ, Eq. (2.10), is a future-directed timelike vector everywhere, it follows that $-P\cdot\chi>0$, which implies $E+yL>0$. Thus a geodesic with $L>0$ can only escape to $y=+\infty$, while a geodesic with $L<0$ can only escape to $y=-\infty$. Since $\dot y\propto y$ asymptotically, these geodesics are also complete and reach infinity in a finite value of τ. Geodesics with $L^2=2\mu^2/3$ satisfy $\dot y^2=8ELy$ asymptotically. So once again if $L>0$, these geodesics can reach $y=\infty$, but not $y=-\infty$. For $L<0$, the situation is reversed. These geodesics are also complete and reach infinity in finite τ.

For product spacetimes such as $AdS_2\times S^2$, one can remove the angular directions and conformally rescale AdS_2 to view infinity as a finite boundary. Then the Killing fields of AdS_2 become conformal symmetries of the boundary. One cannot do this for the throat solution since the geometry does not approach a product metric asymptotically. If one wants to bring infinity in to a finite distance, the best one can do is to rescale the entire metric, although the conformal metric is no longer smooth at the boundary. For convenience, we start with the metric in Poincaré coordinates (2.6) (although we set $r_0=1$). Multiplying by $1/r^2$ and setting $x=1/r$, the metric becomes

$$ds^2=\left(\frac{1+\cos^2\theta}{2}\right)[-dt^2+dx^2+x^2d\theta^2]$$
$$+\frac{2\sin^2\theta}{1+\cos^2\theta}(x\,d\phi+dt)^2. \tag{2.13}$$

Despite its simple appearance, this conformal metric has a curvature singularity at $x=0$. The four Killing fields ξ_i^μ of Eq. (2.6) are

$$\xi_1=\frac{\partial}{\partial\phi},\quad \xi_2=\frac{\partial}{\partial t},\quad \xi_3=t\frac{\partial}{\partial t}-r\frac{\partial}{\partial r},$$
$$\xi_4=\left(\frac{1}{2r^2}+\frac{t^2}{2}\right)\frac{\partial}{\partial t}-\text{tr}\frac{\partial}{\partial r}-\frac{1}{r}\frac{\partial}{\partial\phi}. \tag{2.14}$$

Since they are Killing fields of Eq. (2.6), they are conformal Killing fields of Eq. (2.13) with action $\mathcal{L}_\xi(r^{-2}g_{\mu\nu})$

[3]This is not exactly the same as the original ergosphere, since $\partial/\partial\tau$ is not a Killing field in the full Kerr metric.

$=(\mathcal{L}_{\xi}r^{-2})g_{\mu\nu}=(-2\xi^r/r)(r^{-2}g_{\mu\nu})$. The net result is that ξ_1 and ξ_2 remain Killing fields of the rescaled metric, ξ_3 multiplies the metric (2.13) by 2, and ξ_4 multiplies the metric by $2t$. Thus the last three generate the conformal group on a line with metric $-dt^2$. This can be realized explicitly by introducing a cutoff at $x=\epsilon$ and shifting $\phi=\hat{\phi}-t/\epsilon$. Then the metric on the $x=\epsilon$ surface is essentially the product of a line and a very small sphere.

III. MODES OF A MASSLESS SCALAR FIELD

An important feature of test fields on $AdS_2\times S^2$ is that they naturally obey reflecting boundary conditions at infinity. (This is true for all modes with a nontrivial dependence on S^2.) The wave solutions have a discrete spectrum, with no oscillatory behavior near infinity. We will show that a similar confinement holds for axisymmetric modes in the extremal Kerr throat geometry. However, as might be suspected from the behavior of geodesics with large angular momentum found in Sec. II, some nonaxisymmetric modes do propagate all the way to infinity and transport energy and angular momentum there. These scattering states have a continuous spectrum. This behavior is expected be generic for waves of all spins. We illustrate it with a discussion of solutions of the massless scalar wave equation.

The scalar wave equation for a spacetime described by a metric of the form (2.1) can be written as

$$\frac{1}{\rho^2}\frac{\partial}{\partial\tilde{r}}\left[\Delta\frac{\partial\Psi}{\partial\tilde{r}}\right]+\frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Psi}{\partial\theta}\right]+[e^{-2\nu}(\tilde{\sigma}-m\omega)^2-m^2e^{-2\psi}]\Psi=0, \quad (3.1)$$

assuming harmonic time and axial angle dependence of the form

$$\Psi=\Psi(\tilde{r},\theta)e^{im\tilde{\phi}-i\tilde{\sigma}\tilde{t}}. \quad (3.2)$$

In taking the near-horizon limit (2.5), the throat frequency σ is related to the Kerr frequency $\tilde{\sigma}$ by

$$\tilde{\sigma}-m\Omega_H=\lambda\sigma, \quad (3.3)$$

where Ω_H is the angular velocity of the horizon. Therefore, as $\lambda\to0$, all finite frequencies in the throat correspond to the single frequency $\tilde{\sigma}=m\Omega_H=m/2M$ in the exterior.

Modes in the Kerr geometry with $\tilde{\sigma}=m\Omega_H$ are special since they are totally reflected, and not absorbed by the black hole. This can be seen from the following argument, which applies to all Kerr black holes, and not just extremal ones (see [13]). Let $\xi^\mu=(\partial/\partial\tilde{t})^\mu$ be the usual stationary Killing field. The null vector tangent to the horizon of a Kerr black hole is $l^\mu=\xi^\mu+\Omega_H(\partial/\partial\tilde{\phi})^\mu$. Since $l^\mu\xi_\mu=0$ on the horizon, the energy flux entering the black hole is obtained by integrating $T_{\mu\nu}\xi^\mu l^\nu=(\xi^\mu\partial_\mu\Psi)(l^\nu\partial_\nu\Psi)$ over the horizon. This is proportional to $\tilde{\sigma}(\tilde{\sigma}-m\Omega_H)$. So modes with $\tilde{\sigma}>m\Omega_H$ have a positive energy flux into the black hole and correspond to normal scattering. Modes with $\tilde{\sigma}<m\Omega_H$ have a negative energy flux and correspond to superradiant scattering. The outgoing wave has more energy than the incoming one. Modes with exactly $\tilde{\sigma}=m\Omega_H$ are on the borderline. No energy is absorbed by the black hole. The modes we will study in the throat geometry can be thought of as obtained by starting with a mode in the extreme Kerr metric satisfying $\tilde{\sigma}=m\Omega_H+\lambda\sigma$. For small λ, most of this wave is reflected and only a small part remains near the horizon. After the rescaling (2.5), the wave near the horizon remains nonzero in the limit $\lambda\to0$ and completely decouples from the wave in the asymptotically flat region.

The form of the throat metric in Poincaré and global AdS_2 coordinates becomes identical as the respective radial coordinates r and y become large, so the asymptotic properties of the wave solutions are the same in both coordinate systems. For definiteness, we will use the global coordinates. Since the metric (2.9) is of the general form (2.1), the wave equation in this background takes the form (3.1) for a suitable choice of metric components. This wave equation, after multiplication by ρ^2, is completely separable, just as it is for the original Kerr metric. Setting $\Psi=Y(y)\Theta(\theta)$ splits the partial differential equation into the two ordinary differential equations

$$\frac{d}{dy}\left[(1+y^2)\frac{dY}{dy}\right]+\left[\frac{(\sigma+my)^2}{1+y^2}+m^2-K\right]Y=0 \quad (3.4)$$

and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta}{d\theta}\right]+\left[K-\frac{m^2}{\sin^2\theta}-\frac{1}{4}m^2\sin^2\theta\right]\Theta=0. \quad (3.5)$$

The separation constant K has been defined so that the angular equation is identical to the spheroidal harmonic angular equation familiar from solutions on the Kerr background, with the Kerr frequency set to the unique value implied by Eq. (3.3).

The eigensolutions of Eq. (3.5) with boundary conditions of regularity at $\theta=0$ and $\theta=\pi$ are the usual Legendre functions for $m=0$, so $K=l(l+1)$. But even for nonzero m, the solutions are fairly close to associated Legendre functions, since $\frac{1}{4}m^2\sin^2\theta$ is rather small compared to $l(l+1)$ for $l\geqslant|m|$. Numerical calculations of the eigenvalues K for a few of the lowest modes are listed below. We also include the value of $K_{\rm crit}\equiv2m^2-1/4$, which we will see marks the dividing line between the discrete and continuous part of the spectrum. In general,

$$K_{lm}\approx l(l+1)+cm^2, \quad (3.6)$$

where c is roughly in the range 0.13–0.22 and $l\geqslant|m|$ are integers.

The radial equation for axisymmetric modes is identical to that for AdS_2. The natural boundary condition at infinity is $\Psi=0$. Then $\Psi\sim|y|^{-l-1}$, and normal modes satisfying the boundary conditions at both $y=+\infty$ and $y=-\infty$ exist for discrete real frequencies σ_{ln}. Waves are reflected before they reach infinity.

TABLE I. Some eigenvalues K of the angular equation (3.5).

m	$l=m$	$l=m+1$	$l=m+2$	K_{crit}
1	2.200	6.143	12.133	1.75
2	6.855	12.664	20.597	7.75
3	13.995	21.629	31.459	17.75

The nonaxisymmetric modes are more interesting. The asymptotic solutions for $|y|\gg 1$ are power laws $Y\sim|y|^{\alpha}$, with

$$\alpha=-\frac{1}{2}\pm\left(K-2m^2+\frac{1}{4}\right)^{1/2}. \tag{3.7}$$

For $K>K_{\rm crit}\equiv 2m^2-1/4$ both exponents are real, qualitatively like the axisymmetric modes. Only the more rapidly decaying solution satisfies the boundary condition, and global normal modes exist for discrete frequencies. On the other hand, if $K<2m^2-1/4$, the exponents are complex, and the asymptotic solutions are traveling waves. Positive frequency modes are outgoing in phase velocity if Im $\alpha>0$ and ingoing if Im $\alpha<0$. From Table I, it is clear that for $m=1$, all eigenvalues K are larger than $K_{\rm crit}=2m^2-1/4$, but for all $|m|>1$, at least the $l=|m|$ eigenvalue is less than $K_{\rm crit}$. This is qualitatively different from AdS_2, since in the absence of rotation the condition for confinement is $K=l(l+1)>m^2-1/4$, which is always satisfied.

In a WKB approximation to the traveling waves, the effective wave number $k=(-i/Y)dY/dy$ is

$$k=\pm\frac{1}{(1+y^2)^{1/2}}\left[\frac{(\sigma+my)^2}{1+y^2}+m^2-K\right]^{1/2}. \tag{3.8}$$

The group velocity is then

$$\frac{d\sigma}{dk}=\pm\frac{(1+y^2)^{3/2}}{\sigma+my}\left[\frac{(\sigma+my)^2}{1+y^2}+m^2-K\right]^{1/2}. \tag{3.9}$$

The phase velocity is just σ/k. So, for $m>0$, the phase velocity and group velocity have opposite signs for large negative y. For $m<0$, they have opposite signs for large positive y. This has an important consequence, which we now explain.

Since K is always larger than m^2, the expression inside the brackets in Eq. (3.8) always changes sign around $\sigma+my=0$. This means that the wave encounters a potential barrier and the WKB approximation breaks down. An initial wave with $\sigma>0$ and $m>0$ moving in the negative y direction will be partly reflected and partly transmitted through this barrier. We now have to discuss the physical boundary conditions on these waves. First, note that all modes vanish at infinity. Even when α is complex, we still have $Y\sim|y|^{-1/2}$. But since the volume element on a surface of constant y is proportional to $|y|$ asymptotically, there can still be a nonzero flux of energy and angular momentum to infinity. We demand that the transmitted wave should be purely outgoing, where ‘‘outgoing’’ is defined with respect to the physical group velocity.

To compute the flux of energy and angular momentum, note that each Killing vector field of the spacetime produces a conserved flux four-vector when contracted with the energy-momentum tensor of the wave. The axial Killing field is the unique Killing field with closed orbits and gives the angular momentum flux vector $J^{\mu}=T^{\mu}_{\varphi}$. The average rate (per unit global time τ) of angular momentum transport in the $+y$ direction across a constant-y surface is equal to

$$\int d\theta\, d\varphi(-g)^{1/2}T^{y}_{\varphi}=\int d\theta\, d\varphi \sin\theta(1+y^2)\frac{\partial\Psi}{\partial\varphi}\frac{\partial\Psi}{\partial y}. \tag{3.10}$$

After averaging over time and taking the limit $y\to\infty$ with asymptotically $Y=A_{+}y^{i(2m^2-1/4-K)^{1/2}-1/2}$ for the outgoing positive frequency wave solution, the angular momentum transport rate becomes $m(2m^2-1/4-K)^{1/2}|A_{+}|^2$, assuming the angular harmonics are normalized. Using the Killing field $\partial/\partial\tau$ to define a conserved energy flux, the asymptotic energy transport rate is σ/m times the angular momentum transport rate calculated above and for positive frequency waves is outward for waves with an outward phase velocity.

We now come to the key point. Consider a wave with $\sigma>0$, $m>0$ which starts at large y moving in the negative y direction. After scattering off the potential barrier, there will be a reflected wave and transmitted wave. In order for the transmitted wave to have outgoing group velocity, it must have incoming phase velocity. Thus there is a incoming flux of energy from $y=-\infty$. Since this energy is conserved, there must be a corresponding outgoing flux of energy at $y=\infty$. Since we started with an incoming wave at $y=\infty$, the only way this is possible is if the reflected wave carries more energy than the initial wave. In this sense, waves in the throat geometry exhibit superradiance.

Another way to understand the difference in sign between the group velocity and phase velocity is to consider a local observer who is at rest with respect to the constant-τ surfaces and, hence, has a four-velocity proportional to χ, Eq. (2.10). Such a zero-angular-momentum observer (ZAMO) at constant $y=y_0$ and θ would assign a frequency to the wave

$$\sigma_{\rm ZAMO}=\frac{\sigma+my_0}{[(1+y_0^2)(1+\cos^2\theta)/2]^{1/2}}. \tag{3.11}$$

At large positive y_0 with negative m or large negative y_0 with positive m, the ZAMO would assign a negative frequency to the wave, and according to him, the direction of the phase velocity is the same as the group velocity. The local timelike Killing field tangent to the ZAMO's world line is $\partial/\partial\tau-y_0\,\partial/\partial\varphi$. If this were used to define the energy flux, the energy flux would be in the same direction as the group velocity.

Quantum mechanically, the full extreme Kerr metric radiates particles even though its Hawking temperature is zero. The emitted particles all lie in the superradiant regime $\tilde{\sigma}<m\Omega_H$ [14]. Since the modes we consider correspond to $\tilde{\sigma}=m\Omega_H$, one might wonder if this implies that the throat metric will be quantum mechanically stable. However, the fact that the throat solution itself exhibits superradiance sug-

gests that it will decay quantum mechanically also. Presumably, if one starts with an SL(2,R)$\times$U(1) invariant vacuum state, these symmetries will be preserved. It is then far from clear what the solution (2.9) could decay into. This question requires further investigation.

Classically, the question of stability to linearized perturbations is the question of whether there are unstable quasinormal modes obeying outgoing wave boundary conditions at both infinities. Our expectation, in the absence of a detailed investigation, is that no such unstable quasinormal modes exist, based on the known stability of Kerr black holes [15]. On the other hand, when back reaction is taken into account, the throat geometry is unstable, and nearby solutions are singular. This follows from the singularity theorems [16] since the constant (y,τ) two-spheres are marginally trapped. A generic perturbation will cause the orthogonal null geodesics to start converging, creating trapped surfaces and geodesic incompleteness. If the nearby solutions are all black holes, then this instability need not be a serious problem. But since the singularity theorems do not prove the existence of event horizons, one does not yet know if more serious singularities can arise. In order to apply this instability argument, one only needs to ensure that the nearby spacetimes satisfy Einstein's equation with matter obeying the weak energy condition. The same argument applies to $AdS_2\times S^2$ and shows that this spacetime is also unstable. (For another argument to this effect, see [5].) However, this argument does not apply to products of higher-dimensional AdS spacetimes and spheres, since the null geodesics orthogonal to the sphere consist of an entire light cone in AdS spacetime, which is expanding. So, in the higher-dimensional case, the spheres are not marginally trapped and, e.g., $AdS_5\times S^5$ is stable.

The instability of the throat geometry may not be a serious obstacle to constructing a dual quantum mechanical theory since it has been argued in the case of $AdS_2\times S^2$ that the dual theory may describe just the "ground states" of string theory with these boundary conditions [5]. We will see in Sec. V that a similar argument applies to our throat geometry.

IV. EXTENSION TO EXTREME KERR-NEWMAN THROATS

While our main interest is in vacuum solutions, we note that our results can easily be generalized to include the entire range of extreme Kerr-Newman solutions describing rotating and charged black holes. Their near-horizon geometries smoothly interpolate between the solution (2.9) and $AdS_2\times S^2$. The Kerr-Newman metrics have exactly the same form as Eq. (2.1), except that $\Delta=\tilde{r}^2-2M r+a^2+q^2$, where q is the electric charge and aM is still the angular momentum, and the $2M\tilde{r}$ factor in ω is replaced by $\tilde{r}^2+a^2-\Delta$. The extremal limit corresponds to $M^2=a^2+q^2$, and the horizon is at $\tilde{r}=M$ with area $4\pi(M^2+a^2)$. If we define $r_0^2\equiv M^2+a^2$, the throat metric is similar to Eq. (2.6) with only two modifications: The factor $(1+\cos^2\theta)/2$ becomes $1-(a^2/r_0^2)\sin^2\theta$, and there is a different coefficient in the frame-dragging angular velocity. Expanding ω to first order in $\tilde{r}-M$, we now obtain

$$\omega=\frac{a}{r_0^2}-\frac{2aM}{r_0^4}(\tilde{r}-M), \tag{4.1}$$

so the coefficient of rdt in Eq. (2.6) is $2aM/r_0^4$ instead of $1/r_0^2$. The near-horizon limit is now

$$ds^2=\left(1-\frac{a^2}{r_0^2}\sin^2\theta\right)\left[-\frac{r^2}{r_0^2}dt^2+\frac{r_0^2}{r^2}dr^2+r_0^2d\theta^2\right]+r_0^2\sin^2\theta\left(1-\frac{a^2}{r_0^2}\sin^2\theta\right)^{-1}\left(d\phi+\frac{2arM}{r_0^4}dt\right)^2. \tag{4.2}$$

Notice that when $a=0$, this metric reduces to $AdS_2\times S^2$, as expected. The change in the coefficient of the off-diagonal term carries over to the change of the coefficient in front of the logarithm in the relation (2.8) between ϕ and φ. Otherwise, the transformation to the global AdS_2 coordinates remains the same, and the above changes carry over to the metric of the throat geometry (2.9) in the global coordinates.

An interesting point along the Kerr-Newman sequence is when confinement of modes of scalar waves starts to break down. In the angular harmonic equation $\frac{1}{4}m^2\sin^2\theta$ is replaced by $(a^4/r_0^4)m^2\sin^2\theta$, and a rough estimate of the smallest separation constant for a given m is

$$K_{\min}=|m|(|m|+1)+0.8\frac{a^4}{r_0^4}m^2. \tag{4.3}$$

The m^2 term in the radial equation is replaced by $(2a^2/r_0^2)m^2$, so confinement is broken when

$$\frac{4a^2M^2}{r_0^4}+\frac{2a^2}{r_0^2}>\frac{K_{\min}}{m^2}. \tag{4.4}$$

This happens first for $m^2\gg 1$, when $a^2/M^2\approx 0.242$. The critical points for other types of fields (e.g., electromagnetic or gravitational perturbations) will differ somewhat from this.

V. SEARCHING FOR A HOLOGRAPHIC DUAL

In light of the growing evidence in favor of the AdS/CFT correspondence, it is natural to speculate that string theory on spacetimes that approach Eq. (2.9) will have a dual holographic description. Since the boundary at infinity is effectively one dimensional and SL(2,R) is the conformal group of a line, one expects the dual theory to be a conformal quantum mechanical system. We have already pointed out several ways in which Eq. (2.9) is qualitatively similar to $AdS_2\times S^2$, which is the near-horizon geometry of an extremely charged four-dimensional black hole. So one might expect that the dual theories might be similar, with the SU(2) symmetry of S^2 broken to U(1) in the vacuum case, breaking supersymmetry as well. Unfortunately, $AdS_2\times S^2$ is currently the least well understood example of the AdS/CFT correspondence. We currently have little information about the structure of this quantum mechanical system.

Let us first try to follow the procedure used to discover the original AdS/CFT duality. Starting with the extreme black hole, we can decrease the string coupling g to obtain a weakly coupled string description of the black hole states. The fact that the entropy of an extreme Kerr black hole is independent of Newton's constant and simply given by the angular momentum (which we still assume is positive), $S = 2\pi J$, strongly suggests that it has a simple microscopic description. In fact, it may ultimately be simpler than the Reissner-Nordström solution which requires four different charges from the fundamental string standpoint. Since an extreme Kerr black hole has no Ramond-Ramond fields present, the states at weak coupling must be ordinary excited states of the string. Unfortunately, the lack of supersymmetry makes it difficult at present to give a precise identification of the states.[4] One can instead use the correspondence principle [18] to roughly describe these states as follows. Since angular momentum is quantized, we want to keep J fixed as we slowly decrease the string coupling. The mass of the black hole is $GM^2 = J$, and its horizon radius is $r_+ = GM$, where G is the four-dimensional Newton's constant. The correspondence principle says that the black hole makes a transition to an excited string state when its horizon size is of order the string scale. Setting the mass of the black hole equal to the mass of an excited string state at this point yields

$$M_{\rm bh} \sim \frac{l_s}{G} \sim \frac{\sqrt{N}}{l_s} \sim M_s\,, \tag{5.1}$$

where N is the string level. Since $G \sim g^2 l_s^2$, this implies that $g \sim N^{-1/4}$ at the transition point. The black hole entropy is then

$$S_{\rm bh} \sim GM^2 \sim M r_+ \sim M l_s \sim \sqrt{N}. \tag{5.2}$$

Since $GM^2 = J$ and J is fixed, the appropriate weakly coupled string states are states with $N \sim J^2$ and angular momentum J. (This was also noticed in [19].) The number of such states is approximately $e^{\sqrt{N}}$, which agrees with the entropy of the black hole.

States with $N \sim J^2$ and angular momentum J are far from extremal string states. The minimum mass string state carrying angular momentum J has $N = J$, but there is only one such state, so it could never reproduce the entropy of the black hole. One might ask what happens if one starts with a string state with less energy than the states corresponding to the extreme black hole and increases the coupling. It appears that there might be the danger of forming a naked singularity. But, of course, this is not what happens. If $N < J^2$, then $M\ell_s < J$, but the minimum size of the string is $r_{\rm min} = J/M$, so $r_{\rm min} > l_s$. This means that the string cannot form a black hole at the string scale, but only at a larger scale corresponding to a larger value of the string coupling. Setting the masses equal at this transition point, the resulting black hole will have a Schwarzschild radius at least $r_+ = J/M = a$, which would again correspond to an extreme Kerr black hole.

In terms of trying to construct a dual theory, the obvious problem is that these excited string states are not stable. It is not clear how to take an appropriate limit to decouple the bulk string states and extract the dynamics of the states at a given level $N \sim J^2$.

A clue to the correct description may be the following: Consider the form of the solution (4.2). The area of the horizon at $r = 0$ is $A = 4\pi r_0^2$, where $r_0^2 = 2a^2 + q^2$. Thus the extreme black hole entropy is related to the radius of curvature of the AdS_2 space along the axis in a universal way which is independent of the ratio of charge to angular momentum. This suggests that there may be a unified description of string theory with these boundary conditions.

It was pointed out in [5] that the energy above extremality E of a near-extremal Reissner-Nordström black hole scales with temperature like $E = 2\pi^2 Q^3 T^2 l_p$, where l_p is the Planck length. The fact that l_p enters this formula means that one cannot take $l_p \to 0$ keeping E,Q,T fixed. It is easy to see that the same thing is true for the entire family of near-extreme Kerr-Newman solutions. For simplicity, we restrict ourselves here to the pure Kerr case. Including factors of Newton's constant, the Hawking temperature is

$$T = \frac{(G^2M^2 - a^2)^{1/2}}{4\pi GM[GM + (G^2M^2 - a^2)^{1/2}]}. \tag{5.3}$$

In the extremal limit, $GM = a$. Let $GM = a + \epsilon$, and define $E = M - (a/G) = \epsilon/G$. Then we have $T = (2GM\epsilon)^{1/2}/4\pi G^2 M^2$, so $E = 8\pi^2 T^2 G^2 M^3$. But $GM^2 = J$, so

$$E = 8\pi^2 J^{3/2} T^2 l_p\,. \tag{5.4}$$

VI. FIVE-DIMENSIONAL THROAT GEOMETRIES

Generalizations of the Kerr metric to more than three space dimensions have been discussed thoroughly by Myers and Perry [10]. In N space (plus one time) dimensions there are $[N/2]$ independent planes of rotation in which axial symmetry can be enforced simultaneously, with the same number of independent angular momentum parameters. The spatial coordinates reflecting this symmetry consist of $[N/2]$ axial angles, $[(N-1)/2]$ polar angles, and one radial coordinate. Extremal vacuum black holes with a nonsingular degenerate horizon do not exist in all cases, but when they do, we expect that the near-horizon limit leads to a complete spacetime with an $SL(2,R) \times U(1)^{[N/2]}$ symmetry group. As an illustrative example, we work through some of the details for the 5D $(N = 4)$ case.

In five dimensions, let $\tilde{\phi}$ and $\tilde{\psi}$ denote the two axial angles, a and b the corresponding angular momentum parameters, and define a mass parameter μ with units of length squared. The physical mass of the black hole is $M = 3\pi\mu/8G_5$, and the factor $2M/3$ converts angular momentum parameters to physical angular momenta. The metric in

[4]When charges are present, one can give a precise counting of the states of certain extreme charged and rotating black holes, even when they are far from the supersymmetric state [17].

''Boyer-Lindquist'' coordinates [Eq. (3.18) of [10]] reduces to

$$ds^2=-d\tilde{t}^2+\frac{\mu}{\rho^2}(d\tilde{t}-a\sin^2\theta d\tilde{\phi}-b\cos^2\theta d\tilde{\psi})^2+(\tilde{r}^2+a^2)\sin^2\theta d\tilde{\phi}^2+(\tilde{r}^2+b^2)\cos^2\theta d\tilde{\psi}^2+\rho^2 d\theta^2+\frac{\rho^2\tilde{r}^2}{\Delta}d\tilde{r}^2, \tag{6.1}$$

where

$$\rho^2\equiv\tilde{r}^2+a^2\cos^2\theta+b^2\sin^2\theta,$$
$$\Delta\equiv(\tilde{r}^2+a^2)(\tilde{r}^2+b^2)-\mu\tilde{r}^2. \tag{6.2}$$

The range of the polar angle θ is $0\leqslant\theta\leqslant\pi/2$. Since no odd powers of $\tilde{r}$ are present, it is convenient to work with $\tilde{u}\equiv\tilde{r}^2$ as the radial coordinate, with

$$g_{\tilde{u}\tilde{u}}=\frac{\rho^2}{4\Delta}. \tag{6.3}$$

The horizon is at the outermost zero of Δ. This is a double zero, implying an extremal horizon, if $\mu=(|a|+|b|)^2$. The extremal horizon is at $\tilde{u}=|ab|$. Unless both angular momentum parameters are nonzero, the horizon is singular, since ρ^2 on the horizon is zero at $\theta=0$ if $a=0$ or at $\theta=\pi/2$ if $b=0$, and $\rho^2=0$ implies a curvature singularity. The three-volume of the horizon is $2\pi^2|ab|^{1/2}(|a|+|b|)^2$.

We focus on the inverse metric tensor, since it is the inverse metric tensor that comes into the Carter-Hamilton-Jacobi formulation of the geodesic equations and the scalar field equation. In particular, note that the coordinate angular velocities of a ZAMO are given by $\omega^{\tilde{\phi}}=g^{\tilde{\phi}\tilde{t}}/g^{\tilde{t}\tilde{t}}$ and $\omega^{\tilde{\psi}}=g^{\tilde{\psi}\tilde{t}}/g^{\tilde{t}\tilde{t}}$ or

$$\omega^{\tilde{\phi}}=\frac{a\mu(\tilde{u}+b^2)}{\Sigma},\quad \omega^{\tilde{\psi}}=\frac{b\mu(\tilde{u}+a^2)}{\Sigma}, \tag{6.4}$$

where

$$\Sigma\equiv\mu(\tilde{u}+a^2)(\tilde{u}+b^2)+\Delta\rho^2. \tag{6.5}$$

The inverse square of the lapse function is

$$g^{\tilde{t}\tilde{t}}=-\frac{\Sigma}{\Delta\rho^2}. \tag{6.6}$$

The axial components of the inverse metric tensor are

$$g^{\tilde{\phi}\tilde{\phi}}=\frac{g_{\tilde{\psi}\tilde{\psi}}\rho^2}{\Sigma\sin^2\theta\cos^2\theta}+(\omega^{\tilde{\phi}})^2g^{\tilde{t}\tilde{t}}, \tag{6.7}$$

$$g^{\tilde{\psi}\tilde{\psi}}=\frac{g_{\tilde{\phi}\tilde{\phi}}\rho^2}{\Sigma\sin^2\theta\cos^2\theta}+(\omega^{\tilde{\psi}})^2g^{\tilde{t}\tilde{t}}, \tag{6.8}$$

$$g^{\tilde{\phi}\tilde{\psi}}=-\frac{g_{\tilde{\phi}\tilde{\psi}}\rho^2}{\Sigma\sin^2\theta\cos^2\theta}+\omega^{\tilde{\phi}}\omega^{\tilde{\psi}}g^{\tilde{t}\tilde{t}}. \tag{6.9}$$

Also, we note that

$$(-g)^{1/2}=\frac{1}{2}\rho^2\sin\theta\cos\theta \tag{6.10}$$

using $\tilde{u}$ as the radial coordinate.

The throat limit is obtained by changing the axial coordinates to be corotating with the horizon and then rescaling the $\tilde{u}$ and $\tilde{t}$ coordinates, similar to what was done in Eq. (2.5). Specifically,

$$\tilde{\phi}=\phi+\frac{a}{|ab|+a^2}\tilde{t},\quad \tilde{\psi}=\psi+\frac{b}{|ab|+b^2}\tilde{t},$$
$$\tilde{u}=|ab|+\lambda(|a|+|b|)^2u,\quad \tilde{t}=\frac{\sqrt{|ab|}}{2\lambda}t. \tag{6.11}$$

In the limit $\lambda\to 0$, ρ becomes a function of θ only, $\rho^2=|ab|+a^2\cos^2\theta+b^2\sin^2\theta$, and the components of the inverse metric become

$$g^{tt}=-\frac{4}{\rho^2u^2},\quad g^{uu}=\frac{4u^2}{\rho^2},\quad g^{\theta\theta}=\frac{1}{\rho^2}, \tag{6.12}$$

$$\omega^{\phi}=g^{\phi t}/g^{tt}=-\frac{u}{2}\frac{a}{|a|}\left|\frac{b}{a}\right|^{1/2},$$
$$\omega^{\psi}=g^{\psi t}/g^{tt}=-\frac{u}{2}\frac{b}{|b|}\left|\frac{a}{b}\right|^{1/2}, \tag{6.13}$$

$$g^{\phi\phi}=\frac{|b|+|a|\cos^2\theta}{|a|(|a|+|b|)^2\sin^2\theta}+(\omega^{\phi})^2g^{tt}, \tag{6.14}$$

$$g^{\psi\psi}=\frac{|a|+|b|\sin^2\theta}{|b|(|a|+|b|)^2\cos^2\theta}+(\omega^{\psi})^2g^{tt}, \tag{6.15}$$

$$g^{\phi\psi}=-\frac{ab}{|ab|(|a|+|b|)^2}+\omega^{\phi}\omega^{\psi}g^{tt}. \tag{6.16}$$

Despite its rather complicated appearance, one immediately sees the characteristic structure of AdS_2 in Eq. (6.12), with the horizon at $u=0$. Equations (6.14)–(6.16) are all just functions of θ. This metric has essentially the same set of AdS_2-like Killing vector fields as the 4D Kerr throat metric. An almost identical transformation to global time coordinate τ, radial coordinate y, in which

$$u=[(1+y^2)^{1/2}\cos\tau+y],\quad t=\frac{(1+y^2)^{1/2}\sin\tau}{u}, \tag{6.17}$$

and new axial angle coordinates φ and χ related to ϕ and ψ by expressions like such as Eq. (2.8) with appropriate coefficients in front of the logarithms, gives a globally nonsingular metric describing a geodesically complete spacetime. The total number of Killing fields is 5, since there are two axial Killing fields instead of one.

When the two angular momentum components are equal, the metric simplifies and $\rho^2=2a^2$ no longer depends on θ. Setting $a=b>0$, the metric (not the inverse metric) becomes

$$ds^2=-\frac{a^2}{2}u^2dt^2+\frac{a^2}{2}\frac{du^2}{u^2}+2a^2d\theta^2 +2a^2\left[\sin^2\theta\left(d\phi+\frac{u}{2}dt\right)^2+\cos^2\theta\left(d\psi+\frac{u}{2}dt\right)^2\right] +2a^2\left[\sin^2\theta\left(d\phi+\frac{u}{2}dt\right)+\cos^2\theta\left(d\psi+\frac{u}{2}dt\right)\right]^2. \tag{6.18}$$

Along a ZAMO world line, $d\phi+(u/2)dt=0$ and $d\psi+(u/2)dt=0$.

When one of the angular momenta vanishes, the horizon of the extreme black hole becomes singular. It is a surprising fact that if one takes the near-horizon limit in this case, one finds a spacetime with AdS_3 symmetry.[5] We start with the general solution (6.1) with $b=0$ and set $\mu=a^2$ to obtain the extremal limit. We then shift $\tilde{\phi}=\phi+\tilde{t}/a$ and rescale $\tilde{r}=\lambda r$, $\tilde{t}=t/\lambda$, $\tilde{\psi}=\psi/\lambda$, taking $\lambda\rightarrow 0$. The result is

$$ds^2=\cos^2\theta\left[-\frac{r^2}{a^2}dt^2+\frac{a^2}{r^2}dr^2+r^2d\psi^2\right] +a^2\left[\cos^2 d\theta^2+\frac{\sin^2\theta}{\cos^2\theta}d\phi^2\right]. \tag{6.19}$$

Strictly speaking, ψ should have only infinitesimal extent, since $\tilde{\psi}$ is periodic and $\psi=\lambda\tilde{\psi}$. However, we can clearly extend $-\infty<\psi<\infty$ and obtain a five-dimensional vacuum solution which has the SO(2,2) symmetries of AdS_3. Recall that $0\leqslant\theta\leqslant\pi/2$. The above solution looks like a product of AdS_3 and a disk, but is singular at $\theta=\pi/2$. It is not yet clear whether this singularity justifies throwing this solution away as unphysical. There are many examples of singular solutions which play a prominent role in string theory (e.g., the metric of most *D*-branes). Yet some singular metrics are clearly unphysical (e.g., the negative mass Schwarzschild solution). The prominent role of AdS_3 in this vacuum solution justifies further investigation.

Solutions of the geodesic equations in the general metric (6.12)–(6.16) can be obtained by separation of variables in the Hamiltonian-Jacobi formalism [20]. The generating function S for the canonical transformation to coordinates which are constants of the motion obeys an equation derived by substituting $\partial S/\partial x^\alpha$ for p_α in the expression $g^{\alpha\beta}p_\alpha p_\beta=-\mu^2=\partial S/\partial\lambda$, where λ is the affine parameter. Since $p_t=-E$, $p_\phi=L_\phi$, and $p_\psi=L_\psi$ are trivial conserved quantities, the equation for S becomes

$$4u^2\left(\frac{\partial S}{\partial u}\right)^2-\frac{4}{u^2}\left[E+\frac{u}{2}\frac{a}{|a|}\left|\frac{b}{a}\right|^{1/2}L_\phi+\frac{u}{2}\frac{b}{|b|}\left|\frac{a}{b}\right|^{1/2}L_\psi\right]^2 -\frac{2a^2-b^2}{|a|(|a|+|b|)}L_\phi^2-\frac{2b^2-a^2}{|b|(|a|+|b|)}L_\psi^2-\frac{ab}{|ab|}L_\phi L_\psi +|ab|\mu^2+\left(\frac{\partial S}{\partial\theta}\right)^2+\frac{L_\phi^2}{\sin^2\theta}+\frac{|a|-|b|}{|a|+|b|} \times L_\phi\left(L_\phi+\frac{ab}{|ab|}L_\psi\right)\sin^2\theta+b^2\mu^2\sin^2\theta+\frac{L_\psi^2}{\cos^2\theta} +\frac{|b|-|a|}{|a|+|b|}L_\psi\left(L_\psi+\frac{ab}{|ab|}L_\phi\right)\cos^2\theta+a^2\mu^2\cos^2\theta \tag{6.20}$$

and

$$S=-\mu^2\lambda-Et+L_\phi\phi+L_\psi\psi+\int\Theta(\theta)^{1/2}d\theta +\int R(u)^{1/2}du. \tag{6.21}$$

Using a constant K to separate the u dependence of the first two lines of Eq. (6.20) from the θ dependence of the next three lines gives

$$\Theta(\theta)=K-\frac{L_\phi^2}{\sin^2\theta}-\frac{|a|-|b|}{|a|+|b|}L_\phi\left(L_\phi+\frac{ab}{|ab|}L_\psi\right) \times\sin^2\theta-b^2\mu^2\sin^2\theta-\frac{L_\psi^2}{\cos^2\theta}-\frac{|b|-|a|}{|a|+|b|} \times L_\psi\left(L_\psi+\frac{ab}{|ab|}L_\phi\right)\cos^2\theta-a^2\mu^2\cos^2\theta \tag{6.22}$$

and

$$4u^2R(u)=\frac{4}{u^2}\left[E+\frac{au}{2|a|}\left|\frac{b}{a}\right|^{1/2}L_\phi+\frac{bu}{2|b|}\left|\frac{a}{b}\right|^{1/2}L_\psi\right]^2 +\frac{2a^2-b^2}{|a|(|a|+|b|)}L_\phi^2+\frac{2b^2-a^2}{|b|(|a|+|b|)} \times L_\psi^2+\frac{ab}{|ab|}L_\phi L_\psi-|ab|\mu^2-K. \tag{6.23}$$

Geodesics are obtained by setting the partial derivatives of S with respect to μ,E,L_ϕ,L_ψ,K equal to constants (which reflect initial conditions for the geodesics). This directly gives λ,t,ϕ,ψ as functions of θ and u along the geodesic, as well as the relation between θ and u. The range of radial motion of the test particle is where $R(u)>0$. This extends to infinity if

[5]We thank Kirill Krasnov for pointing this out to us.

$$\frac{2|a|+|b|}{|a|+|b|}L_{\phi}^{2}+\frac{3ab}{|ab|}L_{\phi}L_{\psi}+\frac{2|b|+|a|}{|a|+|b|}L_{\psi}^{2}>K+|ab|\mu^{2}. \tag{6.24}$$

From Eq. (6.22) a trajectory at $\theta=0$ has $L_{\phi}=0$ and $K=[2|b|/(|a|+|b|)]L_{\psi}^{2}+a^{2}\mu^{2}$. A trajectory at $\theta=\pi/2$ has $L_{\psi}=0$ and $K=[2|a|/(|a|+|b|)]L_{\phi}^{2}+b^{2}\mu^{2}$. In both cases there are radially unbounded trajectories if the appropriate angular momentum is sufficiently large. The extension to modes of scalar waves is straightforward, and again there will be some modes with nonzero axial eigenvalues m_{ϕ} and/or m_{ψ} which propagate as oscillating waves to infinity. Superradiant scattering will exist, though the condition for superradiance is more complicated. The qualitative picture is just as it was in the four-dimensional case.

VII. CONCLUSION

We have explored the near-horizon geometry of an extreme rotating black hole in the hope that it will be useful in extending the remarkable duality between string theory and field theory to the vacuum case. It is not yet clear whether this will succeed. While the vacuum solutions we have discussed have striking similarities to the $AdS_2\times S^2$ geometry arising in the near-horizon limit of extreme Reissner-Nordström black holes, there are some crucial differences. These include the fact that in addition to the usual localized modes of a test field with discrete frequencies, there are also traveling waves with continuous frequencies that exhibit a type of superradiance. The quantum analog of this superradiance, and its possible implications for duality, remains to be explored.

Another open question raised by this work is the nature of the singularity in the five-dimensional vacuum solution with AdS_3 symmetry discussed in the previous section. Does this solution have physical interest?

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