Who's afraid of naked singularities? Probing timelike singularities with finite energy waves

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To probe naked spacetime singularities with waves rather than with particles we study the well posedness of initial value problems for test scalar fields with finite energy so that the natural function space of initial data is the Sobolev space. In the case of static and conformally static spacetimes we examine the essential self-adjointness of the time translation operator in the wave equation defined in the Hilbert space. For some spacetimes the classical singularity becomes regular if probed with waves while stronger classical singularities remain singular. If the spacetime is regular when probed with waves we may say that the spacetime is "globally hyperbolic." [S0556-2821(99)00620-7]

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I. INTRODUCTION

In general relativity a singular spacetime is defined by geodesic incompleteness [1]. However, sometimes such a definition gives a very weak singularity which seems almost harmless from a physical point of view. For example, a spacetime from which a single point is taken out is a singular spacetime because there is a geodesic curve which terminates at the point outside of the spacetime with a finite affine time. A stronger "physical singularity" appears for example at the center of a spherically symmetric black hole, where the curvature scalar diverges and therefore the resultant infinite tidal force will tear off any physical object. The classification of singularities is yet under way but has not been completed [2].

The standard definition of a spacetime singularity is physically based on a probe with classical point particles. In this paper we shall discuss a wave probe of timelike singularities which was initiated by Wald [3] and later developed by Horowitz and Marolf [4]. The idea of the probe with waves rather than with classical particles is motivated by quantum field theory because everything should be described by quantum fields. The wave may propagate through the would-be singularity with a definite and unique way. For example, in the case of the hydrogen atom the wave function is finite at the origin, which is a classical singularity. It is known that if the space is geodesically complete the Laplacian operator has a self-adjoint extension and the extension is unique so that the wave propagation is well defined. The converse is not always true. If the geodesic completeness is replaced by the well posedness of initial value problems for test fields the concept of the global hyperbolicity and therefore the cosmic censorship[1] should be drastically changed as Clarke [5] has advocated.

We shall be concerned with a wave propagation dictated by the Klein-Gordon equation in a curved spacetime with timelike singularities. Only for illustration in the introduction we use the simplest case; the Klein-Gordon equation in (1+1)-dimensional Minkowski spacetime, $(-\partial_t^2+\partial_x^2)f=0$ defined in a suitable region of the spacetime. (For a general case, see the following sections.)

For the initial value problem we introduce the following norm on a function space on each $t=$const hypersurface:

$$||f||:=\left(\frac{q^2}{2}\int dx|f|^2+\frac{1}{2}\int dx\left|\frac{df}{dx}\right|^2\right)^{1/2}, \tag{1.1}$$

where q^2 is a positive constant. We call the function space $\mathcal{H}=\{f|\,||f||<\infty\}$ the Sobolev space or H^1. The Sobolev norm has been used in the standard formulation of well-posed initial value problems in a general globally hyperbolic spacetime [1]. We note that in general the well posedness of an initial value problem requires continuous dependence of solutions on initial data, in addition to the existence and the uniqueness of solutions [3]. However, the main issue we will address in this paper is to see the uniqueness of solutions of a wave equation in a nonglobally hyperbolic spacetime, so hereafter we say that the initial value problem is well-posed when the wave propagation is uniquely determined in the whole spacetime.[2]

It is known that the norm is bounded above by the field energy so that the finiteness of the energy implies the finite-

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[1]Here we refer to the "physical formulation" of the strong cosmic censorship rather than the "precise formulation" in Wald's book [6]. If our wave approach can be extended to the initial value problem of the Einstein equations the notion of the cosmic censorship will substantially change.

[2]To show the existence of solutions and to establish an appropriate continuous relation between initial data and solutions, Sobolev norms containing higher order derivatives are chosen to define a topology on the space of initial data. However, to prove the uniqueness of solutions of second order linear hyperbolic equations, it is sufficient to adopt H^1 as our Sobolev space [1], which is larger than H^m with norms containing $m(>1)$th order derivatives. Our results in this paper hold also for $H^{m(>1)}$.

ness of the norm [7]. Since we cannot afford to prepare an infinite energy field configuration as initial data, the function space is naturally limited by the condition:[3]

$$||f||<\infty. \tag{1.2}$$

The corresponding natural inner product in our Hilbert space is defined as

$$(f,g):=\frac{q^2}{2}\int dx f^* g+\frac{1}{2}\int dx \frac{df^*}{dx}\frac{dg}{dx}, \tag{1.3}$$

so that $||f||^2=(f,f)$.

We shall confine ourselves mainly to the case of timelike singularities in static or conformally static spacetimes so that the wave equation becomes of the form

$$\partial_t^2\phi=-A\phi, \tag{1.4}$$

where A is an operator which contains spatial coordinates and spatial derivatives only. In this case the well-posedness of the initial value problem is translated into the essential self-adjointness of the operator A because of the spectral theorem [8]. Namely, we prepare a smooth and nice initial data at some spatial hypersurface by choosing the Sobolev space H^1 as the Hilbert space. The wave will propagate and eventually hit the timelike singularity and will be scattered off in someway. The point of the essential self-adjointness is that any unwanted singular modes which are not contained in the domain of the initial data will not appear after scattering so that the initial value problem is well-posed with no arbitrariness in the choice of the boundary conditions and the prediction is unique. In such a case we say that the spacetime is ''wave regular.''

We would like to emphasize the relevance of the present work to quantum field theory in curved spacetime. It will be natural to expand a quantum field in terms of the normal modes which belong to the Sobolev space rather than L^2. We assign the coefficients of the mode expansion as annihilation and creation operators. The quantum states are constructed by applying the creation operators to the vacuum state which is defined by the condition that the vacuum is annihilated by all the annihilation operators.[4] This construction implies that if the initial value problem is well posed the vacuum expectation value of the energy momentum tensor should be well-behaved near the would-be singularities so that the field energy is finite.

The organization of the rest of the paper is as follows. In Sec. II we propose a natural choice of the function space in which the initial value problem is explored (Sec. II A) and we recapitulate the criterion of essential self-adjointness of operators in the Hilbert space (Sec. II B). In Sec. III we demonstrate how we can probe singularities with waves in Minkowski spacetime with a single point removed and give an intuitive justification of the choice of the Sobolev space as the Hilbert space. Section IV supplies several examples of static spacetimes with timelike singularities. We explicitly show that many of the classical singularities become wave regular, while a single example is wave singular. Section V is the extensions of the discussion of the previous sections to scalar fields with general nonminimal coupling and to conformally static spacetimes. In Sec. VI we discuss how to characterize wave-singular naked singularities in our approach and propose a notion of hair of naked singularities. Section VII is devoted to summary and discussion. In the Appendix some mathematical materials on the essentially self-adjointness are given for the sake of the reader's convenience.

[3]The difference between ours and that of Ref. [4] is in the definition of the norm and therefore of the Hilbert space. In the case of quantum mechanics the natural Hilbert space is the linear function space with the square integrability L^2, because of the probabilistic interpretation of the wave function.

[4]The constructed quantum states belong to L^2 class in the Fock space.

II. THE FUNCTION SPACE OF INITIAL DATA

A. Finite energy field configuration

We consider an $(n+2)$-dimensional static spacetime of the metric form

$$ds^2=-V^2dt^2+h_{ij}dx^i dx^j, \tag{2.1}$$

with a timelike Killing vector field $\xi^\mu=(\partial_t)^\mu$.

We choose a function space on each $t=$const hypersurface Σ as

$$\mathcal{H}=\{f|\,||f||<\infty\} \tag{2.2}$$

with the Sobolev norm $||f||$ being given by

$$||f||^2:=\frac{q^2}{2}\int_\Sigma d\Sigma V^{-1}f^*f+\frac{1}{2}\int_\Sigma d\Sigma V h^{ij}D_i f^* D_j f, \tag{2.3}$$

where q^2 is a positive constant and D_i denotes the covariant derivative with respect to the induced metric h_{ij} on Σ. Here $d\Sigma=d^{n+1}x\sqrt{h}$ is the natural volume element on Σ. The norm is bounded above by a positive constant times the energy integral E,

$$||f||^2<\text{const}\times E, \tag{2.4}$$

where

$$E:=\int_\Sigma d\Sigma\, n^\mu \xi^\nu T_{\mu\nu}[f], \tag{2.5}$$

with n^μ being the unit normal to Σ. Here the energy momentum tensor is given by

$$T_{\mu\nu}[f]:=\frac{1}{2}(\nabla_\mu f^*\nabla_\nu f+\nabla_\nu f^*\nabla_\mu f)-\frac{1}{2}g_{\mu\nu}(\nabla^\sigma f^*\nabla_\sigma f+m^2 f^* f). \tag{2.6}$$

For $n^\mu=V^{-1}(\partial_t)^\mu$ the energy E is expressed by

$$E=\frac{1}{2}\int_{\Sigma}d\Sigma(V^{-1}\partial_t f^*\partial_t f+m^2Vf^*f)$$
$$+\frac{1}{2}\int_{\Sigma}d\Sigma Vh^{ij}D_if^*D_jf, \tag{2.7}$$

which motivated us to choose the norm given by Eq. (2.3). The finiteness of the norm, $||f||<\infty$, is required because we can prepare only a finite energy configuration of the field.[5] This leads us to the Sobolev space as the function space $\mathcal{H}$ on Σ. The energy E is conserved because the energy momentum tensor $T^{\mu\nu}$ satisfies the conservation law $\nabla_\nu T^{\mu\nu}=0$ and ξ^μ satisfies the Killing equation $\mathcal{L}_\xi ds^2=0$. Then the inner product is naturally defined by

$$(f,g):=\frac{q^2}{2}\int_{\Sigma}d\Sigma V^{-1}f^*g+\frac{1}{2}\int_{\Sigma}d\Sigma Vh^{ij}D_if^*D_jg. \tag{2.8}$$

We will consider the massless case only because it is known that the initial value problem is well posed for $m\neq 0$ if it is for $m=0$ [4].

B. Uniqueness of the time translation operator

Let us briefly recapitulate the mathematics on the essential self-adjointness of a linear operator A on the Hilbert space $\mathcal{H}$. For precise definitions see the Appendix, in which we collect relevant mathematical materials.

The wave equation of a massless test scalar field, $\Box\phi=0$, reduces to

$$\partial_t^2\phi=-A\phi, \tag{2.9}$$

where $A:=-VD^iVD_i$ is a positive symmetric operator on $\mathcal{H}$ if the domain of A is suitably chosen, e.g., $C_0^\infty(\Sigma)$, a set of smooth functions with compact support on Σ, so that it is dense in $\mathcal{H}$. In other words we see, by a simple computation

$$(Af,g)=(f,Ag)+\int_{\partial\Sigma}dS^i\{(A+q^2)f^*V\partial_i g$$
$$-V\partial_i f^*(A+q^2)g\} \tag{2.10}$$

so that A is symmetric if $f,g\in C_0^\infty(\Sigma)$ and therefore the surface term above vanishes. In most cases this choice of the domain is not very restrictive.

The domain of A can be further extended by relaxing the boundary condition so that the extended domain coincides with the domain of its adjoint operator. The extended operator in this manner is said to be self-adjoint and its eigenvalues are real and positive. Then, for each self-adjoint extension A_E, the time evolution of the field is uniquely given by [3]

$$\phi(t)=\cos(A_E^{1/2}t)\phi(0)+A_E^{-1/2}\sin(A_E^{1/2}t)\dot{\phi}(0), \tag{2.11}$$

with $\phi(0)$, $\dot{\phi}(0)\in\mathcal{D}(A_E)$ being any initial data. In this sense the self-adjoint extension A_E is a time translation operator.

If there are many possibilities of the self-adjoint extensions, we have to choose one of them by imposing a particular boundary condition, which is normally imposed by some physical requirement. In the case of naked singularities we do not have any criterion to choose the boundary condition. Therefore, if the self-adjoint extension is unique, there remains no ambiguity in the choice of the boundary conditions. A symmetric operator A which has a unique self-adjoint extension is called essentially self-adjoint.

The well-posedness of the initial value problem of Eq. (2.9) is now turned into the essential self-adjointness of the operator A, which can be tested by considering solutions of the equations

$$A^*\psi=\pm i\psi, \tag{2.12}$$

and showing that such solutions do not belong to our Hilbert space [8].

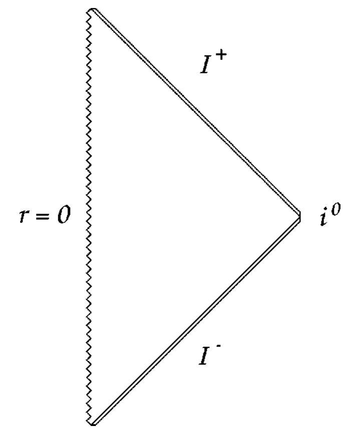

FIG. 1. A conformal diagram of a spacetime with a timelike singularity at the center.

III. SPACE WITH A SINGLE POINT REMOVED

A. Solution of the wave equation

Let us consider a rather artificial model of a timelike singularity which can be fully analyzed. Namely, we consider a spacetime which is locally flat but with a single spatial point removed so that the spacetime has a timelike singularity as illustrated in Fig. 1 and the topology is $(\mathbf{R}^3-\{0\})\times\mathbf{R}$. Our problem in this section is to see the well posedness of the initial value problem of the Klein-Gordon equation

$$-\partial_t^2\phi+\triangle\phi=0, \tag{3.1}$$

[5]Of course, the converse is not necessarily true. That is, $||f||^2<\infty$ does not mean that the energy is finite in general. However, in our present analysis of the spacetime with a timelike (conformal) Killing vector, the Sobolev space implies the finiteness of the field energy.

in this spacetime, which hopefully enhances our understanding of the wave probe for more general timelike singularities in the subsequent sections and partially supports the choice of the Sobolev space.

First we assume that our function space $\mathcal{H}$ on Σ is L^2, i.e.,

$$\mathcal{H}=\left\{\phi\left|\int_\Sigma|\phi|^2d^3x<\infty\right.\right\}, \tag{3.2}$$

and that the tentative domain $\mathcal{D}(\triangle)$ of the Laplacian $\triangle$ is $C_0^\infty(\Sigma)$. Later we consider the case that $\mathcal{H}=H^1$ instead of L^2 and see what is the difference. We do not claim that this analysis is new but we demonstrate this because we believe that this is the most illustrative explicit model in which the choice of the Hilbert space is highlighted.

Separating the time variable t and the angular variables θ,φ we may write solutions in the form

$$\phi_{lm}=e^{-ikt}f_l(r)Y_{lm}(\theta,\varphi)=e^{-ikt}\frac{F_l(r)}{r}Y_{lm}(\theta,\varphi) \tag{3.3}$$

with $Y_{lm}(\theta,\varphi)$ being the spherical harmonics. The reduced wave equation reads

$$\frac{d^2F_l}{dr^2}-\frac{l(l+1)F_l}{r^2}+k^2F_l=0, \tag{3.4}$$

and the L^2 norm squared $\int dx^3|\phi|^2$ reduces to

$$||F||^2=\int_0^\infty dr|F|^2 \tag{3.5}$$

up to an unimportant constant multiple.

The behavior of the radial function near the origin is either

$$\mathcal{F}_l\sim r^{l+1}\quad(f_l\sim r^l), \tag{3.6}$$

or

$$\mathcal{G}_l\sim r^{-l}\quad(g_l\sim r^{-l-1}). \tag{3.7}$$

All the $\mathcal{F}_l$'s belong to the Hilbert space $\mathcal{H}$. The modes $\mathcal{G}_l(l\geqslant1)$ are not square integrable at $r=0$ and therefore are not normal modes. The mode $\mathcal{G}_{l=0}\sim$ const$(g_{l=0}\sim$const$/r)$ is the only mode which requires further care. This mode is square integrable at $r=0$. In the case of $\Sigma\approx\mathbf{R}^3$ this mode does not belong to our Hilbert space because $\triangle(1/r)=-4\pi\delta^3(x)$ is not in L^2 class. However, in the case of $\Sigma\approx\mathbf{R}^3-\{0\}$ this mode is allowed unless one further imposes a boundary condition at $r=0$. However, the boundary condition to be imposed is not unique. Actually a boundary condition

$$aF'-F=0 \tag{3.8}$$

is possible at the origin $r=0$, where a is an arbitrary real parameter. In this sort of simple model one can immediately convince oneself that this is the most general boundary condition at the origin for the self-adjointness of the Laplacian operator but there is a systematic way to get the most general boundary condition, which is powerful for less simpler cases. We defer the demonstration of that method to the following subsections. Let us concentrate on the S-wave solutions ($l=0$). The most general S-wave solution which satisfies the above boundary condition is spanned by

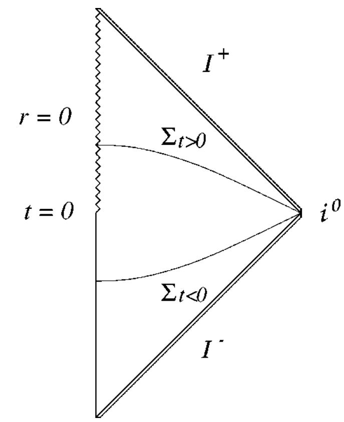

FIG. 2. A conformal diagram of a spacetime $M=\Sigma\times\mathbf{R}$ with $\Sigma_{t<0}\approx\mathbf{R}^3$ and $\Sigma_{t>0}\approx\mathbf{R}^3-\{0\}$ so that a timelike singularity emerges at the center after $t=0$.

$$F_k=\frac{\sin(kr)}{k}+a\cos(kr) \tag{3.9}$$

with a being the constant in Eq. (3.8). In this case we say that the self-adjoint extension is not unique so that the Laplacian with the initial domain $\mathcal{D}(\triangle)=C_0^\infty(\mathbf{R}^3-\{0\})$ is not essentially self-adjoint and therefore the initial value problem is not well-posed. In the case of $\Sigma\approx\mathbf{R}^3$ we have instead

$$F_k=\frac{\sin(kr)}{k}, \tag{3.10}$$

which contains no arbitrary parameter so that the Laplacian with the domain spanned by F_k's of Eq. (3.10) is the only self-adjoint extension.[6] Therefore the Laplacian with the initial domain $C_0^\infty(\mathbf{R}^3)$ is essentially self-adjoint.

This difference may be slightly more dramatic if we consider a spacetime $M=\Sigma\times\mathbf{R}$:

$$\Sigma\approx\begin{cases}\mathbf{R}^3-\{0\} & \text{for } t\geqslant0,\\ \mathbf{R}^3 & \text{for } t<0.\end{cases} \tag{3.11}$$

Namely, a timelike singularity emerges for $t\geqslant0$ as depicted in Fig. 2. In such a spacetime the normal modes do not match at $t=0$ unless $a=0$ so that the initial value problem is ill posed.

[6]The extended domain spanned by F_k's of Eq. (3.10) is the H^1 closure of $C_0^\infty(\mathbf{R}^3)$, and the corresponding self-adjoint extension is the Friedrichs extension [8].

B. A systematic method of self-adjoint extension

From the previous subsection we see that the problem of the function space $L^2(\mathbf{R}^3-\{0\})$ for the field ϕ reduces to the problem of $L^2(0,\infty)$ for the reduced radial wave function F. Let us study the solutions $F_\pm \in L^2(0,\infty)$ of the equations

$$-\frac{d^2F_\pm}{dr^2}=\pm iF_\pm, \tag{3.12}$$

which are reduced from the equations (2.12) concentrated on the S-wave again. The solutions are

$$\mathcal{F}_\pm=\exp\left(-\frac{1\pm i}{\sqrt{2}}r\right), \tag{3.13}$$

$$\mathcal{G}_\pm=\exp\left(\frac{1\mp i}{\sqrt{2}}r\right). \tag{3.14}$$

It is clear that the solutions $\mathcal{G}_\pm$ are not in $L^2(0,\infty)$ class, while $\mathcal{F}_\pm$ are.

The prescription to find the most general boundary condition is to compose

$$F=F_0(r)+\mathcal{F}_+(r)+U\mathcal{F}_+(r), \tag{3.15}$$

where $F_0(r)\in C_0^\infty(0,\infty)$ satisfies the boundary condition $F_0(0)=F_0'(0)=0$ at the origin. U is the isometry of the space $\{\mathcal{F}_+\}$ into the space $\{\mathcal{F}_-\}$ with respect to the $L^2(0,\infty)$ norm, i.e., $U\mathcal{F}_+(r)=e^{i\alpha}\mathcal{F}_-(r)$. An elementary computation shows that

$$\frac{F'(0)}{F(0)}=\frac{-(1+i)/\sqrt{2}-e^{i\alpha}(1-i)/\sqrt{2}}{1+e^{i\alpha}} \tag{3.16}$$

is a real number which we set equal to a^{-1}. This is what we alluded to before.

C. Sobolev space instead of L^2

Let us now change the Hilbert space from $L^2(\mathbf{R}^3-\{0\})$ space to the Sobolev space $H^1(\mathbf{R}^3-\{0\})$. We shall look for the solutions for which the integral

$$\int_0^\infty drr^2|f_\pm'(r)|^2=\int_0^\infty drr^2\left|\frac{d}{dr}\left(\frac{F_\pm(r)}{r}\right)\right|^2 \tag{3.17}$$

is convergent. However, we can see from Eq. (3.13) that the integral (3.17) is divergent for $\mathcal{F}_\pm$. Therefore $f_\pm$ do not belong to the Sobolev space $H^1(\mathbf{R}^3-\{0\})$ so that there remains no room to relax our boundary condition. That is, the Laplacian operator with the initial domain $C_0^\infty(\mathbf{R}^3-\{0\})$ is essentially self-adjoint.

Consider now the previous spacetime model: $M=\Sigma\times\mathbf{R}$ (3.11) and the solutions of the field equation (3.1) in the Sobolev space $H^1(\Sigma)$. It is now clear that the self-adjointly extended domain of the Laplacian agrees in both regions of the spacetime. Therefore the initial value problem is well-posed in the whole spacetime M. Actually the spherical wave propagates with no trace of the would-be singularity at the origin. Of course, this is because the spacetime is almost Minkowski. In a general wave-regular spacetime, the wave would be distorted and scattered by strong curvature there in a definite and unique way.

It is physically assuring to see that the removed point is completely of no effect if the initial field configuration has a finite energy. This also supports that our choice of the Hilbert space is physically sensible.

IV. WAVE PROBE IN STATIC SPACETIMES

A. Spherically symmetric static spacetimes

To illustrate the test of the essential self-adjointness of the operator A in Eq. (2.9) in curved spacetimes, we first study the well-known spherically symmetric spacetimes. In a general $(n+2)$-dimensional spherically symmetric static spacetime, the metric is given by

$$ds^2=-V^2dt^2+V^{-2}dr^2+R^2d\Omega_n. \tag{4.1}$$

Here we assume that V^2 is a positive function of r for $0<r<\infty$ and is singular at $r=0$ so that the causal structure of the spacetime is as shown in Fig. 1. Provided $\psi=f(r)Y(\Omega)$, the equations (2.12) reduce to

$$f''+\frac{(V^2R^n)'}{V^2R^n}f'-\frac{c}{V^2R^2}f\pm i\frac{f}{V^4}=0, \tag{4.2}$$

where the prime denotes the derivative with respect to r and c is the angular momentum quantum number. The norm of f is given by

$$||f||^2=\frac{q^2}{2}\int d\mu drR^nV^{-2}|f|^2+\frac{1}{2}\int d\mu drR^nV^2|f'|^2, \tag{4.3}$$

where $d\mu$ is the volume element on the unit n sphere ($d\Sigma=d\mu drV^{-1}R^n$).

For the essential self-adjointness of A, the norms of the solutions of the equations (4.2) should be divergent for each c and each sign of the imaginary term. We can easily verify that the norm $||f||$ is divergent for $c>0$ if it is for $c=0$, so we will examine the essential self-adjointness for the $c=0$ (S-wave) case.

1. Negative mass Schwarzschild spacetime

The four-dimensional negative mass Schwarzschild metric is given by

$$V^2=1+\frac{2M}{r},\quad R=r,\quad (M>0), \tag{4.4}$$

and a timelike singularity is located at the center $r=0$. Near the singularity, the equations (4.2) become

$$f''+\frac{1}{r}f'=0, \tag{4.5}$$

since the other terms are less singular or even regular at $r=0$. Then the two independent solutions $f=\text{const}$ and $g=\ln r$ are obtained. For the latter solution, the second term of the norm squared (4.3) behaves as

$$\sim \int_0 dr r^2 V^2 |g'|^2 \sim \ln r|_0 \rightarrow \infty. \tag{4.6}$$

Thus the operator A on this spacetime is essentially self-adjoint, hence the spacetime is wave regular.

One might worry that our analysis does not involve the $\pm i$ part and only one of the two solutions for Eq. (4.5) is verified not to be in the Hilbert space. One may be unhappy about the lack of intuition in the test being not completely convinced by the demonstration in Sec. III. Here, we should remark that the solution f which well behaves near the singularity $r=0$ is divergent at infinity $r=\infty$ so that the Sobolev norm is divergent. The point is that if one of the independent solutions fails to behave well near the singularity there are no ways for the other solution to meet the condition at infinity because there is not available other independent solution to superpose.

It is amusing to note that this also holds for the higher-dimensional $(n \geqslant 3)$ negative mass Schwarzschild spacetimes. Note also that if L^2 were chosen as the Hilbert space, the operator A would not be essentially self-adjoint in this case [4].

2. Reissner-Nordström spacetime

For the four-dimensional over extreme Reissner-Nordström metric,

$$V^2 = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad R=r, \ (Q^2 \geqslant M^2), \tag{4.7}$$

where Q denotes the electric (magnetic) charge. Near the timelike singularity at $r=0$, the equations (4.2) for the S wave $(c=0)$ become $f''=0$ $(V^2R^2 \sim Q^2)$ and f behaves as $f\sim r$ or a constant. Then, the norm squared (4.3) is finite. Thus, the classical singularity remains wave-singular. For $c>0$ modes, the norms are divergent. This implies that only the S-wave can fall into the singularity. Our analysis is basically local in time so that our wave approach to singularity probe can also work in the case that timelike singularities are hidden behind horizons by extending the analysis in a straightforward way.

Higher-dimensional generalizations of the Reissner-Nordström solution are given by [9,10]

$$V^2 = 1 - \frac{C}{r^{n-1}} + \frac{D^2}{r^{2(n-1)}}, \quad R=r, \tag{4.8}$$

with the parameters C and D being proportional to the mass and the charge, respectively. It is remarked that the singularities in the higher-dimensional $(n \geqslant 3,\ D^2 \neq 0)$ solutions are also wave singular.

To summarize the two examples above, the timelike singularity in the negative mass Schwarzschild spacetime is wave regular while that of the over extreme Reissner-Nordström spacetime is not. The tendency that the over extreme Reissner-Nordström spacetime is more singular than the negative mass Schwarzschild spacetime sounds natural because the curvature is more divergent for the over extreme Reissner-Nordström spacetime.

This reminds us of the well-known examples in quantum mechanics: the Coulomb potential and the r^{-2} potential problems in three-dimensional space. The former is essentially self-adjoint and the latter is not if the r^{-2} potential is attractive and too strong [8].

One might guess from the two examples above that the quasilocal mass given (in the four-dimensional case) by

$$M_{\text{local}} = -\frac{R}{2}\{g^{\mu\nu}(\partial_\mu R)(\partial_\nu R) - 1\} \tag{4.9}$$

would be finite as $r \rightarrow 0$ in the wave-regular cases while it is infinite in wave-singular cases. However, this is not the case in other models as we shall see below. We shall propose an intuitive criterion for the wave-regularity in Sec. IV C.

In general, if the metric functions of the metric (4.1) behave as

$$R \sim r^p, \quad V^2 R^n \sim r^k, \tag{4.10}$$

near the singularity $r=0$, the equations (4.2) for S wave $(c=0)$ become

$$f'' + \frac{k}{r} f' = 0, \tag{4.11}$$

under the condition $np > k-1$, which holds for all our examples. Then, the solutions are $f=\text{const}$ and $g=\ln r (k=1)$ or $r^{1-k} (k \neq 1)$. Since the norm squared (4.3) for the solution $g=r^{1-k}$ (and similarly for $g=\ln r$) is estimated as

$$||g||^2 \sim \int dr r^{-k+2(np-k+1)} + \int dr r^{-k} \sim \int dr r^{-k}, \tag{4.12}$$

the singularity turns out to be wave regular for the case $k \geqslant 1$. The locally flat example discussed in Sec. III is the case $k=2$, the $(n+2)$-dimensional negative mass Schwarzschild metric is the case $k=1$ and the $(n+2)$-dimensional Reissner-Nordström metric $k=-n+2$.

B. Other spacetimes

We shall consider some less known but hopefully more physical solutions of the Einstein equations coupled to matter fields. The first two solutions below exhibit null naked singularities for some parameter regions. We shall remark on the null naked singularities from our point of view.

1. The Wyman solution

The Wyman solution is a static solution of the four-dimensional Einstein equations coupled to a minimally coupled scalar field [11]. The metric is given by

$$V^2=\left(1-\frac{2\eta}{r}\right)^{m/\eta}=\left(\frac{\rho}{\rho+2\eta}\right)^{m/\eta},$$

$$R^2=r^2\left(1-\frac{2\eta}{r}\right)^{1-m/\eta}=\rho^{1-m/\eta}(\rho+2\eta)^{1+m/\eta}, \tag{4.13}$$

where $\eta=\sqrt{m^2+\sigma^2}$ with a scalar charge σ, so $m/\eta<1$ and $\rho:=r-2\eta$. A curvature singularity is located at $\rho=0$. Since $V^2R^2\sim\rho$, this is the case of the metric functions (4.10) with $k=1$ and thus the spacetime is wave regular.

2. Charged dilaton solution

The four-dimensional charged dilaton solution is given by [10]

$$V^2=\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{(1-a^2)/(1+a^2)}, \tag{4.14}$$

$$R^2=r^2\left(1-\frac{r_-}{r}\right)^{2a^2/(1+a^2)}, \tag{4.15}$$

with a^2 being a positive parameter in the model. Consider the extremal case $r_+=r_-$. In the case $a^2>1$, the central singularity at $\rho:=r-r_+=0$ becomes timelike, while it is null for $a^2\leqslant 1$. For any $a^2>1$, $R^2V^2\sim\rho^2$ this is the case of the metric functions (4.10) with $k=2$ and the singularity is wave regular. The first term of the norm squared (4.12) diverges for $a^2\leqslant 3$ so that we would reproduce the result in Ref. [4] if we chose the L^2 function space as our Hilbert space.

3. String solution

The five-dimensional string solution given by

$$ds^2=V^2(-dt^2+dz^2)+dr^2+r^2d\Omega_2^2, \tag{4.16}$$

$$V^2=\left(1+\frac{M}{r}\right)^{-1}, \tag{4.17}$$

has a curvature singularity at the center $r=0$, which corresponds to a straight string. The operator A which appears in the wave equation $\Box\phi=-V^{-2}(\partial_t^2+A)\phi=0$ is expressed as

$$A=-\partial_z^2-V^2\left\{\partial_r^2+\left(\frac{\partial_rV^2}{V^2}+\frac{2}{r}\right)\partial_r-\frac{c}{r^2}\right\}, \tag{4.18}$$

where c is the angular momentum quantum number on the unit two-sphere. By the separation of the variables $\psi=f(r)e^{ikz}Y(\Omega)$, the equations $(A^*\mp i)\psi=0$ reduce to

$$f''+\frac{3}{r}f'=0,\quad \text{for } S \text{ wave}, \tag{4.19}$$

near $r=0$. In this region, the solutions are $f=\text{const}$ and $g=r^{-2}$, and the norm squared for g

$$||g||^2\sim\int drr^2r^{-4}+\int drr^2V^2r^{-6} \tag{4.20}$$

diverges. Hence the central singularity is wave regular.

C. A simple criterion of wave-regularity for spherically symmetric cases

We may give an intuitive but not necessarily mathematically rigorous explanation of the wave regularity and a simple criterion in what case the classical singularity becomes wave-regular. Take the example of static and spherically symmetric spacetimes. We can see that if we introduce a new radial coordinate X as

$$X:=\int\frac{dr}{R^nV^2} \tag{4.21}$$

the equations in the test of the essential self-adjointness look similar to

$$\frac{d^2f}{dX^2}-cR^{2n-2}V^2f\pm iR^{2n}f=0, \tag{4.22}$$

and the essential part of the Sobolev norm is

$$||f||^2=\int dX\left|\frac{df}{dX}\right|^2. \tag{4.23}$$

Therefore the problem becomes similar to the quantum mechanics in a semi-infinite region (except the norm) if the variable $X\in(a,\infty)$ for $r\in(0,\infty)$ with a being a finite number. As is well known in that case the essential self-adjointness becomes nontrivial. On the other hand, if the variable $X\in(-\infty,\infty)$ for $r\in(0,\infty)$, such a ''half-space problem'' would not appear. For the wave-regular case such as the negative mass Schwarzschild metric the range of X extends to $-\infty$ as the singularity $r=0$ is approached while it is finite for the over extreme Reissner-Nordström metric, which is the wave-singular case. Indeed this holds for all the cases given by the metric functions (4.10). This may suggest that the variable X in the wave mechanics plays a role similar to the affine parameter in the particle mechanics and that for a wave in a wave-regular spacetime the singularity is effectively infinitely far away.

As a by-product of the above observation we can see that if $R^2<\infty$ as $r\to 0$ and the singularity is null, i.e., $\int drV^{-2}\to\infty$ as $r\to 0$, then the singularity is wave regular because $\int drV^{-2}R^{-n}\to\infty$. This can be checked in the Wyman solution replacing σ by $i\sigma$ so that the parameter $m/\eta>1$ and therefore the singularity is null, though the scalar field becomes a ghost and the model becomes unphysical. In the charged dilaton model, the singularity becomes null for $a^2\leqslant 1$ and is wave regular. The wave regularity of the null singularity was also asserted in Ref. [4] but with different reasoning.

V. GENERALIZATIONS

A. Probes with other fields

So far, the examples of timelike singularities have been probed with minimally coupled massless scalar fields. One can think of probing singularities with other fields such as spinor, vector, tensor fields, or metric perturbations. Our procedure of probing singularity is immediately generalized to each case by replacing the inner product (2.8), hence the norm, in an appropriate way for the probing field.

For example, when probing a timelike singularity with a massless scalar field coupled to the scalar curvature, we should adopt the inner product respecting the stress tensor $T^c_{\mu\nu}$ for the field [see Eq. (3.190) in Ref. [12]]. The field equation in the case is $(\Box+\xi\mathcal{R})\phi=0$, so the operator

$$A=-VD^iVD_i-\xi V^2\mathcal{R} \tag{5.1}$$

should be examined for the essential self-adjointness, where ξ is a numerical factor and $\mathcal{R}$ the scalar curvature. In the case of the Wyman solution, the equations $(A^*\mp i)\phi=0$ reduce near $\rho=0$ to

$$f''+\frac{1}{\rho}f'+\frac{\gamma}{\rho^2}f=0, \quad \left(\gamma=\frac{\xi}{2}\frac{\sigma^2}{\eta^2}\right), \tag{5.2}$$

and the solutions are given by $f=\mathcal{A}e^{i\sqrt{\gamma}\ln\rho}+\mathcal{B}e^{-i\sqrt{\gamma}\ln\rho}$, $(\mathcal{A},\mathcal{B}\in\mathbf{C})$. After some calculation, it is observed that the Sobolev norms for the solutions logarithmically diverge near $\rho=0$. Thus, the singularity of the Wyman solution is also wave regular when probed with the scalar field coupled to the scalar curvature. As is well known, in the conformally coupled scalar field case, that is $\xi=(d-2)/4(d-1)$ for any spacetime dimension d, the field equation is invariant under the conformal transformations of the metric and the field, $g_{\mu\nu}(x)\to\overline{g}_{\mu\nu}(x)=C^2(x)g_{\mu\nu}(x)$, $\phi\to\overline{\phi}=C^{(2-d)/2}\phi$. Since $T^c_{\mu\nu}$ transforms as $\overline{T^c}_{\mu\nu}=C^{2-d}T^c_{\mu\nu}$, the corresponding inner product is conformally invariant. Thus, the calculation is as simple as that in the static case when singularities in conformally static spacetimes are probed with conformally coupled scalar fields. This will be seen in the following subsections.

Generalizations to the probes with spin 1/2 and 1 fields are similar. Explicit expressions of $T_{\mu\nu}$ for such fields can be found, for example, in Ref. [12].

B. Conformally static spacetimes

In the previous section, we have examined the would-be naked singularities in static spacetimes, which can be regarded as regular if they are not detectable by waves, i.e., if the initial value problem for the field is well posed. However, physically more interesting cases which can confront with the cosmic censorship are such dynamical spacetimes that naked singularities emerge after gravitational collapse. For example, to get insights into the problem of the final fate of gravitational collapse, the Tolman-Bondi solution has been extensively studied by many people. It has been revealed that for some initial data shell-focusing naked singularities can be formed as a final product of spherical dust collapse [13]. As the wave approach to fully dynamical cases is still far reaching for us at present, we will present a simple model to mimic such a dynamical problem on the basis of our study of the timelike singularity in conformally static spacetimes.

1. Self-similar case

Dynamical problems such as the formation of naked singularities can be made tractable by assuming self-similarity and the nature of naked singularities have been investigated [14]. Self-similar spacetimes also have attracted attention in connection with the critical behavior in gravitational collapse and have been studied in detail by many authors [15]. Probing timelike singularities especially in self-similar spacetimes therefore is an interesting issue concerning the cosmic censorship. A technical advantage of self-similar metrics is that they can be written in the conformally static form so that we can straightforwardly apply our procedure developed for the static spacetime in the previous sections to the self-similar spacetimes when probing with conformally coupled scalar fields.

For a massless scalar field, there is a nonstatic spherically symmetric solution discovered by Roberts [16]. As one of the models of naked singularities in self-similar spacetimes, we will analyze the timelike singularity in the Roberts solution, whose metric can be written in the conformally static form

$$ds^2=e^{2\eta}d\hat{s}^2=e^{2\eta}\{-d\eta^2+dr^2+R^2(r)d\Omega^2\},$$

$$R^2(r):=\frac{1}{4}\{1+p-(1-p)e^{-2r}\}(e^{2r}-1), \tag{5.3}$$

where p is an integration constant and $(\partial_\eta)^\mu$ is the homothetic vector. For the value $0<p<1$, the curvature singularity located at $r=0$ becomes timelike and the global structure is identical to that of the negative mass Schwarzschild spacetime. By using this solution the problem of self-similar scalar field collapse has been discussed [16,17].

When probing the singularity with a conformally coupled scalar field, we can carry out the previous analysis with respect to the static metric $d\hat{s}^2$ instead of ds^2. Since near $r=0$, $R^2\sim r$, this is the case of the metric functions (4.10) with $k=1$. Thus, the norm squared (4.12) logarithmically diverges so that the singularity is wave regular for the conformally coupled scalar field.

2. Conformally flat spacetimes with emerging naked singularity

More physical situations of gravitational collapse require that the spacetime contains a regular initial spacelike hypersurface on which the collapsing matter has a compact support. We can construct a spacetime which has a regular initial hypersurface Σ_{t_0} and a timelike singularity being formed in the future of Σ_{t_0}. For example, let us consider a conformally flat metric

$$ds^2=C^2\{-dt^2+dr^2+r^2d\Omega^2\}, \tag{5.4}$$

with a conformal factor which behaves near the center $r=0$ as

$$C^2 \sim \begin{cases} r^p & (t \geqslant 0), \\ r^p + 1 & (t < 0), \end{cases} \tag{5.5}$$

where $p > -2$ and $C^2(r=0,t) \to 0$ sufficiently smoothly as $t \to 0$. Then, the spacetime has a timelike curvature singularity at the center $r=0$ for $t \geqslant 0$ as depicted in Fig. 2. However, for sufficiently remote past, there are regular hypersurfaces Σ_t. Therefore we can take an initial regular hypersurface Σ_{t_0} at some $t_0 < 0$ and construct the Hilbert space on Σ_{t_0}.

Since all the hypersurfaces Σ_t are isomorphic to the initial hypersurface Σ_{t_0} up to a conformal factor, the Hilbert spaces of a conformally coupled scalar field on Σ_t are the same, even when Σ_t intersects the central singularity. Therefore the spacetime is wave regular for the conformally coupled scalar field.[7] Timelike singularities of the type examined above turn out to be also wave regular when probed with the Maxwell field since it is conformally invariant and Maxwell's equations are reduced to that of a massless Klein-Gordon field under a suitable gauge condition.

C. Cylindrically symmetric case

So far, we have studied spherically symmetric examples, whose central singularities are thus considered to be pointlike. Here, as another example, we will probe a singularity of a cylindrically symmetric spacetime given by the metric

$$ds^2 = -\rho^{2\sigma_1} dt^2 + \rho^{2\sigma_2} dz^2 + \rho^{2\sigma_3} d\varphi^2 + d\rho^2, \tag{5.6}$$

where the parameters σ_i satisfy $\Sigma_i \sigma_i = \Sigma_i (\sigma_i)^2 = 1$. This metric is known as the timelike Kasner solution and describes a cylindrical vacuum spacetime with a timelike curvature singularity along the line $\rho = 0$ for the parameter values $|\sigma_i| < 1$.

On each hypersurface Σ_t, the operator A in Eq. (2.9) is written as

$$A = -\rho^{2\sigma_1} \left(\partial_\rho^2 + \frac{1}{\rho} \partial_\rho + \frac{1}{\rho^{2\sigma_2}} \partial_z^2 + \frac{1}{\rho^{2\sigma_3}} \partial_\varphi^2 \right). \tag{5.7}$$

Since $|\sigma_i| < 1$, the equations (2.12) reduce to

$$\left(\partial_\rho^2 + \frac{1}{\rho} \partial_\rho \right) \psi = 0, \tag{5.8}$$

near the singularity $\rho = 0$. In this region, the solution behaves as $f = \text{const}$ or $g = \ln\rho$ and the norm squared becomes

$$\|\psi\|^2 \sim \int d\rho \rho^{1-2\sigma_1} |\psi|^2 + \int d\rho \rho |\partial_\rho \psi|^2. \tag{5.9}$$

For the solution $g = \ln\rho$, the second term logarithmically diverges, thus the singularity is wave regular.

For $\sigma_i = (0,0,1)$, the spacetime is flat and, if the angle coordinate φ has a deficit, the line $\rho = 0$ becomes a cone singularity, which thus expresses a thin cosmic string in a locally flat spacetime. Also in this case the spacetime is wave regular.

VI. HAIR OF WAVE-SINGULAR NAKED SINGULARITIES

In the previous sections, we have seen that most of the timelike "singularities" are wave regular and that the singularity of the Reissner-Nordström spacetime is the only exception in our examples. In the wave-singular case, since probing waves feel the existence of the singularity in some sense, we naively expect that the waves scattered by the singularity will inform us of some feature of the singularity. Here, we shall discuss how to characterize wave-singular naked singularities.

In the wave-singular case, the symmetric operator A in the wave equation has many different self-adjoint extensions. As discussed in Sec. II B, each self-adjoint extension corresponds to a different boundary condition at the singularity and accordingly describes a different time evolution of the wave for the same initial data. In other words, a wave-singular naked singularity has degrees of freedom for the possible choice of the time evolution operator. The degrees of freedom can be interpreted as the character or the *hair* of the wave-singular naked singularity. Since the self-adjoint extensions are in one-to-one correspondence with the set of partial isometries between the deficiency subspaces $\mathcal{K}_\pm$ of A (see the Appendix for the definition), the set of isometries U describes the degrees of freedom.

In general, when probing timelike singularities in static spacetimes with massless Klein-Gordon waves, we will find a positive real symmetric operator of the form $A = -VD^iVD_i$ with domain $C_0^\infty(\Sigma)$. Then the deficiency indices n_+, n_- of A are equal and self-adjoint extensions can be made. If the naked singularity is wave singular, then $n_+ = n_- = N \neq 0$ and hence the partial isometry U is represented by an $N \times N$ unitary matrix $U(N)$. Then, we say that the singularity has a $U(N)$ hair.

Let us consider a wave-singular spacetime which has an asymptotically flat region and a static region in a neighborhood of the central timelike singularity. More precisely, we will consider such a spacetime that the metric form is given by Eq. (4.1) and the metric functions behave as Eq. (4.10) with $k < 1$ near $r = 0$. Consider the wave probe with a massless Klein-Gordon wave, in particular the S wave, in this

[7] An everywhere smooth example can be constructed by choosing the conformal factor as

$$C^2 = \begin{cases} U_{\geqslant} \times e^{-1/r^2} & (\text{for } t \geqslant 0), \\ U_{<} \times (e^{-1/r^2} + e^{-1/t^2}) & (\text{for } t < 0), \end{cases}$$

where $U_{\geqslant}$ and $U_<$, the analytic functions of r and t, should be chosen so that $C^2 \to 1$ as $r \to \infty$. By taking U's appropriately, we have regular hypersurfaces at $t < 0$ whose most of the portions are flat.

spacetime. The solutions of the equations $(A^* \mp i)\psi_\pm = 0$ near $r=0$ are given by

$$f_\pm \sim a_\pm r^{1-k} + b_\pm, \tag{6.1}$$

where $\psi_\pm = f_\pm(r) Y(\Omega)$ and $a_\pm$, $b_\pm$ are constants. Both $f_\pm$ well behave near $r=0$. On the other hand, in the asymptotic region, the equations $(A^* \mp i)\psi_\pm = 0$ reduce to

$$f''_\pm + \frac{2}{r} f'_\pm + \left\{ \pm i - \frac{l(l+1)}{r^2} \right\} f_\pm = 0, \tag{6.2}$$

and the solutions for the S wave $(l=0)$ are $\exp\{(1 \mp i) r/\sqrt{2}\}/r$ and $\exp\{(-1 \mp i) r/\sqrt{2}\}/r$. Clearly the former solutions diverge in the asymptotic region and hence do not belong to the Hilbert space while the latter span the deficiency subspaces. Therefore the deficiency indices for $l=0$ mode are (1,1) and the isometry is $U(1)$. Thus, the wave-singular naked singularity of this spacetime has a $U(1)$ hair. More detailed study will be given in the future work, in which we shall investigate what might occur in such a wave-singular spacetime in quantum field theory.

VII. SUMMARY AND DISCUSSION

We have studied the well posedness of the initial value problem of the scalar wave equation in the case of static and conformally static spacetimes with timelike singularities choosing the Sobolev space as the natural Hilbert space. The physical idea behind the choice is that we can prepare an initial data only with finite energy. We have examined in detail the essential self-adjointness of the operator A in the wave equation defined in the Hilbert space in various models of spacetimes which contain timelike singularities in the conventional sense. In spacetimes such as the negative mass Schwarzschild spacetime the classical singularity becomes regular if probed with waves while more stronger classical singularities such as the over extreme Reissner-Nordström spacetime remain singular.

We should comment that the wave regularity of a spacetime does not guarantee that the spacetime is physically realizable. For example, a negative mass Schwarzschild spacetime is wave regular but allowance of negative mass solutions would make Minkowski spacetime unstable as pointed out by Horowitz and Myers [18]. Probably there is a physics which rules out the negative mass Schwarzschild solution. In contrast, we may say that the over extreme Reissner-Nordström spacetime is unphysical on the ground that it is wave singular.

We have briefly touched upon the case that the timelike singularity emerges at some point in spacetime in rather artificial models which are the Minkowski spacetime with a single spatial point removed and the spacetime model which is conformal to Minkowski spacetime. However, timelike singularities of general spacetimes without any (conformal) timelike Killing vector will be more interesting from the view point of the cosmic censorship especially in the case of gravitational collapse. In the case that spacetimes with ''naked singularities'' as illustrated by Fig. 3 are found to be wave regular, such spacetimes can be said to be ''globally hyperbolic'' in the sense that the initial value problem is well posed with the unique boundary value at the classical singularity. Clarke [5] gave a sufficient condition for the well posedness of the initial value problem for test fields. However, it turns out that the curve integrability condition for a naked singularity in Ref. [5] is not satisfied for almost all the wave-regular cases discussed in the present work. We suspect that the condition is too demanding for the well posedness of the initial value problem and his theorem can be further sharpened.

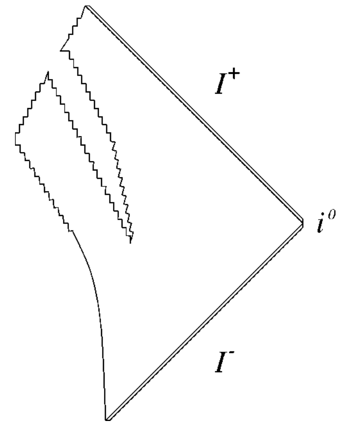

FIG. 3. A spacetime with emerging would-be naked singularities.

The application of the present work to quantum field theory in curved spacetime is most interesting. The normal modes are solutions of the wave equation and an analog of the Klein-Gordon inner product exists; $i\{(f, \partial_t g) - (\partial_t f, g)\}$ with (f,g) being the inner product providing the Sobolev norm, which conserves if the spacetime has a (conformal) timelike Killing vector.

We may carry out similar analyses to spinor, vector, and tensor fields. A natural question will be: does the wave-regularity depend on with what fields we probe? At the moment what we can say is that in principle yes, i.e., it depends and it is nothing wrong from the physical point of view. If initial value problems are well-posed for all fields, ''naked singularities'' are harmless and nothing to be afraid of.

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APPENDIX: EXTENSIONS OF SYMMETRIC OPERATORS

We briefly review some definitions of linear operators on a Hilbert space with an inner product $(\cdot,\cdot)$. An operator on a Hilbert space $\mathcal{H}$ is a pair of a linear mapping $A:\mathcal{H}\to\mathcal{H}$ and its domain of definition $\mathcal{D}(A)$. The pair $(A,\mathcal{D}(A))$ is often abbreviated by A. If an operator A with $\mathcal{D}(A)$ densely defined in $\mathcal{H}$ satisfies

$$(\phi, A\psi) = (A\phi, \psi), \quad \forall \phi, \psi \in \mathcal{D}(A), \tag{A1}$$

then A is called *symmetric*. In the case, any vector $v \in \mathcal{H}$ can be approximated by vectors in $\mathcal{D}(A)$ as close as possible. An operator A' is called an *extension* of A, if $\mathcal{D}(A) \subset \mathcal{D}(A')$ and $A'\psi = A\psi, \ \forall \psi \in \mathcal{D}(A)$. Extensions of an operator A are obtained by the relaxation of the boundary condition on $\mathcal{D}(A)$. Consider sequences $\{\psi_n\} \subset \mathcal{D}(A)$ such that there exist limits $\lim_{n\to\infty}\psi_n =: \xi \in \mathcal{H}$ and $\lim_{n\to\infty} A\psi_n =: \zeta \in \mathcal{H}$. If, for every such sequence, $\xi \in \mathcal{D}(A)$ and $A\xi = \zeta$, then $(A, \mathcal{D}(A))$ is said to be closed. If a nonclosed operator A has a closed extension it is called closable. Every closable operator has a smallest closed extension, which is called its closure. Consider a symmetric operator $(A, \mathcal{D}(A))$. Define $\mathcal{D}(A^*)$ to be the set of all $\phi \in \mathcal{H}$ for which there exists $\chi \in \mathcal{H}$ such that

$$(\phi, A\psi) = (\chi, \psi), \quad \forall \psi \in \mathcal{D}(A). \tag{A2}$$

Then, since $\mathcal{D}(A)$ is dense, χ is uniquely determined by $\phi \in \mathcal{D}(A^*)$ and Eq. (A2). An operator $(A^*, \mathcal{D}(A^*))$ defined by $A^*\phi = \chi$ for every $\phi \in \mathcal{D}(A^*)$ is called the adjoint of $(A, \mathcal{D}(A))$. $\mathcal{D}(A^*)$ may be larger than $\mathcal{D}(A)$, in which case A^* is a proper extension of A. If $(A^*, \mathcal{D}(A^*)) = (A, \mathcal{D}(A))$, an operator $(A, \mathcal{D}(A))$ is said to be self-adjoint.

Now let us see an example of extensions of symmetric operators to self-adjoint ones. Take $\mathcal{H} = L^2(0,1)$ and consider an operator T:

$$T\psi = -i\frac{d\psi(x)}{dx}, \tag{A3}$$

$$\mathcal{D}(T) = \{\psi | \psi(0) = \psi(1) = 0, \ \ \psi \in AC[0,1]\}, \tag{A4}$$

where $AC[0,1]$ expresses the set of absolutely continuous functions on $[0,1]$ whose derivatives are in $L^2(0,1)$. It can be verified that T is symmetric. For $\phi(x) = \exp(ikx) \in \mathcal{D}(T^*)$, $T^*\phi = \chi = k\exp(ikx) \in \mathcal{H}$ is not an element of $\mathcal{D}(T)$, thus $\mathcal{D}(T^*) \supset \mathcal{D}(T)$; T is not self-adjoint.

Next, consider an operator T_α with the same action as T in $\mathcal{D}(T)$ and with the domain

$$\mathcal{D}(T_\alpha) := \{\psi | \psi(0) = e^{i\alpha}\psi(1), \ \ \psi \in AC[0,1]\}, \tag{A5}$$

where α is a real number. Clearly, this is an extension of T. For $\phi \in \mathcal{D}(T_\alpha{}^*)$, there exists $\chi \in \mathcal{H}$ such that

$$\forall \psi \in \mathcal{D}(T_\alpha), \quad (\phi, T_\alpha\psi) = (\chi, \psi), \quad T_\alpha{}^*\phi = \chi. \tag{A6}$$

Namely,

$$\int_0^1 dx\, \phi^* \left(-i\frac{d\psi}{dx}\right) = \int_0^1 dx (T_\alpha{}^*\phi)^*\psi. \tag{A7}$$

On the other hand, by partial integration the following is obtained:

$$\int_0^1 dx\, \phi^* \left(-i\frac{d\psi}{dx}\right) = -i[\phi^*\psi]_0^1 + \int_0^1 dx (T_\alpha{}^*\phi)^*\psi. \tag{A8}$$

Thus, $\phi^*(1)\psi(1) - \phi^*(0)\psi(0) = 0$. From the boundary condition for ψ, it follows that $\phi^*(1) = e^{i\alpha}\phi^*(0) = [e^{-i\alpha}\phi(0)]^*$, hence $\phi(0) = e^{i\alpha}\phi(1)$. Therefore $\mathcal{D}(T_\alpha{}^*) = \mathcal{D}(T_\alpha)$; T_α is self-adjoint. Since α is arbitrary, it turns out that T has infinitely many different self-adjoint extensions.

In general, for a closed symmetric operator A, the closed symmetric extensions can be carried out in a more systematic way. Consider solutions of $A^*\phi_\pm = \pm i\phi_\pm$ and the sets $\mathcal{K}_\pm$ of the solutions $\phi_\pm$, respectively, which are called the deficiency subspaces of A. The pair of numbers $(n_+, n_-) := (\dim\mathcal{K}_+, \dim\mathcal{K}_-)$ is called the deficiency indices of A. If A is a closed symmetric operator with the deficiency indices $n_+ = n_-$, then A has self-adjoint extensions, and if $n_+ = n_- = 0$, A is self-adjoint. Let U be the partial isometries $\mathcal{K}_+ \to \mathcal{K}_-$. Then, the self-adjoint extensions A_E can be obtained by taking the domains as

$$\mathcal{D}(A_E) := \{\phi_0 + \phi_+ + U\phi_+ | \phi_0 \in \mathcal{D}(A), \ \ \phi_+ \in \mathcal{K}_+\}. \tag{A9}$$

In the example above, the normalized solutions $\phi_\pm$ of the equations $T^*\phi_\pm = \pm i\phi_\pm$ are

$$\phi_\pm(x) = \frac{\sqrt{2}e^{\mp x}}{\sqrt{\pm(1 - e^{\mp 2})}}. \tag{A10}$$

The deficiency subspaces of T are $\mathcal{K}_\pm = \{\beta\phi_\pm | \beta \in \mathbf{C}\}$ and thus the deficiency indices are $(1,1)$. Then, the partial isometries are taken as $U: \mathcal{K}_+ \to \mathcal{K}_- : \phi_+ \mapsto \gamma\phi_-$, where $|\gamma| = 1$. The symmetric extension with respect to $\gamma \in U$ is given by $T_\gamma = -i\,d/dx$ with the domain

$$\mathcal{D}(T_\gamma) := \{\phi_o + \beta(\phi_+ + \gamma\phi_-) | \phi_0 \in \mathcal{D}(T), \ \ \beta \in \mathbf{C}\}. \tag{A11}$$

It turns out that the phase factor $e^{i\alpha}$ is given by

$$e^{i\alpha} = \frac{\phi(1)}{\phi(0)} = \frac{1 + \gamma e}{e + \gamma}. \tag{A12}$$

If the closure of a closable symmetric operator A is self-adjoint, A is called essentially self-adjoint. In this case, A has a unique self-adjoint extension. The basic criterion for essential self-adjointness is to verify that its deficiency indices are both zero, namely, the solutions $\phi_\pm$ of the equations $(A^* \mp i)\phi_\pm = 0$ are not in the considering Hilbert space. An example of essentially self-adjoint operators is the Laplacian operator on $L^2(\mathbf{R}^n)$ with the domain $C_0^\infty(\mathbf{R}^n)$: a set of smooth functions with compact support. Detailed studies of extensions of symmetric operators and further examples can be found in the text book [8].

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