Quasilocal energy for rotating charged black hole solutions in general relativity and string theory

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(Received 3 May 1999; published 26 October 1999)

We explore the (non-)universality of Martinez's conjecture, originally proposed for Kerr black holes, within and beyond general relativity. The conjecture states that the Brown-York quasilocal energy at the outer horizon of such a black hole reduces to twice its irreducible mass, or equivalently, to $\sqrt{A}/2\sqrt{\pi}$, where A is its area. We first consider the charged Kerr black hole. For such a spacetime, we calculate the quasilocal energy within a two-surface of constant Boyer-Lindquist radius embedded in a constant stationary-time slice. Keeping with Martinez's conjecture, at the outer horizon this energy equals $\sqrt{A}/2\sqrt{\pi}$. The energy is positive and monotonically decreases to the ADM mass as the boundary-surface radius diverges. Next we perform an analogous calculation for the quasilocal energy for the Kerr-Sen spacetime, which corresponds to four-dimensional rotating charged black hole solutions in heterotic string theory. The behavior of this energy as a function of the boundary-surface radius is similar to the charged Kerr case. However, we show that it does not approach the expression conjectured by Martinez at the horizon. [S0556-2821(99)06018-X]

PACS number(s): 04.20.Cv, 05.30.Ch, 97.60.Lf

I. INTRODUCTION

A geometric theory of gravity, such as general relativity, is known to lack a meaningful notion of local energy density [1–3]. This is essentially due to the absence of an unambiguous prescription for decomposing the spacetime metric into ''background'' and ''dynamical'' components. The low-energy effective field theory for heterotic string theory is no exception. Such theories, however, admit several alternative prescriptions for computing quasilocal energies (see Refs. [3,4] and the references therein). In Ref. [3], Brown and York (BY) introduced a way to define the quasilocal energy (QLE) of a spatially bounded system in general relativity in terms of the total mean curvature of the boundary. In this paper, we first apply this definition to the case of a charged Kerr family of black hole spacetimes, which represent rotating, charged black hole solutions in general relativity. Next we apply it to study the behavior of quasilocal energy of the Kerr-Sen family, which represents rotating, charged black hole solutions in heterotic string theory [5]. Since the static, charged (or neutral) solution is a special case of this family, the QLE of such a solution is obtained in the limit of vanishing angular momentum (or charge). We perform these calculations in the regime of the slow-rotation approximation, where the expressions for the QLE are obtainable in closed form.

The motivation for the analysis in this paper is as follows. It is of interest to explore the form of the classical laws of black hole mechanics and the ensuing picture of black hole thermodynamics in alternative theories of gravity such that they can be compared with the corresponding scenario in general relativity. But the study of the thermodynamical laws entails the knowledge of the energy and entropy associated with these spacetimes. Moreover, equilibrium thermodynamics of a black hole (specifically, in the case of an asymptotically flat solution), requires that it be put in a finite-sized ''box,'' just as one does in general relativity [7]. Thus, such a study requires the knowledge of the quasilocal energy of these ''finite-sized'' systems. Moreover, in a study of the quasilocal energy corresponding to different types of two-boundaries embedded in constant stationary-time slice of the Kerr spacetime (in the slow-rotation approximation), Martinez has conjectured that the QLE approaches twice the irreducible mass of such a black hole. In this paper, we study the (non-)universality of this conjecture. We find that it remains valid for the charged Kerr black hole. In the case of the Kerr-Sen black hole, however, we show that the QLE evaluated on a two-surface of constant Boyer-Lindquist radius embedded in a constant stationary-time slice does not approach the expression conjectured by Martinez at the horizon.

We introduce the BY quasilocal energy in Sec. II. We use this definition in Sec. III to find the QLE of the charged Kerr black hole. Similarly, in Sec. IV we evaluate the QLE of the Kerr-Sen black hole. We present our observations on the results of these sections and discuss the status of Martinez's conjecture for these two cases in Sec. V. Finally, in the Appendix we derive the integral expression for the QLE associated with a certain class of quasilocal two-surfaces embedded in an axisymmetric spatial three-slice. This expression is extensively used in this paper for the QLE computations for different spacetimes mentioned above. We follow the conventions of Ref. [1] and employ geometrized units $G=1=c$.

II. BROWN-YORK QUASILOCAL ENERGY

The BY derivation of the quasilocal energy, as applied to a four-dimensional (4D) spacetime solution of Einstein gravity can be summarized as follows. The system one considers is a 3D spatial hypersurface Σ bounded by a 2D spatial sur-

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face B in a spacetime region that can be decomposed as a product of a 3D hypersurface and a real line-interval representing time. The time-evolution of the boundary B is the surface ^{3}B. One can then obtain a surface stress-tensor on ^{3}B by taking the functional derivative of the action with respect to the 3D metric on ^{3}B. The energy surface density is the projection of the surface stress-tensor normal to a family of spacelike surfaces such as B that foliate ^{3}B. The integral of the energy surface density over such a boundary B is the quasilocal energy associated with a spacelike hypersurface Σ whose *orthogonal* intersection with ^{3}B is the boundary B. Here we assume that there are no inner boundaries, such that the spatial hypersurfaces Σ are complete. In the case where horizons form, one simply evolves the spacetime inside as well as outside the horizon. Under these conditions, the QLE is defined as:

$$E=\frac{1}{8\pi}\oint_{\mathsf{B}} d^2x\sqrt{\sigma}(k-k_0), \tag{2.1}$$

where σ is the determinant of the 2-metric on B, k is the trace of the extrinsic curvature of B, and k_0 is a reference term that is used to normalize the energy with respect to a reference spacetime, not necessarily flat. To compute the QLE for asymptotically flat solutions, we will choose the reference spacetime to be flat as well. In that case, k_0 is the trace of the extrinsic curvature of a two-dimensional surface embedded in flat spacetime, such that it is isometric to B.

Interestingly, the foregoing analysis can be applied in a straightforward way to compute QLE of spatial sections of solutions of scalar-tensor theories as well. This is because, for spacetime dimensions higher than two, a scalar-tensor theory can be cast in the ‘‘Einstein-Hilbert’’ form by performing a conformal transformation. The solutions of this conformally related action are given by the Einstein-frame metrics, which themselves are related to solutions of the scalar-tensor theory by a conformal transformation. It was shown in Ref. [4] that the quasilocal mass (QLM), which is a construct related to QLE, is invariant under such a transformation. When there is a timelike Killing vector field ξ^μ on the boundary ^{3}B, such that it is also hypersurface forming, one can define an associated conserved QLM for the bounded system [6,3]:

$$M=\int_{\mathsf{B}} d^{(D-1)}x\sqrt{\sigma}N\varepsilon, \tag{2.2}$$

where N is the lapse function related to ξ^μ by $\xi^\mu=Nu^\mu$. Further, if $\xi\cdot u=-1$, then $N=1$ and consequently the QLM is the same as the QLE. Unlike QLE, the quasilocal mass is independent of any foliation of the bounded system. Moreover, owing to its conformal invariance, a frame of convenience can be chosen for the computation of QLM without affecting its value. The QLE, in general, is not a conformal invariant. Thus, the frame in which it is evaluated needs to be clearly specified. In this paper, all computations of the QLE will be carried out in the Einstein frame. This is the frame in which Sen analyzed the properties of his rotating charged black hole solution.

III. QLE FOR CHARGED KERR BLACK HOLES

The charged Kerr solution in general relativity represents the spacetime of a rotating, charged black hole. Its spacetime metric and electromagnetic vector potential are given by [9]

$$\begin{aligned} ds^2=&-\left(\frac{\Delta-a^2\sin^2}{\Sigma}\right)dt^2-\frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dt\,d\phi\\ &+\left[\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta\,d\phi^2\\ &+\frac{\Sigma}{\Delta}dr^2+\Sigma\,d\theta^2, \end{aligned} \tag{3.1}$$

$$A_a=-\frac{Qr}{\Sigma}[(dt)_a-a\sin^2\theta(d\phi)_a], \tag{3.2}$$

where

$$\Sigma=r^2+a^2\cos^2\theta, \tag{3.3}$$

$$\Delta=r^2+a^2+Q^2-2Mr. \tag{3.4}$$

The above fields define a black hole solution with mass M, charge Q, and specific angular momentum a.

A. Unreferenced QLE for charged Kerr black holes

Consider a constant stationary-time hypersurface Σ embedded in the charged Kerr spacetime with the metric (3.1). Define the two-surface B to be a surface with some constant value for the Boyer-Lindquist radial coordinate, $r=r_0$, embedded in Σ. The line-element on this surface in the slow-rotation approximation, i.e., $a\ll r_0$, is:

$$ds^2\approx r_0^2\left(1+\frac{a^2}{r_0^2}\cos^2\theta\right)d\theta^2+r_0^2\left\{1+\frac{a^2}{r_0^2}\left[1+\left(\frac{2M}{r_0}-\frac{Q^2}{r_0^2}\right)\sin^2\theta\right]\right\}\sin^2\theta\,d\phi^2. \tag{3.5}$$

Terms of order $O(a^4/r_0^4)$ and higher have been neglected.

We now calculate the unreferenced QLE within the two-surface B, defined by $r=r_0$. We assume that $r_0\geqslant r_+$, where r_+ represents the outer horizon of the charged Kerr black hole. Using Eq. (A6) of the Appendix, the unreferenced QLE ε for the surface $r=r_0$ can be written explicitly as

$$\varepsilon=-\frac{r_0}{2}\sqrt{1-\frac{2M}{r_0}+\frac{a^2+Q^2}{r_0^2}}\int_0^\pi d\theta\sin\theta\left\{1-\frac{a^2}{2r_0^2}\left[\cos^2\theta+\left(\frac{M}{r_0}-\frac{Q^2}{r_0^2}\right)\sin^2\theta\right]+O\left(\frac{a^4}{r_0^4}\right)\right\}, \tag{3.6}$$

where we have retained only terms of leading order in the parameter a/r_0. The above integration can be performed in a straightforward way to yield

$$\varepsilon=-r_0\sqrt{1-\frac{2M}{r_0}+\frac{a^2+Q^2}{r_0^2}}\times\left[1-\frac{a^2}{6r_0^2}\left(1+\frac{2M}{r_0}-\frac{Q^2}{r_0^2}\right)+O\left(\frac{a^4}{r_0^4}\right)\right]. \tag{3.7}$$

Note that in the limit $Q\to 0$, the above expression simplifies to

$$\varepsilon=-r_0\sqrt{1-\frac{2M}{r_0}+\frac{a^2}{r_0^2}}\left[1-\frac{a^2}{6r_0^2}\left(1+\frac{2M}{r_0}\right)+O\left(\frac{a^4}{r_0^4}\right)\right], \tag{3.8}$$

which is the unreferenced QLE of the neutral Kerr black hole [10]. It also has the expected limit when $a\to 0$, in which case Eq. (3.7) gives

$$\varepsilon=-r_0\sqrt{1-\frac{2M}{r_0}+\frac{Q^2}{r_0^2}}, \tag{3.9}$$

which is the unreferenced QLE of the Reissner-Nordstrom black holes [3].

Note that no approximations have been made inside the square-root appearing in ε. As $r_0\to\infty$, we have $\varepsilon\to M-r_0$, which is divergent. This prompts the need for a subtraction term to renormalize the unreferenced QLE. Below, we compute such a reference term.

B. Reference term

To obtain the reference term in the QLE expression, Eq. (2.1), we first find a 2D surface isometric to Eq. (3.5), which is embeddable in a flat 3D slice with the line element

$$ds^2=d\mathcal{R}^2+\mathcal{R}^2d\Theta^2+\mathcal{R}^2\sin^2\Theta\, d\Phi^2, \tag{3.10}$$

where $\mathcal{R}$, Θ, Φ are the spherical polar coordinates. Let the desired 2D surface in the flat slice be characterized by $\mathcal{R}=f(\Theta)$, where f is a function of the azimuthal angle Θ and the parameters (M,a,r_0). Its intrinsic metric is obtained from Eq. (3.10) as follows. We assume that $\Theta=\Theta(\theta)$ and $\Phi=\phi$. Then, on the two-surface B, we find that $\mathcal{R}$ is a function of θ, i.e., $\mathcal{R}=R(\theta)$, say. Hence, the line-element on B is

$$ds^2=[\dot{R}^2+R^2\dot{\Theta}^2]d\theta^2+R^2\sin^2\Theta\, d\phi^2, \tag{3.11}$$

where an overdot denotes derivative with respect to θ.

Requiring the above line-element to be isometric to Eq. (3.5) implies that the following couple of equations be obeyed:

$$\dot{R}^2+R^2\dot{\Theta}^2=r_0^2\left[1+\frac{a^2}{r_0^2}\cos^2\Theta\right], \tag{3.12}$$

and

$$R^2\sin^2\Theta=r_0^2\sin^2\theta\left[1+\frac{a^2}{r_0^2}\left(1+\frac{2M}{r_0}\sin^2\theta-\frac{Q^2}{r_0^2}\sin^2\theta\right)\right]. \tag{3.13}$$

Assuming that $\dot{R}^2\simeq O(a^4/r_0^4)$ (this condition will be justified *a posteriori*), the above equations can be combined to yield the following first-order ordinary differential equation:

$$\frac{d\Theta}{\sin\Theta}=\frac{d\theta}{\sin\theta}\left[1-\frac{a^2}{2r_0^2}\sin^2\theta\left(1+\frac{2M}{r_0}-\frac{Q^2}{r_0^2}\right)\right], \tag{3.14}$$

which is easily solved to give:

$$\sin\Theta=\sin\theta\left[1+\frac{a^2}{2r_0^2}\left(1+\frac{2M}{r_0}-\frac{Q^2}{r_0^2}\right)\cos^2\theta\right]. \tag{3.15}$$

Putting this back in Eq. (3.13), we find $R(\theta)$. In the resulting expression, using Eq. (3.15) to substitute for θ in terms of Θ, we finally get

$$f(\Theta)=r_0\left[1+\frac{a^2}{2r_0^2}\sin^2\Theta-\frac{a^2}{2r_0^2}\left(\frac{2M}{r_0}-\frac{Q^2}{r_0^2}\right)\cos^2\Theta\right]. \tag{3.16}$$

The two-surface $\mathcal{R}=f(\Theta)$ then describes an oblate spheroid, which bulges out near the equator (i.e., near $\Theta=\pi/2$). Note that the above equations can be used to prove that indeed $\dot{R}^2\simeq O(a^4/r_0^4)$.

C. Referenced QLE for charged Kerr black hole

In the slow-rotation approximation the intrinsic metric on B, as embedded in flat space, is

$$ds^2\approx r_0^2\left[1+\frac{a^2}{r_0^2}\sin^2\Theta-\frac{a^2}{r_0^2}\left(\frac{2M}{r_0}-\frac{Q^2}{r_0^2}\right)\cos^2\Theta\right]\times(d\Theta^2+\sin^2\Theta\, d\Phi^2). \tag{3.17}$$

The extrinsic curvature k_0 of this surface embedded in flat space can be evaluated using the method detailed in the Appendix. This in turn can be used to compute the renormalization integral in the QLE (2.1):

$$\varepsilon^0=-r_0\left[1+\frac{a^2}{3r_0^2}\left(1+\frac{M}{r_0}-\frac{Q^2}{2r_0^2}\right)\right]. \tag{3.18}$$

The referenced QLE is, therefore, obtained to be

$$E=r_0\left[1-\sqrt{1-\frac{2M}{r_0}+\frac{a^2+Q^2}{r_0^2}}\right]+\frac{a^2}{6r_0}\left[2\left(1+\frac{M}{r_0}-\frac{Q^2}{2r_0^2}\right)+\left(1+\frac{2M}{r_0}-\frac{Q^2}{r_0^2}\right)\sqrt{1-\frac{2M}{r_0}+\frac{a^2+Q^2}{r_0^2}}\right]. \quad (3.19)$$

As $r_0\to\infty$, we have

$$E\to r_0\left\{1-\left[1-\left(\frac{M}{r_0}-\frac{a^2+Q^2}{2r_0^2}\right)\right]\right\}\to M+O\left(\frac{1}{r_0}\right), \quad (3.20)$$

which is indeed the Arnowitt-Deser-Misner (ADM) mass [11] of the charged Kerr spacetime.

Near the outer horizon $r_0=r_+$, the energy is

$$E(r_0=r_+)=r_+\left[1+\frac{a^2}{2r_+^2}\right]. \quad (3.21)$$

In this limit,

$$E(r_0=r_+)\simeq\left[\frac{1}{4\pi}A\right]^{1/2}=2M_{\rm irr}\left[1+O\left(\frac{a^4}{r_+^4}\right)\right] \quad (3.22)$$

to leading order in a/r_+. Above, we have defined

$$M_{\rm irr}^2=\frac{1}{2}\left[Mr_+-\frac{Q^2}{2}\right] \quad (3.23)$$

for the charged Kerr black hole, which is the generalization of the irreducible mass of a neutral Kerr black hole [12].

IV. THE KERR-SEN SOLUTION

Consider the following string-inspired low-energy effective action in four dimensions:

$$S=-\frac{1}{16\pi}\int d^4x\sqrt{-\det G}e^{-\Phi}\left(-R+\frac{1}{12}H_{\mu\nu r}H^{\mu\nu r}-G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi+\frac{1}{8}F_{\mu\nu}F^{\mu\nu}\right). \quad (4.1)$$

Here $G_{\mu\nu}$ is the metric, R is the scalar curvature, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the field strength corresponding to the Maxwell field A_μ, Φ is the dilaton field, and

$$H_{\mu\nu r}=\partial_\mu B_{\nu r}+\text{cyclic permutations}-(\Omega_3(A))_{\mu\nu r}, \quad (4.2)$$

where $B_{\mu\nu}$ is the antisymmetric tensor gauge field. Above we have defined

$$(\Omega_3(A))_{\mu\nu r}=\frac{1}{4}(A_\mu F_{\nu r}+\text{cyclic permutations}), \quad (4.3)$$

which is the gauge Chern-Simons term. It must be noted that the above theory is one where 6 of the 10 dimensions have been compactified to a six-torus. The massless fields arising from compactification have not been included in the effective action. Only a $U(1)$ component of the full set of non-Abelian gauge fields present in the theory has been included above. Consequently, the corresponding solutions carry a $U(1)$ charge only. The metric $G_{\mu\nu}$ used here is the metric that arises naturally in the σ-model, and is related to the Einstein metric $g_{\mu\nu}$ through the relation:

$$g_{\mu\nu}=e^{-\Phi}G_{\mu\nu}. \quad (4.4)$$

Finally, the action was truncated to contain only those terms that contain two or less number of derivatives.

Sen showed that the above theory has rotating charged black hole solutions given by the following field configuration [5]:

$$ds^2=-\frac{(r^2+a^2\cos^2\theta-2mr)(r^2+a^2\cos^2\theta)}{\left(r^2+a^2\cos^2\theta+2mr\sinh^2\frac{\alpha}{2}\right)^2}dt^2-\frac{4mra\cosh^2\frac{\alpha}{2}(r^2+a^2\cos^2\theta)\sin^2\theta}{\left(r^2+a^2\cos^2\theta+2mr\sinh^2\frac{\alpha}{2}\right)^2}dt\,d\phi+\frac{r^2+a^2\cos^2\theta}{r^2+a^2-2mr}dr^2+(r^2+a^2\cos^2\theta)d\theta^2+\left\{(r^2+a^2)(r^2+a^2\cos^2\theta)+2mra^2\sin^2\theta+4mr(r^2+a^2)\sinh^2\frac{\alpha}{2}+4m^2r^2\sinh^4\frac{\alpha}{2}\right\}\frac{(r^2+a^2\cos^2\theta)\sin^2\theta}{\left(r^2+a^2\cos^2\theta+2mr\sinh^2\frac{\alpha}{2}\right)^2}d\phi^2, \quad (4.5a)$$

$$\Phi=-\ln\frac{r^2+a^2\cos^2\theta+2mr\sinh^2\frac{\alpha}{2}}{r^2+a^2\cos^2\theta}, \quad (4.5b)$$

$$A_\phi = -\frac{2mra \sinh\alpha \sin^2\theta}{r^2 + a^2\cos^2\theta + 2mr\sinh^2\frac{\alpha}{2}}, \tag{4.5c}$$

$$A_t = \frac{2mr\sinh\alpha}{r^2 + a^2\cos^2\theta + 2mr\sinh^2\frac{\alpha}{2}}, \tag{4.5d}$$

$$B_{t\phi} = \frac{2mra\sinh^2\frac{\alpha}{2}\sin^2\theta}{r^2 + a^2\cos^2\theta + 2mr\sinh^2\frac{\alpha}{2}}. \tag{4.5e}$$

The other components of A_μ and $B_{\mu\nu}$ vanish. The Einstein metric $ds_E^2 \equiv e^{-\Phi}ds^2$ is given by

$$ds_E^2 = -\left(1 - \frac{2mr\cosh^2\frac{\alpha}{2}}{\Upsilon}\right)dt^2 + \frac{\Upsilon}{\Gamma}dr^2 + \Upsilon d\theta^2 - \frac{4mra\cosh^2\frac{\alpha}{2}\sin^2\theta}{\Upsilon}dtd\phi$$
$$+\left[\frac{\left(r^2 + a^2 + 2mr\sinh^2\frac{\alpha}{2}\right)^2 - \Gamma a^2\sin^2\theta}{\Upsilon}\right]\sin^2\theta d\phi^2, \tag{4.6}$$

where

$$\Upsilon = r^2 + a^2\cos^2\theta + 2mr\sinh^2\frac{\alpha}{2},$$

$$\Gamma = r^2 + a^2 - 2mr.$$

This metric describes a black hole solution with mass M, charge Q, angular momentum J, and magnetic dipole moment μ given by

$$M = \frac{m}{2}(1 + \cosh\alpha), \qquad Q = \frac{m}{\sqrt{2}}\sinh\alpha,$$
$$J = \frac{ma}{2}(1 + \cosh\alpha), \qquad \mu = \frac{1}{\sqrt{2}}ma\sinh\alpha. \tag{4.7}$$

This is often called the Kerr-Sen black hole solution. The location of the horizon is given by the coordinate singularities, which occur on the surfaces

$$r^2 - 2mr + a^2 = 0. \tag{4.8}$$

This has the following two roots:

$$r = m \pm \sqrt{m^2 - a^2} = M - \frac{Q^2}{2M} \pm \sqrt{\left(M - \frac{Q^2}{2M}\right)^2 - \frac{J^2}{M^2}} \equiv r_\pm, \tag{4.9}$$

which correspond to the outer and inner horizons, respectively. The area of the outer event horizon with the metric given in Eq. (4.6) is found to be

$$A = 8\pi M\left[M - \frac{Q^2}{2M} + \sqrt{\left(M - \frac{Q^2}{2M}\right)^2 - \frac{J^2}{M^2}}\right]. \tag{4.10}$$

Equation (4.9) shows that the horizon disappears unless

$$|J| \leqslant M^2 - \frac{Q^2}{2}. \tag{4.11}$$

Thus the extremal limit of the black hole corresponds to $|J| \to M - Q^2/2M$. In this limit, $A \to 8\pi|J|$. (We note that rotating black hole solutions in a related theory of dilaton gravity were also found in Ref. [8].)

A. Unreferenced QLE for Kerr-Sen black hole

Consider a constant stationary-time hypersurface Σ embedded in the Kerr-Sen spacetime with the metric (4.5a). Define the two-surface B to be the surface with some constant value for the Boyer-Lindquist radial coordinate $r = r_0$, embedded in Σ. The line element on this surface in the slow-rotation approximation, i.e., $a \ll r_0$, is:

$$ds^2 = r_0^2\kappa\left[1 + \frac{a^2\cos^2\theta}{r_0^2\kappa}\right]d\theta^2 + z^2 d\phi^2, \tag{4.12}$$

where

$$z=r_0\sqrt{\kappa}\sin\theta\left\{1+\frac{a^2}{2r_0^2\kappa}+\frac{ma^2}{r_0^3\kappa^2}\cosh^2\frac{\alpha}{2}\sin^2\theta\right\}, \tag{4.13}$$

and $\kappa=1+(2m/r_0)\sinh^2\alpha/2$. Terms of order $O(a^4/r_0^4)$ and higher have been neglected.

We now calculate the unreferenced QLE within the two-surface B, defined by $r=r_0$. We assume that $r_0\geqslant r_+$, where r_+ represents the outer horizon of the Kerr-Sen black hole. Using Eq. (A6) of the Appendix, the integral ε for the surface $r=r_0$ can be written explicitly as

$$\varepsilon=-\frac{r_0\kappa'}{2\kappa}\sqrt{1-\frac{2m}{r_0}+\frac{a^2}{r_0^2}}\int_0^\pi d\theta\sin\theta\left[1-\frac{a^2}{2r_0^2\kappa}\left(\cos^2\theta+\frac{m}{r_0\kappa}\cosh^2\frac{\alpha}{2}\right)+\frac{m^2a^2}{2r_0^4\kappa^2\kappa'}\sinh^2\frac{\alpha}{2}\cosh^2\frac{\alpha}{2}\sin^2\theta+O\left(\frac{a^4}{r_0^4}\right)\right], \tag{4.14}$$

where $\kappa'=1+(m/r_0)\sinh^2\alpha/2$. On performing the above integration we get

$$\varepsilon=-\frac{r_0\kappa'}{\sqrt{\kappa}}\sqrt{1-\frac{2m}{r_0}+\frac{a^2}{r_0^2}}\left[1-\frac{a^2}{6r_0^2\kappa}\left(1+\frac{2m}{r_0\kappa}\cosh^2\frac{\alpha}{2}-\frac{2m^2}{r_0^2\kappa\kappa'}\sinh^2\frac{\alpha}{2}\cosh^2\frac{\alpha}{2}\right)+O\left(\frac{a^4}{r_0^4}\right)\right], \tag{4.15}$$

which is the unreferenced QLE for Kerr-Sen black hole.

In the limit $\alpha\to 0$, the expression (4.15) gives

$$\varepsilon=-r_0\sqrt{1-\frac{2m}{r_0}+\frac{a^2}{r_0^2}}\left\{1-\frac{a^2}{6r_0^2}\left[1+\frac{2m}{r_0}+O\left(\frac{a^4}{r_0^4}\right)\right]\right\}. \tag{4.16}$$

This is the unreferenced QLE of the neutral Kerr black hole [10]. It also has the expected limit when $a\to 0$, in which case

$$\varepsilon=-\frac{r_0\kappa'}{\sqrt{\kappa}}\sqrt{1-\frac{2m}{r_0}+\frac{a^2}{r_0^2}}, \tag{4.17}$$

which is the unreferenced QLE of the static, charged dilatonic black holes [4].

As $r_0\to\infty$, we have $\varepsilon\to M-r_0$, which is again divergent. Below, we compute the reference term required to renormalize this QLE.

B. Two-surface isometric to B embedded in flat space

To obtain the reference term in the QLE expression, Eq. (2.1), we first find a 2D surface isometric to Eq. (4.12), which is embeddable in a flat 3D slice with the line element

$$ds^2=d\mathcal{R}^2+\mathcal{R}^2d\Theta^2+\mathcal{R}^2\sin^2\Theta\,d\Phi^2. \tag{4.18}$$

The equation for the desired 2D surface in the flat slice is denoted by $\mathcal{R}=g(\Theta)$, where g is a function of the azimuthal angle Θ and the parameters (M,a,r_0) of the surface in Σ. Its intrinsic metric is obtained from Eq. (4.18). We assume that $\Theta=\Theta(\theta)$ and $\Phi=\phi$. Then, on the two-surface $\mathcal{R}$ is a function of θ, i.e., $\mathcal{R}=R(\theta)$, say. Hence, the line-element on it is given by Eq. (3.11).

Similar to the case of the charged Kerr black hole discussed in Sec. III B, even for the Kerr-Sen spacetime, requiring the line-element (3.11) to be isometric to Eq. (4.12) yields the following first-order ordinary differential equation:

$$\frac{\dot{\Theta}}{\sin\Theta}=\frac{r_0\sqrt{\kappa}}{z}\left[1+\frac{a^2\cos^2\theta}{2r_0^2\kappa}\right]. \tag{4.19}$$

This equation is easily solved to give:

$$\sin\Theta=\sin\theta\left[1+\frac{a^2}{2\kappa r_0^2}\left(1+\frac{2m}{\kappa r_0}\cosh^2\frac{\alpha}{2}\right)\cos^2\theta\right]. \tag{4.20}$$

As in the case of the charged Kerr black hole, the above expression can be used to find

$$g(\Theta)=\sqrt{\kappa}r_0\left[1+\frac{a^2}{2\kappa r_0^2}\sin^2\Theta-\frac{ma^2}{k^2r_0^3}\cosh^2\frac{\alpha}{2}\cos^2\Theta\right]. \tag{4.21}$$

The two-surface $\mathcal{R}=g(\Theta)$ once again describes an oblate spheroid.

C. The referenced QLE

In the slow-rotation approximation, Eq. (4.21) implies that the intrinsic metric on B, as embedded in flat space, is

$$ds^2\approx\sqrt{\kappa}r_0\left[1+\frac{a^2}{2\kappa r_0^2}\sin^2\Theta-\frac{ma^2}{\kappa^2r_0^3}\cosh^2\frac{\alpha}{2}\cos^2\Theta\right]\times(d\Theta^2+\sin^2\Theta\,d\Phi^2). \tag{4.22}$$

The extrinsic curvature k_0 of this surface embedded in flat space can be evaluated using the method detailed in the Appendix. This in turn can be used to compute the renormalization integral in the QLE (2.1):

$$\varepsilon^0=\frac{1}{\kappa}\int_{^2\mathrm{B}}k^0\sqrt{\sigma}d\Theta d\Phi=-r_0\sqrt{\kappa}\left[1+\frac{a^2}{3\kappa r_0^2}\left(1+\frac{m\cosh^2\frac{\alpha}{2}}{\kappa r_0}\right)\right]. \tag{4.23}$$

The referenced QLE is, therefore, obtained to be

$$E=\frac{r_0}{\kappa}\left[\kappa-\kappa'\sqrt{1-\frac{2m}{r_0}+\frac{a^2}{r_0^2}}\right]+\frac{a^2}{6r_0\kappa^{3/2}}\left[2\kappa\left(1+\frac{m\cosh^2\frac{\alpha}{2}}{\kappa r_0}\right)+\kappa'\sqrt{1-\frac{2m}{r_0}+\frac{a^2}{r_0^2}}\left(1+\frac{2m}{r_0\kappa}\cosh^2\frac{\alpha}{2}\right.\right.$$
$$\left.\left.-\frac{2m^2}{r_0^2\kappa\kappa'}\sinh^2\frac{\alpha}{2}\cosh^2\frac{\alpha}{2}\right)+O\left(\frac{a^4}{r_0^4}\right)\right]. \tag{4.24}$$

As $r_0\to\infty$, we have

$$E\to\frac{r_0}{\kappa}\left[\kappa-\kappa'\left(1-\frac{m}{r_0}\right)\right]\to m\cosh^2\frac{\alpha}{2}, \tag{4.25}$$

which is indeed the ADM mass (4.7) of the Kerr-Sen solution.

The QLE (4.24) has the correct limit for vanishing charge, namely, for $\alpha=0$. In that case, one obtains the expression in Eq. (3.19), which is the QLE for the Kerr black hole. Similarly, for vanishing rotation, i.e., for $a=0$, Eq. (4.24) reduces to

$$E=\frac{r_0}{\kappa}\left[\kappa-\kappa'\sqrt{1-\frac{2m}{r_0}}\right], \tag{4.26}$$

which is the QLE for charged black holes in string theory [3].

Near the outer horizon $r_0=r_+$, the energy is

$$E(r_0=r_+)=r_+\left[1+\frac{a^2}{3\kappa r_+^2}\left(1+\frac{\cosh^2\frac{\alpha}{2}}{2\kappa}\right)+O\left(\frac{a^4}{r_+^4}\right)\right], \tag{4.27}$$

$$=2m\left\{1-\frac{a^2}{4m^2}\left[1-\frac{1}{3\kappa}\left(1+\frac{\cosh^2\frac{\alpha}{2}}{2\kappa}\right)\right]+O\left(\frac{a^4}{r_+^4}\right)\right\}. \tag{4.28}$$

In the limit of vanishing charge, which is given by $\alpha=0$, the above quantity goes over to

$$E(r_0=r_+)=r_+\left[1+\frac{a^2}{2r_+^2}+O\left(\frac{a^4}{r_0^4}\right)\right], \tag{4.29}$$

$$=2m\left[1-\frac{a^2}{8m^2}+O\left(\frac{a^4}{r_0^4}\right)\right]. \tag{4.30}$$

which are the expected values for the Kerr black hole [10]. Note that in this latter case,

$$E(r_0=r_+)\simeq\left[\frac{1}{4\pi}A\right]^{1/2} \tag{4.31}$$

to leading order in a/r_+. Whether such a relation continues to hold even after the slow-rotation approximation is dropped, is not known. For the Kerr-Sen black hole, however,

$$\left[\frac{1}{4\pi}A\right]^{1/2}=r_+\cosh\frac{\alpha}{2}\left[1+\frac{a^2}{2r_+^2}+O\left(\frac{a^4}{r_+^4}\right)\right], \tag{4.32}$$

which is not the same as $E(r_0=r_+)$ given in Eq. (4.27) for such a black hole.

V. DISCUSSION

The Brown-York quasilocal energy of a Kerr black hole, for the type of slice Σ and quasilocal surface B considered in this paper, has not been evaluated yet for the exact case (i.e., beyond the slow-rotation approximation). One of the main hurdles in this computation is the determination of the two-surface $\mathcal{R}=f(\Theta)$, isometric to B, to be embedded in a flat three-slice. It is nevertheless interesting to explore the slow-rotation regime of such black holes in general relativity and alternative theories of gravity, for the results obtained can often tell us about the behavior of certain physical quantities in the exact case. One such quantity is the value of QLE at the (outer) horizon of such black holes, which for the Kerr black hole approaches twice its irreducible mass. In this paper, we find that this result continues to hold even for the charged Kerr black hole. Thus, within general relativity, this identity appears to bear a universal quality as far as its applicability to stationary black hole solutions is concerned.

It is known that the mechanical laws of stationary black hole solutions in general relativity are extendible also to other alternative theories of gravity, especially ones connected with a diffeomorphism invariant Lagrangian. One such leading alternative to general relativity is a string-inspired four-dimensional low-energy effective theory. It is of interest to ask if other identities related to black hole mechanics in general relativity continue to hold for black holes in string theory. This motivated us to study the status of Martinez's conjecture in the context of the Kerr-Sen family of black holes, which arise as solutions in heterotic string theory. We first find the QLE of such black holes for a choice of three-slice Σ quasilocal surface B identical to the ones used to evaluate the QLE of charged Kerr black holes. This expression is found to have the correct limits when the radial coordinate is made to diverge, or when the charge is made to vanish. However, at the outer horizon, the QLE for such black holes does not reduce to twice its irreducible mass. It is important to note that the value of the QLE is influenced by the choice of a reference term. In fact, at the outer horizon it is solely this reference term that contributes to the QLE. It may, therefore, be possible to motivate an alternative reference term that is concomitant with the applicability of Martinez's conjecture even for such black holes. We hope to return to this issue elsewhere.

ACKNOWLEDGMENTS

We thank Naresh Dadhich and Mohammad Nouri-Zonoz for helpful discussions. One of us (T.Z.N.) would like to express his gratitude to IUCAA for the hospitality and financial support, which he received during his stay in IUCAA.

APPENDIX

Here we show how to compute the trace of the extrinsic curvature, k, corresponding to a closed two-surface B embedded in a 3D axisymmetric Riemannian manifold. The result can then be used in Eq. (2.1) to evaluate the energy integrals appearing in several places in this paper. An analogous calculation is given in Ref. [10]. It is discussed here for the sake of completeness only.

Consider a 3D axisymmetric spacelike hypersurface Σ described by the line element

$$h_{ij}dx^i dx^j = x^2 dr^2 + y^2 d\vartheta^2 + z^2\, d\varphi^2, \tag{A1}$$

where $i=(r,\vartheta,\varphi)$, x^i denote arbitrary coordinates adapted to the symmetry. The metric coefficients x, y, and z depend only on the ''radial'' and ''azimuthal'' coordinates r and ϑ, respectively. An arbitrary 2D axisymmetric surface B having the topology of a two-sphere, and embedded in 3D space Σ is defined by the relation $r=R(\vartheta)$, where R is a function of the azimuthal angle and the parameters of the solution. Its two-dimensional line element is

$$\sigma_{ab}dx^a dx^b = (x^2 R'^2 + y^2) d\vartheta^2 + z^2\, d\varphi^2, \tag{A2}$$

where a prime denotes differentiation with respect to the coordinate ϑ. The functions x, y, and z in Eq. (A2) are evaluated at the two-surface $r=R(\vartheta)$.

Let n^i denote the unit outward-pointing spacelike normal to B as embedded in Σ. Its components are

$$(n^r, n^\vartheta, n^\varphi) = \frac{1}{\sqrt{y^2 + x^2 R'^2}} (y/x\,, -x\,R'/y\,, 0). \tag{A3}$$

The extrinsic curvature of the two-surface B as embedded in Σ is denoted by $k_{\mu\nu}$. Its trace k can be written as

$$k = -\frac{\partial_\mu (n^\mu \sqrt{h})}{\sqrt{h}}, \tag{A4}$$

where h denotes the determinant of the three-metric $h_{\mu\nu}$. Using the coordinate-components of the unit normal (A3) we find that the trace is

$$k = -\frac{1}{xyz}[(\alpha\,\gamma^{-1/2})_{,r} - (\beta\,\gamma^{-1/2})_{,\vartheta}]\bigg|_{r=R(\vartheta)}, \tag{A5}$$

where $\alpha \equiv y^2 z$, $\beta \equiv x^2 z\, R'$, $\gamma \equiv y^2 + x^2 R'^2$, and $\delta \equiv \ln\gamma$. Its proper surface integral yields

$$\frac{1}{8\pi}\int_{{}^2\mathsf{B}} k\,\sqrt{\sigma}\,d\vartheta d\varphi = -\frac{1}{4}\int_0^\pi d\vartheta\,\frac{1}{xy}\left(\alpha_{,r} - \beta_{,\vartheta} - \frac{\alpha}{2}\delta_{,r} + \frac{\beta}{2}\delta_{,\vartheta}\right)\bigg|_{r=R(\vartheta)}. \tag{A6}$$

This integral is evaluated at the surface $r=R(\vartheta)$. Finally note that both integrals in Eq. (2.1) are of the general form (A6). It is, however, important to note that each one of these integrals involves different values for the functions x, y, and z. Also, the coordinates on the two-surface, ϑ and φ, which appear in these two integrals, may not always have identical definitions.

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