Exact inhomogeneous cosmologies whose source is a radiation-matter mixture with consistent thermodynamics

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We derive a new class of exact solutions of Einstein's equations providing a physically plausible hydrodynamical description of cosmological matter in the radiative era, between nucleosynthesis and decoupling. The solutions are characterized by the Lemaître-Tolman-Bondi metric with a viscous fluid source, subjected to the following conditions: (a) the equilibrium state variables satisfy the equation of state of a mixture of an ultrarelativistic and a nonrelativistic ideal gases, where the internal energy of the latter has been neglected, (b) the particle numbers of the mixture components are independently conserved, (c) the viscous stress is consistent with the transport equation and entropy balance law of extended irreversible thermodynamics, with the coefficient of shear viscosity provided by kinetic theory. The satisfaction of (a), (b), and (c) restricts initial conditions in terms of an initial value function $\Delta_i^{(s)}$, which in the limit of small density contrasts becomes the average of spatial gradients of the fluctuations of photon entropy per baryon in the initial hypersurface. For $\Delta_i^{(s)}\neq 0$ and choosing the phenomenological coefficients of the ''radiative gas'' model, we have an interactive photon-baryon mixture under local thermal equilibrium, with radiation dominance and temperatures characteristic of the radiative era (10^6 K$>T>10^3$ K). Constraints on the observed anisotropy of the microwave cosmic radiation and the condition that decoupling occurs at $T=T_D\approx 4\times 10^3$ K yield an estimated value $|\Delta_i^{(s)}|\approx 10^{-8}$ which can be associated with a bound on promordial entropy fluctuations. The Jeans mass at decoupling is of the same order of magnitude as that of baryon dominated perturbation models ($\approx 10^{16}M_\odot$). [S0556-2821(99)07818-2]

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I. INTRODUCTION

The radiative era of cosmic evolution comprises the period from the end of primeval nucleosynthesis to the decoupling of matter and radiation (see Refs. [1–11]). A gross description of cosmological matter sources in this period is given by an interactive mixture of ideal relativistic and nonrelativistic gases (''radiation'' and ''matter'') in local thermal equilibrium (LTE).

The standard approach to this type of matter source is either a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime with equilibrium kinetic theory distributions [5–7], gauge invariant perturbations on a FLRW background [4–6,8–10], or various types of hydrodynamical models, [12–14] which, in general, fail to incorporate a physically plausible description of the interaction between matter and radiation. Even if we argue that the universe is ''almost FLRW'' or ''almost in thermal equilibrium,'' the small deviations from equilibrium are extremely important, [1,2,4–6,8,9], to account for most interesting phenomena of cosmic evolution: nucleosynthesis, structure formation, abundance of relic gases, etc. Models with perfect fluid sources, whether hydrodynamic [12,13], or based on kinetic theory [15], necessarily assume a quasistatic adiabatic and reversible evolution and thus, fail to incorporate into the resulting picture even small deviations from equilibrium.

Dissipative sources have been incorporated numerically within a purely FLRW geometry [16] or following a perturbative approach [17]. However, the literature still lacks an alternative hydrodynamical treatment, based on inhomogeneous exact solutions of Einstein's equations with dissipative sources and fully complying with the thermodynamics of a radiative gas within a transient regime. Ideally, such exact models should include all dissipative agents (heat flux, bulk, and shear viscosity) and should be consistent with the theoretical framework of extended irreversible thermodynamics (see Refs. [17–29]), thus satisfying suitable transport equations complying with causality, with phenomenological coefficients given by kinetic theory for this type of source. Since this general treatment would be mathematically untractable, we aim at the best possible approach based on exact solutions of Einstein's equations. Therefore, we have made the following simplifying assumptions: (a) the matter source is a fluid with shear viscosity but without heat conduction nor bulk viscosity, (b) the equilibrium state variables satisfy the equation of state of a mixture of relativistic and nonrelativistic ideal gases, where the internal energy and pressure of the latter have been neglected, (c) the particle numbers of each mixture components is independently conserved, (d) we exclude dark matter and/or exotic particles and assume instead a tight coupling between photons (radiation) and baryons and electrons (matter), hence there is a common temperature for the mixture (LTE), while the microscopical interaction models are the various processes of radiative transfer [2,6,8,23,25–27]: Thomson scattering, brehmstrallung, free-free absortion, etc. Although this type of interactions involve mostly photons and electrons, the dynamics of the matter component is governed by the baryons since the latter provide most of the rest mass content of nonrelativistic matter (without dark matter).

Restrictions (b) and (c) are easy to justify: since the ratio of photons to baryons is such a large number ($\approx 10^9$), we can truly ignore the pressure of nonrelativistic matter. Also, after nucleosynthesis, in the temperature range $T<10^6$, matter creation and anhilitation processes balance each other and effectively cease to be dynamically important [5,6]. On the other hand, the lack of heat conduction [restriction (a)] is more difficult to justify. It can be associated with an adiabatic (zero heat flux) but still irreversible evolution (nonzero viscosity), and can be a reasonable approximation on specific conditions. For example, for a radiative gas at higher temperatures shear viscosity dominates over heat conduction, but the latter becomes significant as the mixture cools [2]. The lack of bulk viscosity is a better approximation: it is negligible for a radiative gas in the temperature range 10^3 K$<T<10^6$ K that we are interested in [2,17,21,23,25], becoming important for higher temperatures (the mid relativistic regime where $k_B T\approx mc^2$ [17,28,29]). However, we accept that ignoring these dissipative fluxes weakens the scope and validity of the models, but we argue that this is compensated by the simplification of the field equations, leading to exact forms for the equilibrium state variables and shear viscosity that still satisfy (under the restrictions mentioned) thermodynamically consistent relations.

The models we present are based on the spherically symmetric Lemaître-Tolman-Bondi metrics, usually associated with dust sources [30,31]. However, this metric is compatible with a comoving fluid source with zero heat flux but with anisotropic stresses, which we describe as shear viscosity. Obviously, the lack of heat flux and four-acceleration necessarily implies a very special shear viscous tensor whose divergence exactly balances the nonzero spatial gradient of the equilibrium pressure. Considering this metric and this source, we impose on the equilibrium state variables the equation of state for a mixture of ideal gases [under the restriction (b)]. The field equations can be solved up to a quadrature, without having to make any assumption on the form of the shear viscous pressure. The latter, as well as all equilibrium state variables can be determined from the solution of the quadrature, up to two initial value functions that can be identified with the initial energy densities of the matter and radiation components. We consider only the case that would be equivalent to spacelike sections of zero curvature. A generalization of this class of exact solutions to the more general Szekeres-Szafron metrics admitting no isometries has been published recently [32], while the study of a nonrelativistic ideal gas is considered in Ref. [33].

Once the field equations have been integrated, we define a set of initial value functions that gauge the deviation from homogeneity of the average of initial density contrasts. The terms involving various gradients of metric functions can be given in terms of these gauges, so that in the limit when the latter vanish a FLRW spacetime can always be obtained as the homogeneous (and reversible) subcase. In Sec. VII we derive the conditions that the models must satisfy in order to be consistent with the theoretical framework of irreversible extended thermodynamics, in the case where shear viscosity is the only dissipative agent and the coefficient of shear viscosity is that given by kinetic theory for the radiative gas [2,23,25–27,29]. This leads to an entropy balance law and a suitable transport equation for shear viscosity that is satisfied for a specific functional form of the relaxation time. Conditions are given so that the latter quantity behaves as a relaxation parameter for an interactive cosmological mixture of matter and radiation. These conditions of thermodynamical consistency are then explicitly tested on the models, leading to a set of restrictions on the initial conditions (the latter given in terms of the gauges of initial density constrasts). The most relevant result is that thermodynamical consistency constrains an initial value adimensional function $\Delta_i^{(s)}$, which in the limit of small density contrasts is approximately the average gradient of the photon entropy per baryon along the initial hypersurface $t=t_i$. An analogy is provided with the theory of perturbations on a FLRW background, whereby $\Delta_i^{(s)}=0$ is formally analogous to the definition of initialy adiabatic perturbations in the sincronous gauge [4,6,9,10]. The constraints on the observed anisotropy of the microwave cosmic background, as well as the condition that decopling occurs at $T=T_D\approx 4\times 10^3$ K, leads to the estimated value $|\Delta_i^{(s)}|\approx 10^{-8}$. Since initial conditions of the radiative era should be traced to previous periods of cosmic evolution, this constraint can be related to maximal bounds on entropy fluctuations in primordial perturbations. Finally, we compute the Jeans mass associated with the thermodynamically consistent models, leading to a value similar to that obtained for baryon dominated perturbation models: $M_J\approx 10^{16} M_\odot$.

II. INTERACTING MIXTURE OF RADIATION AND NON-RELATIVISTIC MATTER

A radiation-matter mixture can be described by a mixture of two ideal gases: one an ultrarelativistic gas of massless particles, the other a nonrelativistic ideal monatomic gas with m being the mass of the particles. This is characterized by the total matter energy ρ and pressure p:

$$\rho = mc^2 n^{(m)} + \frac{3}{2} n^{(m)} k_B T^{(m)} + 3 n^{(r)} k_B T^{(r)}, \tag{1a}$$

$$p = n^{(m)} k_B T^{(m)} + n^{(r)} k_B T^{(r)}, \tag{1b}$$

where k_B is Boltzmann's constant and n,T are particle number densities and temperatures of the two components, distiguished by the superindices (m) (matter) and (r) (radiation). If there is local thermal equilibrium (LTE) between the components, the latter interact and evolve with the same temperature: $T^{(r)}=T^{(m)}=T$. If the components are decoupled, each gas evolves with a different temperature.

Assuming LTE, if $n^{(m)}\ll n^{(r)}$, but the ratio $mc^2/k_B T$ is not negligible, then Eq. (1) can be approximated by

$$\rho \approx mc^2 n^{(m)} + 3n^{(r)} k_B T, \tag{2a}$$

$$p \approx n^{(r)} k_B T, \tag{2b}$$

an equation of state describing a radiation dominated mixture in which the presence of nonrelativistic matter is dynamically important. If we assume nonrelativistic matter to be made up of baryons (with m being a protonic mass) and since the ratio of baryons to photons $n^{(m)}/n^{(r)} \approx 10^{-9}$ is a small number, the equation of state (2) is a reasonable approximation in the temperature range $10^3 \leqslant T \leqslant 10^6$ K, characteristic of the ''radiative era'' from the end of nucleosynthesis to the transition between radiation to matter dominance, including the recombination and decoupling eras. At such temperatures, it is also safe to assume [1,3–6] that electrons and photons interact mostly through Thomson scattering but creation and annihilation processes (bremsstrahlung and free-free absorption) roughly compensated one another so that particle number densities of the components of the mixture satisfy independent conservation laws. Once the decoupling of the matter-radiation mixture takes place at about $T \approx 4 \times 10^3$ K, the assumption of LTE is no longer valid and interaction between components ceases. Equation of state (1) can also be approximated by a form similar to Eq. (2) with the internal energy of radiation taking approximately the Stefan-Boltzmann law $\rho^{(r)} = a_B T^4$, where a_B denotes the radiation constant. However, out of thermal equilibrium the Steffan-Boltzmann law is incompatible with the ideal gas equation of state.

Having in mind the conditions justifying Eq. (2), we will describe a matter-radiation mixture evolving along adiabatic but irreversible processes by the fluid tensor

$$T^{ab} = \rho u^a u^b + p h^{ab} + \Pi^{ab}, \tag{3}$$

$$h^{ab} = c^{-2} u^a u^b + g^{ab}, \quad u_a \Pi^{ab} = 0, \quad \Pi^a_e = 0,$$

where ρ, p satisfy Eq. (2), u^a is the four-velocity shared by radiation and matter, Π^{ab} is the shear viscous pressure tensor (a symmetric traceless tensor) which arises because of the matter-radiation interaction, and particle number densities satisfy the conservation laws

$$(n^{(m)} u^a)_{;a} = 0, \quad (n^{(r)} u^a)_{;a} = 0. \tag{4}$$

As mentioned previously, bulk viscosity is negligible within the temperature range we are interested in [23,2,25–28], while even if neglection of heat conduction can be justified for relativistic temperatures [2], it does weaken the scope of the models. However, this restriction is compensated by the obtention of exact solutions that are still thermodynamically consistent.

III. THE LEMAÎTRE-TOLMAN-BONDI METRICS

Consider Eq. (3) as the source of the Lemaître-Tolman-Bondi (LTB) metric ansatz, usually associated with spherically symmetric Lemaître-Tolman-Bondi dust solutions [30,31]

$$ds^2 = -c^2 dt^2 + \frac{Y'^2}{1-F} dr^2 + Y^2 [d\theta^2 + \sin^2(\theta) d\phi^2], \tag{5}$$

where $Y = Y(t,r)$, $F = F(r)$, and a prime denotes partial derivative with respect r. Just as in the LTB dust solutions, we assume the coordinates in Eq. (5) to be comoving and the four-velocity of the fluid source to be $u^a = c\,\delta^a_t$, a geodesic vector field, since $\dot{u}_a \equiv u_{a;b} u^b = 0$. Other kinematic invariants associated with Eq. (5) are the scalar expansion $\Theta \equiv u^a_{;a}$, and the shear tensor $\sigma_{ab} \equiv u_{(a;b)} - (\Theta/3) h_{ab}$, given in the coordinates of Eq. (5) by

$$\Theta = \frac{\dot{Y}'}{Y'} + \frac{2\dot{Y}}{Y}, \tag{6a}$$

$$\sigma^a_b = \mathrm{diag}[0, -2\sigma, \sigma, \sigma], \quad \sigma \equiv \frac{1}{3}\left(\frac{\dot{Y}}{Y} - \frac{\dot{Y}'}{Y'}\right), \tag{6b}$$

while the most general form of Π^a_b for the metric (5) is given by

$$\Pi^a_b = \mathrm{diag}[0, -2P, P, P], \tag{6c}$$

where $\dot{Y} \equiv u^a Y_{,a} = Y_{,t}$ and $P = P(t,r)$ is an arbitrary function. Notice that a comoving and nonaccelerating four-velocity does not imply $p' = 0$, as in the perfect fluid case ($G^\theta_\theta - G^r_r = 0$). As revealed by the momentum balance law: $h_{ca} T^{ab}{}_{;b} = 0$, applied to the viscous fluid source (3), we have

$$h^b_a (p_{,b} + \Pi_{bc;d} h^{cd}) = 0 \;\Rightarrow\; (p - 2P)' + 6P \frac{Y'}{Y} = 0 \tag{7a}$$

showing how the divergence of the shear viscous tensor exactly balances the nonzero pressure gradient. The energy balance $u_a T^{ab}{}_{;b} = 0$, is given by

$$\dot{\rho} + (\rho + p)\Theta + \sigma_{ab} \Pi^{ab} = 0 \;\Rightarrow\; \dot{\rho} + \frac{4}{3}\Theta p + 6\sigma P = 0 \tag{7b}$$

illustrating how the term $\sigma_{ab} \Pi^{ab} = 6\sigma P$ can be understood as an interaction term responsible for local energy exchange between matter and radiation.

Integration of the conservation laws (4) for (5) yields

$$n^{(m)} = n_i^{(m)} \left(\frac{Y_i}{Y}\right)^3 \frac{Y_i'/Y_i}{Y'/Y}, \quad n^{(r)} = n_i^{(r)} \left(\frac{Y_i}{Y}\right)^3 \frac{Y_i'/Y_i}{Y'/Y}, \tag{8}$$

where $n_i^{(m)}, n_i^{(r)}$ depend only on r and are the particle number densities of nonrelativistic matter and radiation, evaluated along a suitable initial hypersurface labeled by $t = t_i$. The subindex i affixed to any quantity, as Y_i, will denote henceforth initial value functions (functions of t,r evaluated along $t = t_i$). It is important to state that our initial conditions do not refere to present cosmic time (usually labeled as $t = t_0$), and so we will not use the subindex 0.

The spherically symmetric LTB metrics (5) are contained within a larger class of more general metrics (the Szekeres-Szafron metrics [30,31]), admitting in general no isometries. The integration of the field equations for Eq. (5), given a source (3) satisfying Eq. (2), is examined in the next section. For the case of more general Szekeres-Szafron metrics, see Ref. [32].

IV. INTEGRATION OF THE FIELD EQUATIONS

Einstein's field equations for Eqs. (5) and (3) are

$$\kappa\rho = -\frac{[Y(\dot{Y}^2+Fc^2)]'}{Y^2Y'} = -G^t{}_t, \tag{9a}$$

$$\kappa p = -\frac{[Y(\dot{Y}^2+Fc^2)+2Y^2\ddot{Y}]'}{3Y^2Y'} = \frac{1}{3}(2G^\theta{}_\theta + G^r{}_r), \tag{9b}$$

$$\kappa P = \frac{Y}{6Y'}\left[\frac{Y(\dot{Y}^2+Fc^2)+2Y^2\ddot{Y}}{Y^3}\right]' = \frac{1}{3}(G^\theta{}_\theta - G^r{}_r), \tag{9c}$$

where $\kappa \equiv 8\pi G/c^2$. Imposing on Eqs. (9a) and (9b) the equation of state (2), using Eq. (8) and integrating with respect to r yields the following constraint:

$$2Y(\dot{Y}^2+Fc^2)+Y^2\ddot{Y} - \kappa mc^2\int n_i^{(m)}Y_i^2Y_i'\,dr = \lambda(t), \tag{10}$$

where $\lambda(t)$ is an arbitrary integration function. It is important to remark that Eq. (10) follows only from Eqs. (9a) and (9b) without involving Eq. (9c), i.e., it was not necessary to make any assumption regarding the form of P in order to obtain Eq. (10). A second integration of the field equations necessarily requires setting $\lambda(t)=0$ in Eq. (10), leading to

$$\dot{Y}^2 = \frac{\kappa}{Y}\left[M + W\left(\frac{Y_i}{Y}\right)\right] - Fc^2, \tag{11}$$

where

$$M = \int \rho_i^{(m)}Y_i^2Y_i'\,dr, \quad \rho_i^{(m)} \equiv mc^2 n_i^{(m)}, \tag{12a}$$

$$W = \int \rho_i^{(r)}Y_i^2Y_i'\,dr, \quad \rho_i^{(r)} \equiv 3n_i^{(r)}k_BT_i, \tag{12b}$$

so that $\rho_i^{(m)},\rho_i^{(r)}$ respectively define the initial densities of the nonrelativistic and relativistic components of the mixture.

In the remaining of the paper we restrict ourselves to $F=0$, similar to the choice of spacelike sections of zero curvature in FLRW geometry, leaving the case $F\neq 0$ for a future analysis. An explicit integral of Eq. (11) in this case is given by

$$\frac{3}{2}\sqrt{\mu}(t-t_i) = \sqrt{y+\epsilon}(y-2\epsilon) - \sqrt{1+\epsilon}(1-2\epsilon), \tag{13a}$$

where

$$\mu \equiv \frac{\kappa M}{Y_i^3}, \quad \epsilon \equiv \frac{W}{M}, \quad y \equiv \frac{Y}{Y_i}. \tag{13b}$$

It is possible to invert Eq. (13a), thus obtaining $y=y(t,r)$ as a complicated, but closed analytic form, where the r dependence is contained in the functions μ,ϵ appearing in Eq. (13b). However it turns out to be more convenient to use Eqs. (11) and (13) to simplify the field equations and radial gradients of y in order to express all state and geometric variables in terms of y and suitable initial value functions related to those of Eqs. (12) and (13b).

V. THE STATE VARIABLES

From Eq. (8) and Eqs. (11)–(13) it is possible to obtain the state variables $n^{(m)},n^{(r)},T,\rho^{(m)},\rho^{(r)},p,P$. However, before doing so it is useful to define the averaged initial densities

$$\langle\rho_i^{(m)}\rangle \equiv \frac{\int \rho_i^{(m)}d(Y_i^3)}{Y_i^3} = \frac{3M}{Y_i^3},$$

$$\langle\rho_i^{(r)}\rangle \equiv \frac{\int \rho_i^{(r)}d(Y_0^3)}{Y_i^3} = \frac{3W}{Y_i^3}, \tag{14}$$

averaged over the volume Y_i^3. Since the solutions allow for an arbitrary rescaling of the radial coordinate, without loss of generality we can select $Y_i = rR_i$, where R_i is a characteristic constant length scale. Therefore, the volume Y_i^3, evaluated from the symmetry center $r=0$, to an arbitrary fluid layer r, can be characterized invariantly as the volume of the orbits of the rotation group SO(3) in the hypersurface $t=t_i$. In the Newtonian limit, the distance Y_i becomes the radius of the circular keplerian orbit in the field of Eq. (5).

Together with the averaged initial densities, we shall define the quantities $\Delta_i^{(m)},\Delta_i^{(r)}$ given by

$$\Delta_i^{(m)} = \frac{\int [\rho_i^{(m)}]'Y_0^3dr}{3M} = \frac{\rho_i^{(m)}}{\langle\rho_i^{(m)}\rangle} - 1$$

$$\Rightarrow \quad \rho_i^{(m)} = \langle\rho_i^{(m)}\rangle[1+\Delta_i^{(m)}], \tag{15a}$$

$$\Delta_i^{(r)} = \frac{\int [\rho_i^{(r)}]'Y_i^3dr}{3W} = \frac{\rho_i^{(r)}}{\langle\rho_i^{(r)}\rangle} - 1$$

$$\Rightarrow \quad \rho_i^{(r)} = \langle\rho_i^{(r)}\rangle[1+\Delta_i^{(r)}], \tag{15b}$$

whose interpretation as effective initial density contrasts is discussed in the following section. Using Eqs. (14) and (15) we can rewrite μ,ϵ in Eq. (13b) as

$$\mu=\frac{\kappa}{3}\langle\rho_i^{(m)}\rangle=\frac{\kappa\rho_i^{(m)}}{3(1+\Delta_i^{(m)})},\quad \epsilon=\frac{\langle\rho_i^{(r)}\rangle}{\langle\rho_i^{(m)}\rangle}=\frac{\rho_i^{(r)}}{\rho_i^{(m)}}\frac{1+\Delta_i^{(m)}}{1+\Delta_i^{(r)}}. \tag{16}$$

The state variables $n^{(m)},n^{(r)},\rho,p,P$ now follow by inserting Eq. (11) into Eqs. (8) and (9), while T is obtained with the help of Eq. (2). This yields the following forms:

$$n^{(m)}=\frac{n_i^{(m)}}{y^3\Gamma},\quad n^{(r)}=\frac{n_i^{(r)}}{y^3\Gamma}, \tag{17a}$$

$$T=\frac{T_i}{y}\Psi, \tag{17b}$$

$$\rho=\rho^{(m)}+\rho^{(r)}=\left[\frac{\rho_i^{(m)}}{y^3}+\frac{\rho_i^{(r)}}{y^4}\Psi\right]\frac{1}{\Gamma}, \tag{17c}$$

$$p=\frac{\rho_i^{(r)}}{3y^4}\frac{\Psi}{\Gamma}, \tag{17d}$$

$$P=\frac{\rho_i^{(r)}}{6y^4}\frac{\Phi}{\Gamma}, \tag{17e}$$

where the functions Γ, Ψ, and Φ are given by

$$\Gamma\equiv\frac{Y'/Y}{Y_i'/Y_i}\quad \Psi\equiv1+\frac{(1-\Gamma)}{3(1+\Delta_i^{(r)})},\quad \Phi\equiv1+\frac{(1-4\Gamma)}{3(1+\Delta_i^{(r)})}. \tag{18}$$

The solutions characterized by Eqs. (2),(4),(5)–(18) become determinate once Γ above is obtained in terms of y from Eq. (13) for given initial value functions $\rho_i^{(m)}$, $\rho_i^{(r)}$ (re-expressed in terms of the quantities $\Delta_i^{(m)},\Delta_i^{(r)},\epsilon$). This transforms Eq. (18) into

$$\Gamma=1+3A\Delta_i^{(m)}+3B\Delta_i^{(r)}, \tag{19a}$$

$$\Psi=1-\frac{A\Delta_i^{(m)}+B\Delta_i^{(r)}}{1+\Delta_i^{(r)}}, \tag{19b}$$

$$\Phi=\frac{-4A\Delta_i^{(m)}+(1-4B)\Delta_i^{(r)}}{1+\Delta_i^{(r)}}, \tag{19c}$$

where

$$A=\frac{1}{3y^2}\left[y^2-4\epsilon y-8\epsilon^2-\frac{\sqrt{y+\epsilon}}{\sqrt{1+\epsilon}}(1-4\epsilon-8\epsilon^2)\right], \tag{20a}$$

$$B=\frac{\epsilon}{y^2}\left[y+2\epsilon-(1+2\epsilon)\frac{\sqrt{y+\epsilon}}{\sqrt{1+\epsilon}}\right], \tag{20b}$$

with $\Delta_i^{(m)},\Delta_i^{(r)}$ given by Eq. (15). The kinematic parameters σ,Θ follow by inserting Eq. (11) with $y=Y/Y_i$ and Eq. (19a) into Eqs. (6a) and (6b):

$$\sigma=-\frac{\sqrt{\mu}\sqrt{y+\epsilon}[A_{,y}\Delta_i^{(m)}+B_{,y}\Delta_i^{(r)}]}{y^2[1+3(A\Delta_i^{(m)}+B\Delta_i^{(r)})]}, \tag{21}$$

$$\frac{\Theta}{3}=\frac{\sqrt{\mu}\sqrt{y+\epsilon}[1+(3A+yA_{,y})\Delta_i^{(m)}+(3B+yB_{,y})\Delta_i^{(r)}]}{y^2[1+3(A\Delta_i^{(m)}+B\Delta_i^{(r)})]}, \tag{22}$$

where $A_{,y},B_{,y}$ are the derivatives of A,B in Eq. (20) with respect to y. Given a set of initial conditions specified by $\epsilon,\Delta_i^{(m)},\Delta_i^{(r)}$, Eqs. (17) and (19)–(22) provide fully determined forms of the state and geometric variables as functions of y and the chosen initial conditions.

The solutions presented so far contain a FLRW particular case, obtained by setting in Eqs. (11) and (12) $n_i^{(m)}=\overline{n_i^{(m)}}$, $n_i^{(r)}=\overline{n_i^{(r)}}$, and $T_i=\overline{T_i}$, where $\overline{n_i^{(m)}},\overline{n_i^{(r)}},\overline{T_i}$ are arbitrary positive constants. Under this parameter specialization, Eq. (13) holds with $Y=R(t)f(r)$ [so that $y=R(t)$] and Eq. (5) becomes a FLRW metric. This leads to

$$\langle\rho_i^{(m)}\rangle=\overline{\rho_i^{(m)}},\quad \langle\rho_i^{(r)}\rangle=\overline{\rho_i^{(r)}},\quad \Gamma=\Psi=1,\quad \Phi=0,$$

where $\overline{\rho_i^{(m)}}=mc^2\overline{n_i^{(m)}}$, $\overline{\rho_i^{(r)}}=3\overline{n_i^{(r)}}k_B\overline{T_i}$, so that $T=T(t)$, $\rho=\rho(t)$, $p=p(t)$, and $P=0$, with Eq. (3) becoming a perfect fluid tensor where ρ and p satisfy (2). The FLRW limit can also be characterized by $\Delta_i^{(m)}=\Delta_i^{(r)}=0$, and so, from Eq. (11), Eqs. (21) and (22) become $\sigma=0$ and $\Theta/3=\dot{y}/y=\dot{R}/R$. Another limit is that of LTB dust solutions, obtained by setting $T_i=0$ in Eqs. (12b) and (11), so that Eq. (17) becomes $T=p=P=0$ and $\rho=mc^2n^{(m)}$.

VI. DENSITY CONTRASTS AND REGULARITY CONDITIONS

Since the radial dependence of all state and geometric variables is sensitive to $\Delta_i^{(r)}$ and $\Delta_i^{(m)}$ defined in Eq. (15), it is important to provide an interpretation for these quantities. From Eqs. (14) and (15), it is evident that $\Delta_i^{(r)}$ and $\Delta_i^{(m)}$ are effective ‘‘gauges’’ of the deviation of $\rho_i^{(m)},\rho_i^{(r)}$ from their volume averages for every closed interval in the range of the integration variable r along the initial hypersurface $t=t_i$. The signs of these quantities characterize initial density profiles, with ‘‘density lumps’’ as these densities decrease ($[\rho_i^{(m)}]'<0$, $[\rho_i^{(r)}]'<0$) or ‘‘density voids’’ as they increase ($[\rho_i^{(m)}]'>0$, $[\rho_i^{(r)}]'>0$). Also, with the help of Rolle’s theorem applied to Eq. (14) we find that $\Delta_i^{(r)}$ and $\Delta_i^{(m)}$, as adimensional functions of r, are constrained by the maximal density contrasts in terms of

$$|\Delta_i^{(m)}| \leqslant \frac{\rho_i^{(m)\max}}{\rho_i^{(m)\min}} - 1,$$

$$|\Delta_i^{(r)}| \leqslant \frac{\rho_i^{(r)\max}}{\rho_i^{(r)\min}} - 1,$$

where the superindices ''max'' and ''min'' respectively indicate the maximal and minimal values of $\rho_i^{(m)}, \rho_i^{(r)}$ in any interval $0 \leqslant r$ along the hypersurface $t = t_i$. Small initial density contrasts obviously imply

$$\rho_i^{(m)\max} \approx \rho_i^{(m)\min}, \quad \rho_i^{(m)} \approx \langle \rho_i^{(m)} \rangle \;\Rightarrow\; |\Delta_i^{(m)}| \ll 1, \tag{23a}$$

$$\rho_i^{(r)\max} \approx \rho_i^{(r)\min}, \quad \rho_i^{(r)} \approx \langle \rho_i^{(r)} \rangle \;\Rightarrow\; |\Delta_i^{(r)}| \ll 1, \tag{23b}$$

$$\mu \approx \frac{\kappa}{3}\rho_i^{(m)}, \quad \epsilon \approx \frac{\rho_i^{(r)}}{\rho_i^{(m)}} \approx \frac{3n_i^{(r)}}{n_i^{(m)}} \frac{k_B T_i}{mc^2} \approx 10^{-9}\frac{k_B T_i}{mc^2}, \tag{23c}$$

allowing us to consider a formal analogy between $\Delta_i^{(r)}$ and $\Delta_i^{(m)}$ and energy density ''exact'' initial perturbations. This is further reinforced from the definitions in Eq. (15), and by remarking that the FLRW ''background'' follows by ''turning the perturbations off,'' that is, setting $\Delta_i^{(m)} = \Delta_i^{(r)} = 0$.

An important restriction that the solutions must satisfy is the following regularity condition

$$\Gamma \equiv \frac{Y'/Y}{Y_i'/Y_i} > 0, \tag{24}$$

which prevents negative densities $n^{(r)}, n^{(m)}$, as well as the occurrence of a shell crossing singularity [35]. This singularity is characterized by unphysical behavior because Γ appears in the denominator of Eqs. (17a), (17c), (17d), and (17e), but does not appear in (17b). Therefore, if $\Gamma = 0$, the densities, pressure and viscous pressure diverge with T finite (in general), a totally unacceptable situation that can be avoided by considering only the range of evolution of the models to spacetime seccions with $t \geqslant t_i$ satisfying Eq. (24). The fulfilment of Eq. (24) depends on the functions A, B and on the magnitudes of the initial density contrasts gauged by $\Delta_i^{(r)}$ and $\Delta_i^{(m)}$. This will be examined further ahead together with the conditions for thermodynamical consistency.

VII. THERMODYNAMICAL CONSISTENCY

The models derived and presented in the previous sections must be compatible with a suitable thermodynamical formalism. For this purpose, it is advisable to leave aside the ''conventional'' theory of irreversible thermodynamics [36,37], whose transport equations are unphysical as they violate relativistic causality of the dissipative signals as well as stability of the equilibrium states (see, e.g., Refs. [18,19,24]). We shall consider instead ''extended irreversible thermodynamics'' (EIT) [20–23], a theory free of such serious drawbacks [24], and so a more adequate theoretical framework for the models. According to EIT, the entropy of a system away from thermodynamical equilibrium depends not only on the ''conserved'' (equilibrium) variables (i.e., particle number densities, energy density and so on), but also into the non-equilibrium fluxes (i.e., heat flux, and bulk and shear dissipative stresses). This theory is supported by kinetic theory of gases, information theory, and by the theory of hydrodynamical fluctuation—see Ref. [23] for a detailed description. When shear viscosity is the only dissipative agent, the corresponding generalized entropy s of radiation plus matter obeying the usual balance law with non-negative divergence, up to second order in Π^{ab} takes the form

$$s = s^{(e)} + \frac{\alpha}{nT}\Pi_{ab}\Pi^{ab}, \;\Rightarrow\; (snu^a)_{;a} \geqslant 0, \tag{25}$$

where $s^{(e)}$ is obtained from the integration of the equilibrium Gibbs equation $n = n^{(r)} + n^{(m)}$ and α is a phenomenological coefficient to be specified later. The evolution of the viscous pressure is, in turn, governed by the transport equation

$$\tau\dot{\Pi}_{cd}\, h_a^c h_b^d + \Pi_{ab}\left[1 + \frac{1}{2}T\eta\left(\frac{\tau}{T\eta}u^c\right)_{;c}\right] + 2\eta\sigma_{ab} = 0, \tag{26}$$

where η and τ are the coefficient of shear viscosity and the relaxation time of shear viscosity, respectively. The former as well as other related quantities can be obtained by a variety of means [2] including kinetic theory, statistical mechanics, or both [25–28]. The relaxation time τ, is related to and larger than the mean collision time between particles and it may, in principle, be estimated by collision integrals provided the interaction potential is known. As a physical reference to infere the form these coefficients might take, consider the ''radiative gas,'' with p, ρ satisfying Eq. (1) [or the approximation (2)]. For the radiative gas the forms of η, α in terms of the relaxation time of the dissipative process τ are

$$\eta_{(rg)} = \frac{4}{5}p^{(r)}\tau, \quad \alpha_{(rg)} = -\frac{\tau}{2\eta_{(rg)}} = -\frac{5}{8p^{(r)}}, \tag{27}$$

where $p^{(r)}$ is either $n^{(r)}k_B T$ or $a_B T^4$ and the subscript ''(rg)'' emphasizes that these quantities are specific to the radiative gas.

We verify now the compatibility of Eqs. (13)–(22) with Eqs. (25)–(27). Integrating the equilibrium Gibbs equation and substituting Eq. (27) into Eq. (25), we obtain

$$\begin{aligned} s &= \frac{4a_B T_i^3}{3n_i^{(r)}} + k_B \ln\left[\frac{n_i^{(r)}}{n^{(r)}}\left(\frac{T}{T_i}\right)^3\right] - \frac{15k}{8}\left(\frac{P}{p}\right)^2 \\ &= \frac{4a_B T_i^3}{3n_i^{(r)}} + k_B \ln\left[\frac{\Psi^3}{\Gamma}\right] - \frac{15k}{32}\left(\frac{\Phi}{\Psi}\right)^2, \end{aligned} \tag{28}$$

where the approximation: $n = n^{(m)} + n^{(r)} \approx n^{(r)}$ was used and the initial value of $s^{(e)}$ has been set to be the equilibrium entropy per photon. Equation (28) reflects the fact that we are neglecting the contribution of the entropy due to nonrelativistic particles, a justified approximation since the latter are much less abundant than the photons.

Ideally, the transport equation (26) should be satisfied for η having the form (27), associated to the radiative gas, and the relaxation time, τ, given by collision integrals obtained from kinetic theory. However, as mentioned in the Introduction, and in order to obtain exact expressions for all thermodynamical parameters, we will assume η given by Eq. (27) and deduce τ from the fulfilment of Eq. (26). This yields

$$\tau=\frac{-\Psi\Phi}{\sigma}\,\frac{\frac{9}{4}[1+\Delta_i^{(r)}]^2}{(4/5)[3+4\Delta_i^{(r)}+(13/32)\Gamma]^2+(171/256)\Gamma^2}. \tag{29}$$

While there is no need to justify η given by Eq. (27), this form of τ is acceptable as long as Eq. (29) satisfies the requirement of a relaxation parameter: it must be a positive quantity and must comply with a positive entropy production law $\dot{s}\geqslant 0$. It is desirable that τ should somehow relate or approach its definition as a collision integral and that its behavior be qualitatively analogous to a suitable mean collision time, therefore it should be an increasing (decreasing) function if the fluid is expanding (collapsing). Evaluating $\dot{s}$ from Eq. (28) and comparing with Eq. (29), we obtain the following relation between $\dot{s}$ and τ:

$$\dot{s}=\frac{15k_B}{4\tau}\left(\frac{P}{p}\right)^2=\frac{15k_B}{16\tau}\left(\frac{\Phi}{\Psi}\right)^2, \tag{30}$$

consistent with the general relation [22] $(nsu^a)_{;a}=\Pi_{ab}\Pi^{ab}/(2\eta nT)$ associated with Eqs. (25) and (26). As a consequence of Eq. (30), $\dot{s}>0$ and $\tau>0$ imply each other. Also, from the form of Eq. (29), necessary and sufficient conditions for positive $\tau,\dot{s},p,T$ can be given by

$$\Psi>0, \tag{31a}$$

$$\sigma\Phi<0, \tag{31b}$$

while the condition ensuring concavity and stability of s can be phrased for an expanding fluid configuration as the requirement that $\dot{s}$ decreases for increasing τ ($\dot{\tau}>0\Leftrightarrow\ddot{s}<0$). From Eqs. (30),(31), this follows as

$$\dot{\tau}>0,\quad \frac{\ddot{s}}{\dot{s}}=\frac{2\sigma\Gamma}{3\Psi\Phi}\,\frac{\langle\rho_i^{(r)}\rangle}{\rho_i^{(r)}}\left[1+\frac{\langle\rho_i^{(r)}\rangle}{3\rho_i^{(r)}}\right]-\frac{\dot{\tau}}{\tau}<0. \tag{31c}$$

So that, if Eqs. (24), (31a), and (31b) hold, then Eq. (31c) reduces to $\dot{\tau}>0$. Since τ is a thermodynamic relaxation parameter, it is important to compare it with another natural timescale of the models: the Hubble expansion time defined by

$$t_H=\frac{3}{\Theta}, \tag{32a}$$

where Θ follows from Eq. (22). Such comparison should provide an insight into the time scales associated with the mixture interaction and decoupling. However, strictly speaking, the criterion for interaction and decoupling in cosmological gas mixtures is not given by comparing τ and t_H but by comparing the latter with the timescales associated with the various reaction rates of the radiative processes involved, particularly the photon mean collision time t_γ obtained from Thomson scattering [5,6]. Hence, we can consider this relaxation time as approximately gauging the interactivity of the matter mixture by demanding that for a range of the evolution of the mixture, approximating its interactive range, we must have

$$\tau<t_H \tag{32b}$$

while, as the mixture evolves and the components decouple, eventually

$$\tau>t_H. \tag{32c}$$

Since τ in Eqs. (27) and (29) must behave qualitatively similar to t_γ [29], the comparison in Eqs. (32b),(32c) should yield qualitatively analogous results as a similar comparison between t_H and t_γ. The temperature associated with the passage from Eqs. (32b) and (32c) [obtained from Eq. (17b)] should approximate the decoupling temperature obtained by the condition $t_\gamma=t_H$. This point is examined in Sec. XI.

Equations (31),(32), together with the regularity condition (24), provide the necessary and sufficient conditions for a theoretically consistent thermodynamical description of the solutions within the framework of EIT and kinetic theory applied to the radiative gas. We examine the effect of these conditions in the following section.

VIII. THERMODYNAMICALLY CONSISTENT MODELS

A. Conditions (24) and (31a)

From Eqs. (19a) and (19b), the fulfilment of Eqs. (24) and (31a) is equivalent to the following condition:

$$-\frac{1}{3}<A\Delta_i^{(m)}+B\Delta_i^{(r)}<1+\Delta_i^{(r)}. \tag{33}$$

From Eq. (20), since the functions A and B diverge as $y\to 0$ there are necessarily values of y for which Eq. (33) is violated. However, from Eqs. (19a),(19b) we have $\Gamma_i=\Psi_i=1$, and the range of y we are interested is $y\geqslant 1$. As shown by Figs. 1(a) and 1(b), displaying the implicit plots of $\Gamma=0$ and $\Psi=0$ for an initial temperature $T_i\approx 10^6$ K [from Eq. (23c), $\epsilon\approx 10^3$], there are no zeros of Γ and Ψ for $y\geqslant 1$, $|\Delta_i^{(m)}|\leqslant 1$ and $|\Delta_i^{(r)}|\leqslant 1$. Therefore, Eq. (33) holds in the range of interest for a large class of initial conditions.

B. Conditions (31b), (31c), and (32)

For the examination of these conditions we will assume that Eq. (33) holds for $y\geqslant 1$, $|\Delta_i^{(m)}|\leqslant 1$, and $|\Delta_i^{(r)}|\leqslant 1$ [see Figs. 1(a) and 1(b)]. Then, with the help of Eqs. (19)–(22), condition (31b) is equivalent to

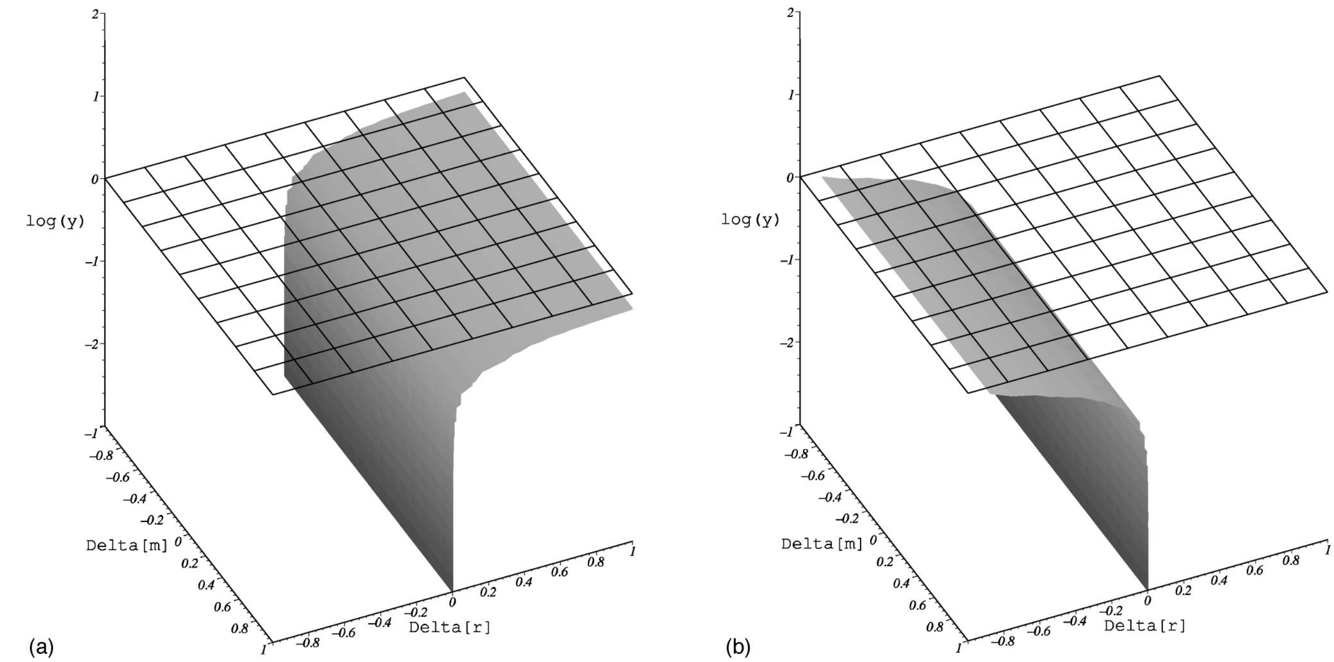

FIG. 1. The equations $\Gamma=0$ and $\Psi=0$. Implicit plots of the solution of $\Gamma=0$ (a) and $\Psi=0$ (b), in terms of $\Delta_i^{(m)}$, $\Delta_i^{(r)}$, and $\log_{10}(y)$, with Γ,Ψ given by Eq. (19). The grid marks the initial surface $y=1$, and is not intersected by the surfaces $\Gamma=0,\Psi=0$, illustrating that Γ and Ψ have no zeroes in the evolution range $y>1$ for density contrasts bound by $-1\leqslant\Delta_i^{(m)}\leqslant 1$ and $-1\leqslant\Delta_i^{(r)}\leqslant 1$. We have considered $n_i^{(r)}/n_i^{(m)}\approx 10^9$, m to be the mass of a proton and $T_i\approx 10^6$ K, hence we have made the approximation (23c): $\epsilon\approx 10^{-3}T_i\approx 10^3$. These (and the remaining) plots were obtained with the help of the symbolic computing package MAPLE V [38].

$$C\equiv 4AA_{,y}[\Delta_i^{(m)}]^2+(4B-1)B_{,y}[\Delta_i^{(r)}]^2 +[4AB_{,y}+(4B-1)A_{,y}]\Delta_i^{(m)}\Delta_i^{(r)}<0. \tag{34a}$$

An insight into this expression follows by looking at its initial value

$$C_i=\frac{-\Delta_i^{(r)}}{2(1+\epsilon)}[\Delta_i^{(r)}+\epsilon\Delta_i^{(m)}] \tag{34b}$$

and its asymptotical behavior as $y\gg 1$

$$\begin{aligned}C\approx{}& \frac{4\epsilon[\Delta_i^{(s)}]^2}{9y^2}+\frac{4\lambda\Delta_i^{(s)}}{\sqrt{1+\epsilon}y^{5/2}}+\frac{35}{9}\,\frac{\epsilon\lambda\Delta_i^{(s)}}{\sqrt{1+\epsilon}y^{7/2}}\\ &-\frac{\lambda_1[\Delta_i^{(s)}]^2+24\lambda_2\Delta_i^{(r)}\Delta_i^{(s)}+9[\Delta_i^{(r)}]^2}{24(1+\epsilon)y^4}+\frac{64}{63}\,\frac{\epsilon^2\lambda\Delta_i^{(s)}}{\sqrt{1+\epsilon}y^{9/2}}\\ &-\frac{\epsilon}{18}\,\frac{\lambda_1[\Delta_i^{(s)}]^2+24\lambda_2\Delta_i^{(r)}\Delta^{(s)}+9[\Delta_i^{(r)}]^2}{(1+\epsilon)y^4}\\ &-\frac{55}{8}\,\frac{\epsilon^3\lambda\Delta_i^{(s)}}{\sqrt{1+\epsilon}y^{11/2}}+\mathcal{O}(y^{-6}),\end{aligned} \tag{34c}$$

where

$$\Delta_i^{(s)}\equiv\frac{3}{4}\Delta_i^{(r)}-\Delta_i^{(m)}, \tag{35}$$

and

$$\lambda\equiv\Delta_i^{(m)}\lambda_2-3\epsilon(1+2\epsilon)\Delta_i^{(r)},$$

$$\lambda_1\equiv 1-8\epsilon+2\epsilon^3+2\epsilon^4,\quad \lambda_2\equiv 8\epsilon^2+4\epsilon-1.$$

As shown by Eq. (34c), the quantity $\Delta_i^{(s)}$ defined in Eq. (35) plays a fundamental role with regards to the fulfilment of Eqs. (31b) and (31c). This suggest classifying initial conditions in terms of $\Delta_i^{(s)}$. We shall examine Eqs. (31b), (31c), and (32) for each case $\Delta_i^{(s)}=0$ and $\Delta_i^{(s)}\neq 0$ separately.

1. The case $\Delta_i^{(s)}=0$

From Eq. (34c), the condition $\Delta_i^{(s)}=0$ is necessary and sufficient for having $C<0$ asymptotically. In fact, substituting $\Delta_i^{(s)}=0$ into Eq. (34a), leads to the following dramatic simplification of C:

$$C=-\frac{[\Delta_i^{(r)}]^2(3y+4\epsilon)}{8(1+\epsilon)y^5}, \tag{36}$$

so that $\Delta_i^{(s)}=0$ is a sufficient condition for the fulfilment of Eqs. (24), (31a), and (31b) along the range $y\geqslant 1$, for $|\Delta_i^{(r)}|\leqslant 1$ and $|\Delta_i^{(m)}|<1$.

Condition (31c) is also satisfied, since τ increases monotonously along the fluid world lines, behaving asymptotically as $\tau\approx y^{3/2}$. Hence, those models whose initial conditions satisfy $\Delta_i^{(s)}=0$ can be characterized as the subclass of

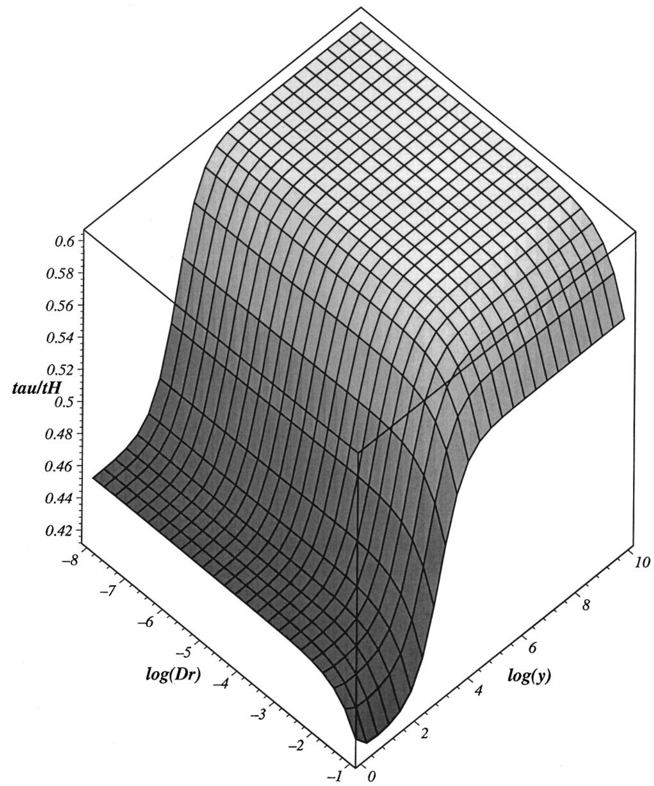

FIG. 2. The ratio of relaxation vs Hubble times for the case $\Delta_i^{(s)}=0$. This plot displays the ratio τ/t_H, as a function of $\log_{10}(y)$ and $\log_{10}(|\Delta_i^{(r)}|)$ [denoted as ''log(Dr)''], where τ is the relaxation time and $t_H=3/\Theta$ is the Hubble expansion time, given by Eqs. (22) and (29) under the condition $\Delta_i^{(s)}=0$. This ratio is less than unity for all y, and $|\Delta_i^{(r)}|<1$ thus violating Eqs. (32b) and (32c).

models complying with the conditions of thermodynamical consistency in the asymptotic range of y. This point is consistent with the fact that $\Delta_i^{(s)}=0$ is a necessary and sufficient condition for having $|P|/p=|\Phi|/(2\Psi)\to 0$ and $\dot{s}\to 0$ as $y\to\infty$, so that the fluid layers evolve towards an asymptotic equilibrium state. However, using $t_H=3/\Theta$ calculated by inserting $\Delta^{(s)}=0$ into Eq. (22), we have $\tau/t_H<1$ for all the evolution of the fluid, thus failing to comply with Eqs. (32b),(32c). This is illustrated by Fig. 2, and implies that the relaxation time τ cannot be associated with a radiation matter mixture whose components interact and then decouple.

2. The case $\Delta^{(s)}\neq 0$

If $\Delta^{(s)}\neq 0$, condition (31b) cannot hold along the full range of y [because of Eq. (34c)], but might hold along a restricted range of physical interest $1\leq y\leq y_*$ for which the mixture could be in the interactive stage. This situation is not incompatible with the thermodynamical arguments of the previous section, since the phenomenology of the radiative gas model strictly applies if the mixture components interact. The fact that condition (31b) can hold for $1\leq y\leq y_*$ follows from evaluating the sign of C_i given by Eq. (34b), a quantity that can be negative (so that $\tau_i>0$) for a wide range of acceptable situations (for example, if $\Delta_i^{(r)}$ and $\Delta_i^{(m)}$ have the same sign). However, as y increases, τ either changes sign or diverges positively, depending on the zeroes of Φ and σ. Since we are assuming that Eqs. (24) and (31a) hold, the sign of τ, as given by Eq. (29), depends only on the quotient Φ/σ, and so the behavior of τ is strongly related to the signs and zeroes of these functions. If $\tau_i>0$ and, as y increases, there is a zero of Φ for $\sigma\neq 0$, then τ passes from positive to negative, but if the zero of σ appears first, then τ diverges positively. A zero of Φ (with $\sigma\neq 0$) might be compatible with Eq. (31b) ($\tau>0$), but violates Eq. (31c) and so is unacceptable. However, a zero of σ (with $\Phi\neq 0$) is acceptable, since τ would diverge positively (as $y\to y_*$) and so would be a positive and increasing function along the range $1\leq y\leq y_*$. In order to verify if this type of evolution is possible, it is necessary to gather information on the zeros of Φ and σ. Since these expressions are cumbersome, it is convenient to examine their zeros graphically, and so we have plotted in Figs. 3(a) and 3(b) the solutions of the implicit equations $\Phi=0$ and $\sigma=0$, in terms of $\Delta_i^{(m)}$, $\Delta_i^{(r)}$ and $\log_{10}(y)$, while Fig. 4 displays the sectors in the plane $\Delta_i^{(m)}$,$\Delta_i^{(r)}$ where $\Phi=0$ and $\sigma=0$ occur. As these figures reveal, sufficient conditions for the desired evolution are given by

$$\Delta_i^{(s)}>0 \quad \text{for} \quad \Delta_i^{(r)}>0, \tag{37a}$$

$$\Delta_i^{(s)}<0 \quad \text{for} \quad \Delta_i^{(r)}<0. \tag{37b}$$

Therefore, if $\Delta_i^{(s)}\neq 0$, conditions (37) are sufficient for the satisfaction of Eqs. (31b) and (31c) along the range $1\leq y\leq y_*$, where y_* is a zero of σ. Conditions (32) are satisfied under the restrictions (37). This will be discussed further ahead in Sec. X.

IX. INITIAL CONDITIONS AND EXACT INITIAL PERTURBATIONS

The behavior of the quantity $\Delta_i^{(s)}$ given by Eq. (35) determines the set of initial conditions that characterize the thermodynamical consistency of the models. Since this quantity plays such an important role, it must be related to a physically significant property along the initial hypersurface $t=t_i$. From Eq. (35), using Eqs. (12), (14), and (15), we obtain

$$\Delta_i^{(s)}=\frac{3}{4}\Delta_i^{(r)}-\Delta_i^{(m)}=\frac{1}{4}+\frac{[\log(W^{3/4}/M)]'}{[\log(Y_i^3)]'}=\frac{d[\log\Sigma_i]}{d[\log Y_i^3]}, \tag{38a}$$

where

$$\Sigma_i\equiv\frac{\langle\rho_i^{(r)}\rangle^{3/4}}{\langle\rho_i^{(m)}\rangle}. \tag{38b}$$

In order to provide an interpretation for (38), we remark [from Eqs. (17), (19), and (28)] that the entropy per photon along the initial hypersurface

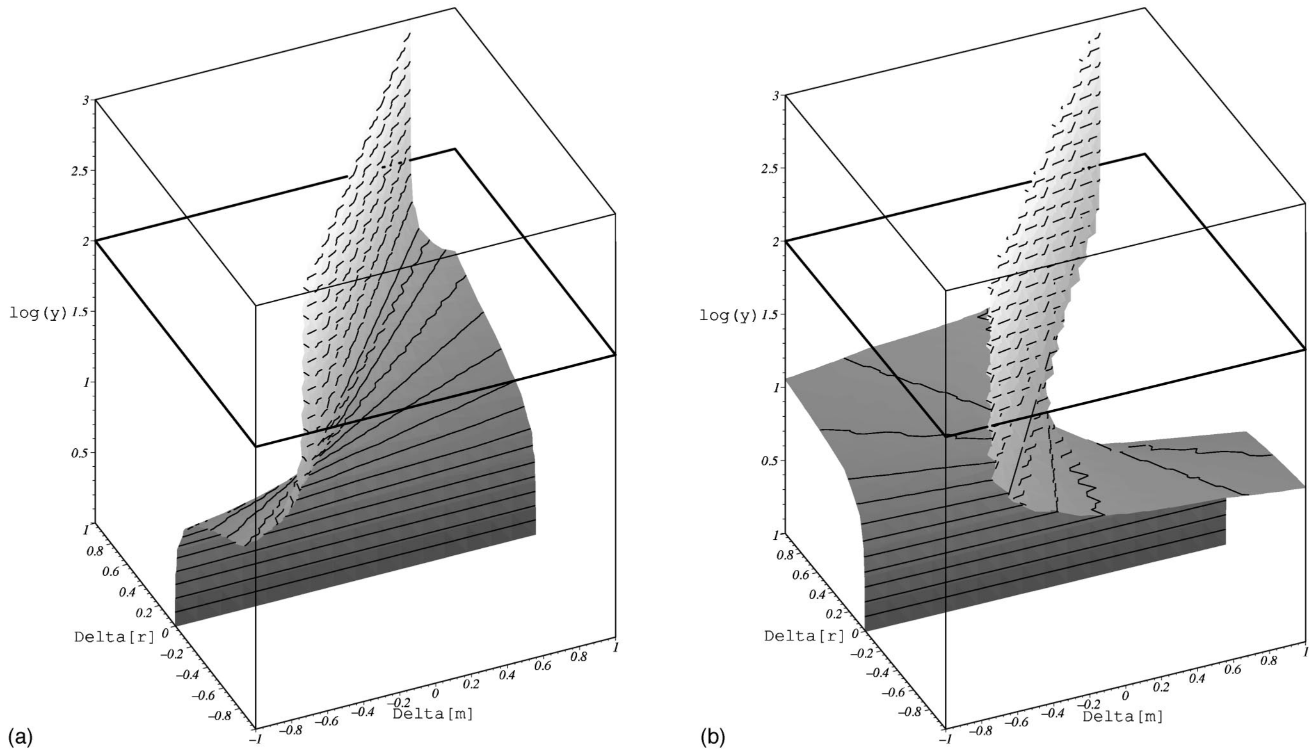

FIG. 3. The equations $\Phi=0$ and $\sigma=0$. Implicit plots of the solution of $\Phi=0$ (a) and $\sigma=0$ (b), in terms of $\Delta_i^{(m)}$, $\Delta_i^{(r)}$, and $\log_{10}(y)$. The ''wall'' where the surfaces $\sigma=0$ and $\Phi=0$ occur for values greater than $y=10^2$ corresponds to the line $\frac{3}{4}\Delta_i^{(r)}=\Delta_i^{(m)}$ (or $\Delta_i^{(s)}=0$). The plots also illustrate that the values of $\Delta_i^{(r)}$, $\Delta_i^{(m)}$ where a zero of σ occurs are clearly distinct from those associated with a zero of Φ, leading to conditions (37). The vertical height $y=10^2$ is displayed in (a) and (b), illustrating that the values of $\Delta_i^{(r)},\Delta_i^{(m)}$ with $10^2<y_*<10^3$ are very close to $\Delta_i^{(s)}=0$.

$$s_i=\frac{4a_BT_i^3}{3n_0^{(r)}}-\frac{15}{32}k_B\left[\frac{\Phi_i}{\Psi_i}\right]^2=\frac{4a_BT_i^3}{3n_i^{(r)}}-\frac{15}{32}k_B\left[\frac{\Delta_i^{(r)}}{1+\Delta_i^{(r)}}\right]^2$$

is very close to its equilibrium value, since its off equilibrium correction is proportional to $[\Delta_i^{(r)}]^2$, a very small quantity since $\Delta_i^{(r)}$ is already assumed to be small. Therefore, using the approximations associated with small density contrasts [Eqs. (23a)–(23c)], we obtain

$$\rho_i^{(r)}\approx 3a_BT_i^4,\qquad \Sigma_0\propto\frac{\langle a_BT_i^4\rangle^{3/4}}{mc^2\langle n_i^{(m)}\rangle}\propto\frac{\langle s_i\rangle}{\langle n_i^{(m)}\rangle}\approx\left\langle\frac{s_i}{n_i^{(m)}}\right\rangle, \tag{39}$$

implying that Σ_i is proportional to the ratio of the averages of photon entropy and baryon number density, which (for small density contrasts) is roughly equivalent to the averaged ratio of these quantities. Hence, condition $\Delta_i^{(s)}=0$ roughly means an initial hypersurface with constant averages of photon entropy per baryon, while condition (37) roughly means [see Eqs. (20)] that the sign of the spatial gradient of the average of photon entropy per baryon must agree with the sign of the gradient of the initial photon energy density: as r increases along $t=t_i$, it must increase for a density void ($[\rho_i^{(r)}]'>0$) and decrease for a density lump ($[\rho_i^{(r)}]'<0$).

The assumption of small density contrasts leads to a natural comparison with the theory of perturbations of a FLRW background, in the isochronous gauge and considering a mixture of radiation and nonrelativistic matter, see Refs. [4–6,8–10]. Under these assumptions, matter and radiation densities are given by

$$\rho^{(m)}=\overline{\rho^{(m)}}[1+\delta^{(m)}], \tag{40a}$$

$$\rho^{(r)}=\overline{\rho^{(r)}}[1+\delta^{(r)}], \tag{40b}$$

where $\overline{\rho^{(m)}},\overline{\rho^{(r)}}$ are the respective densities in a FLRW background and $\delta^{(m)}\ll 1$, $\delta^{(r)}\ll 1$ are the perturbations. The gauge invariant quantity

$$\delta^{(s)}=\frac{3}{4}\delta^{(r)}-\delta^{(m)}, \tag{41a}$$

which formally resembles Eq. (35), defines the fluctuations of photon entropy per baryon and leads to the classification of perturbations as

$$\text{Adiabatic}\quad \delta^{(s)}=0, \tag{41b}$$

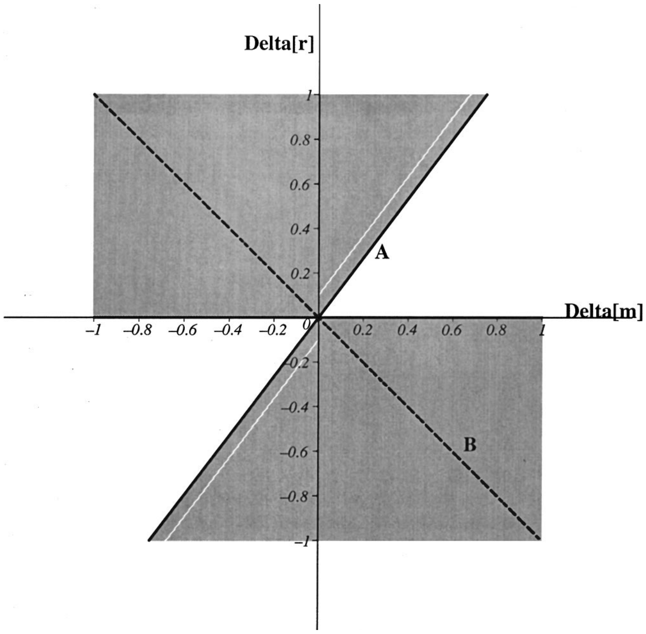

FIG. 4. Allowed values of $\Delta_i^{(r)},\Delta_i^{(r)}$ for $\sigma=0$. The figure displays the plane $\Delta_i^{(r)},\Delta_i^{(r)}$ associated with Fig. 3. The regions for which a zero of σ occurs for $\Phi\neq 0$ are shown with a gray shadow, while the blank regions are ''forbidden'' areas corresponding to zeroes of Φ. The values of $\Delta_i^{(r)},\Delta_i^{(r)}$ in the gray areas comply with conditions (37) and so satisfy conditions (31a), (31b), and (31c). The level curve $\log_{10}(y)=2.4$ has been qualitatively sketched. This curve is extremely close ($\approx 10^{-8}$) to the diagonal line corresponding to $\frac{3}{4}\Delta_i^{(r)}=\Delta_i^{(m)}$, marked by the letter A (the letter B marks the line $\Delta_i^{(r)}=-\Delta_i^{(m)}$. By following the analogy with perturbation theory (see Sec. VIII), the line A can be associated with ''adiabatic'' initial conditions, while ''isocurvature'' initial conditions would be characterized by the horizontal axis $\Delta_i^{(r)}=0$. However, the values of $y=y_D$ for the latter condition are much smaller than $y=10^{2.4}$ for all $|\Delta_i^{(m)}|\neq 0$ [see Fig. 3(b)]. Hence, initial conditions with $\Delta_i^{(s)}\neq 0$ and leading to the right decoupling temperature [close to the level curve $\log_{10}(y)=2.4$] cannot be termed ''isocurvature,'' but rather ''quasiadiabatic'' initial conditions.

$$\text{Isocurvature}\quad \delta^{(s)}=\delta^{(m)}\left[1+\frac{3\rho^{(m)}}{4\rho^{(r)}}\right]. \tag{41c}$$

In order to establish a comparison with perturbation theory, consider defining the initial densities along the lines of Eq. (40):

$$\rho_i^{(m)}=\overline{\rho_i^{(m)}}[1+\delta_i^{(m)}] \quad\Rightarrow\quad \langle\rho_i^{(m)}\rangle=\overline{\rho_i^{(m)}}[1+\langle\delta_i^{(m)}\rangle], \tag{42a}$$

$$\rho_i^{(r)}=\overline{\rho_i^{(r)}}[1+\delta_i^{(r)}] \quad\Rightarrow\quad \langle\rho_i^{(r)}\rangle=\overline{\rho_i^{(r)}}[1+\langle\delta_i^{(r)}\rangle], \tag{42b}$$

where $\overline{\rho_i^{(m)}},\overline{\rho_i^{(r)}}$ are now the constant values of the initial densities in a FLRW background and $|\delta_i^{(m)}|\ll 1,|\delta_i^{(r)}|\ll 1$ are exact initial perturbations. Inserting Eq. (42) into Eq. (38) leads to

$$\Sigma_i\approx\frac{\overline{s_i}}{\overline{n_i^{(m)}}}\,\frac{1+\frac{3}{4}\langle\delta_i^{(r)}\rangle}{1+\langle\delta_i^{(m)}\rangle} \quad\Rightarrow\quad \frac{\Delta_i^{(s)}}{Y_i^3}\approx\frac{d}{dY_i^3}\left\langle\frac{3}{4}\delta_i^{(r)}-\delta_i^{(m)}\right\rangle$$
$$\approx\frac{d}{dY_i^3}\langle\delta_i^{(s)}\rangle, \tag{42c}$$

where $\overline{s_i}\propto[\overline{\rho_i^{(r)}}]^{3/4}$ and $\overline{n_i^{(r)}}$ are the photon entropy and baryon number density of the FLRW background in the hypersurface $t=t_i$. The right-hand side of Eq. (42c) illustrates that, under the assumptions (42a)–(42b) of a perturbative treatment, $\Delta_i^{(s)}$ approximately reduces to the radial gradient of $\delta^{(s)}$ evaluated in $t=t_i$. However, the comparison with perturbations must be handled with caution, since in the case of Eq. (35) and (36)–(39) we are dealing with quantities and relations strictly defined in the initial hypersurface, and so determining initial conditions in terms of averages of initial value functions. The theory of perturbations, on the other hand, traces the evolution of $\delta^{(m)},\delta^{(r)}$ for all t. In particular, we must be specially careful if we borrow the terminology of perturbations in Eq. (41) to characterize the cases $\Delta_i^{(s)}=0$ and $\Delta_i^{(s)}\neq 0$, since in our case a relation such as $\Delta_i^{(s)}=0$ specifies initial conditions and will not be satisfied (in general) for $t>t_i$. Another delicate point refers to the current use of the term ''adiabatic'' for the case (41b), meaning perturbations that conserve photon entropy. Strictly speaking, these perturbations should be denoted as ''isentropic'' or ''reversible,'' since an adiabatic process need not be isentropic (or reversible). It is quite possible that this conceptual vagueness follows from the fact that most papers in perturbation theory, either assume thermal equilibrium, or have incorporated dissipative processes without the necessary rigour. Hopefully, recent work [17] on these lines might be helpful to clarify these issues. Raising this point is relevant because the models presented in this paper assume a fluid evolving along adiabatic but irreversible (nonisentropic) processes, therefore initial conditions such as $\Delta_i^{(s)}=0$ [formally analogous to Eq. (41b)] do not imply a conserved photon entropy, but a roughly constant average of photon entropy at $t=t_i$, but not (in general) for $t>t_i$. Never the less, for the sake of mantaining continuity with currently established terminology, we shall use the formal analogy between Eqs. (41) and (42) in the examination of the cases $\Delta_i^{(s)}=0$ and $\Delta_i^{(s)}\neq 0$. Developing further the comparison with perturbations on a FLRW background is relevant and interesting. However, such a task requires a comprehensive and detailed elaboration and so will be carried on elsewhere.

X. THE CASE $\Delta_I^{(s)}\neq 0$, AN INTERACTIVE MIXTURE

From Sec. VIII we know that if Eqs. (37a),(37b) hold, then τ diverges positively as $y\to y_*$ where y_* is a zero of σ. We still need to know if $\tau/t_H<1$ along the range $1\leq y<\tilde{y}$, where the set of values $y=\tilde{y}<y_*$ are characterized by $\tau/t_H=1$, approximately marking the decoupling of matter and radiation. A sufficient condition for this type of evolution follows from evaluating $[\tau/t_H]_i$ at the initial hypersurface

$$\left[\frac{\tau}{t_H}\right]_i = \frac{5\Delta_i^{(r)}[1+\Delta_i^{(r)}][(3+4\varepsilon)\Delta_i^{(r)}-4\Delta_i^{(s)}+8(1+\varepsilon)]}{[(3+4\varepsilon)\Delta_i^{(r)}-4\Delta_i^{(s)}(3+4)][36+47\Delta_i^{(r)}+16(\Delta_i^{(s)})^2]} \approx \frac{10(1+\epsilon)}{9(3+4\epsilon)} \approx \frac{5}{18} < 1,$$

where we have eliminated $\Delta_i^{(m)}$ from Eq. (35) and assumed $\Delta_i^{(s)} \ll 1$, $\Delta_i^{(r)} \ll 1$, and $\epsilon \approx 10^{-3} T_i \approx 10^3$. Since τ diverges as $y \to y_*$ but Θ remains finite in this limit, the ratio τ/t_H, initially smaller, necessarily becomes larger than unity for $1 < y < \tilde{y} < y_*$.

The currently accepted value of matter and radiation decoupling is $T_D \approx 4\times 10^3$ K. Assuming $T_i \approx 10^6$ K, the set of values $y = y_D$ associated with this temperature follow from Eq. (17b) by solving for $y = y_D$ the equation

$$T_D \approx 4\times 10^3 = \frac{10^6}{y_D}\Psi(y_D, \Delta_i^{(s)}, \Delta_i^{(r)}), \tag{43}$$

where Ψ is given by Eq. (19b). Assuming small density contrasts, Fig. 5 illustrates that this value of T_D is closely associated with $y_D \approx 10^{2.4}$. Also, it is evident from Fig. 3(b), that having a zero of σ for values $10^2 < y_* < 10^3$ requires a very small deviation from $\Delta_i^{(s)} = 0$. This is illustrated by the approximated sketch of the level curve $y = y_D = 10^{2.4}$ that appears in Fig. 4, implying that $\tilde{y}, y_*$ and y_D lie in a a very narrow sector of the plane $\Delta_i^{(m)}, \Delta_i^{(r)}$, very close to the line $\Delta_i^{(s)} = 0$. Therefore, for these values we must have $y_* \approx \tilde{y} \approx y_D$. However, it is not clear from Figs. 3(b) and 4 how small the deviation from $\Delta_i^{(s)} = 0$ should be. Hence, we have

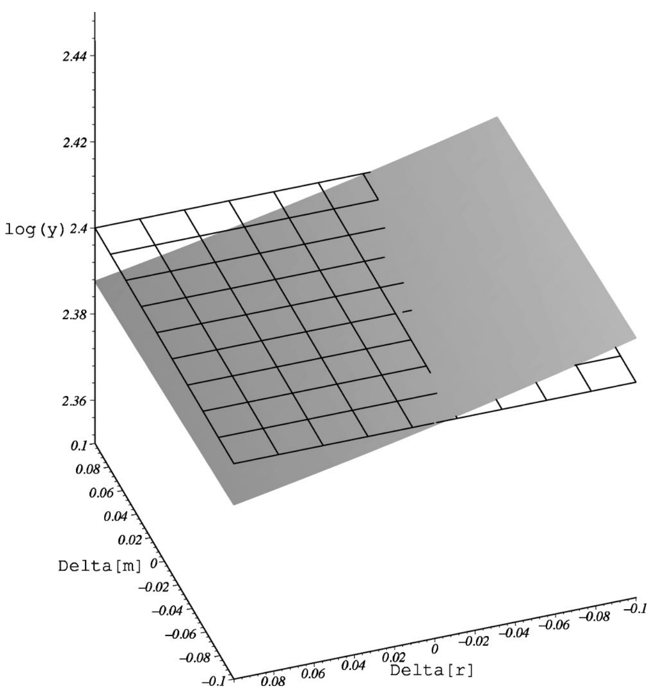

FIG. 5. The decoupling temperature $T = T_D = 4\times 10^3$ K for $y = y_D$. This figure displays the implicit plot of Eq. (43) in terms of $\Delta_i^{(m)}, \Delta_i^{(r)}$ and $\log_{10}(y)$. The values of y for this temperature value are clearly shown to be very close to $y = y_D = 10^{2.4}$.

plotted in Fig. 6 the implicit equation $\sigma = 0$, showing a more precise relation between the orders of magnitude that the occurrence of the zero of σ for $y_D \approx y_* \approx 10^{2.4}$ implies for $|\Delta_i^{(r)}|, |\Delta_i^{(s)}|$.

Another constraint the models have to comply with is the observational bounds [5,6] on the anisotropy of the cosmic microwave radiation (CMR), given by the maximal photon temperature contrast $[\delta T/T]_D \approx 10^{-5}$. For $y \approx 10^{2.4}$ and $T_i \approx 10^6$, so that $\epsilon \approx 10^3$, we have $A \approx 1.1\times 10^{-1}$ and $B \approx 2.2\times 10^{-1}$, were A and B are given by Eq. (20). Also, from Eqs. (17b) and (19b), the deviation from the equilibrium FLRW form $T_{eq} = T_i/y$ at $y = y_D$ is given by

$$\left[\frac{\delta T_{eq}}{T_{eq}}\right]_D = \left[\frac{T - T_{eq}}{T_{eq}}\right]_D = [\Psi - 1]_D \approx -\frac{1.1\times 10^{-1}\Delta_i^{(m)} + 2.2\times 10^{-1}\Delta_i^{(r)}}{1+\Delta_i^{(r)}}, \tag{44}$$

indicating that compliance with the maximal observed temperature contrast of the CMR constrains the maximal values of $\Delta_i^{(m)}, \Delta_i^{(r)}$ to about 10^{-4}. Therefore, from Fig. 6, the corresponding variation range of $|\Delta_i^{(s)}|$ is $|\Delta_i^{(s)}| < 10^{-8}$, and so compatibility with acceptable values of T_D and the CMR anisotropy implies $|\Delta_i^{(s)}| \approx |\Delta_i^{(r)}|^2 \ll |\Delta_i^{(r)}|$.

Under the analogy between $\Delta_i^{(m)}, \Delta_i^{(r)}$ with perturbations in a FLRW background, the case $\Delta_i^{(s)} \neq 0$ seems to correspond to isocurvature initial perturbations. However, since $\rho^{(m)} \ll \rho^{(r)}$ before decoupling, the latter are characterized [see Eq. (41c)] by $\delta^{(s)} \approx \delta^{(m)}$ and $\delta^{(r)} \ll \delta^{(s)}$, and so, following the analogy between Eqs. (41) and (42) would disqualify $\Delta_i^{(s)} \neq 0$ as comparable to an isocurvature initial perturbation. In fact, the maximal bounds on $\Delta_i^{(r)}$ and $\Delta_i^{(s)}$ obtained in the previous paragraph yield exactly the opposite behavior. Borrowing the terminology of perturbation theory, the analogy between Eqs. (41) and (42) would characterize ''isocurvature'' initial conditions by $\Delta_i^{(r)} = 0$. From Figs. 3 and 4, this condition is not incompatible with Eqs. (31) and (32), but yields a decoupling surface $y_D \approx 10$ with a decoupling temperature much larger than the accepted value. Hence, the acceptable values of the case $\Delta_i^{(s)} \neq 0$ cannot be associated with this type of initial perturbations, but since perturbations are in general combinations of adiabatic and isocurvature components and we have $\Delta_i^{(s)} \ll \Delta_i^{(r)}$ and $\Delta_i^{(s)} \ll \Delta_i^{(m)}$, a more accurate analogy for $\Delta_i^{(m)}, \Delta_i^{(r)}$ is that of ''quasiadiabatic'' initial perturbations. Following this analogy, and bearing in mind that $\Delta_i^{(m)}$ and $\Delta_i^{(r)}$ are defined for an initial hypersurface characterized by $T_i \approx 10^6$ K (still outside the horizon for perturbations of all wave numbers), the bounds on the magnitude of $\Delta_i^{(s)}$ can be related to bounds on the amplitude of ''nearly adiabatic'' fluctuations of photon entropy per

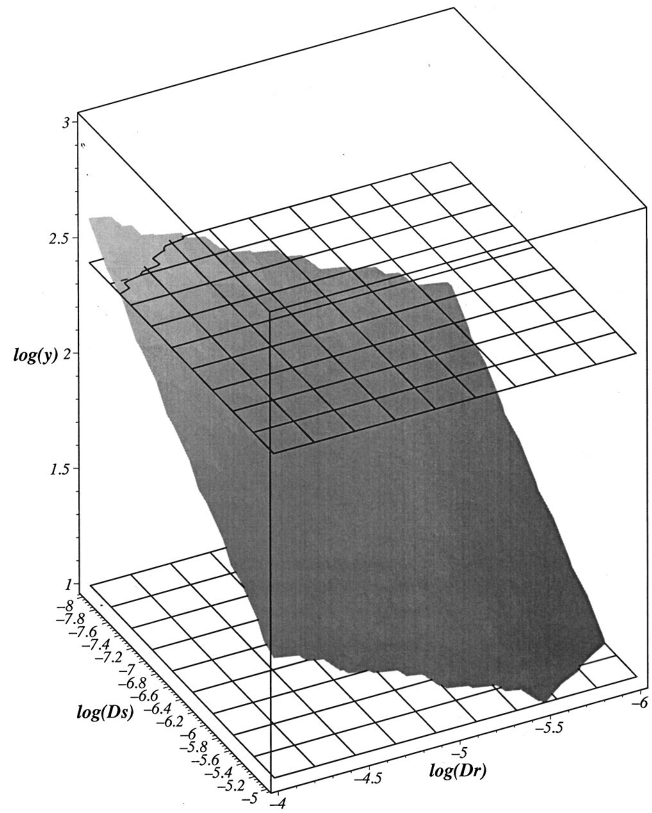

FIG. 6. The equation $\sigma=0$ for $y=y_D$. Implicit plot of the solution to the equation $\sigma=0$ in terms of $\log_{10}(|\Delta_i^{(s)}|)$ and $\log_{10}(|\Delta_i^{(r)}|)$, denoted as ''log(Ds)'' and ''log(Dr).'' This plot illustrates the constraints on the orders of magnitude of these quantities under the condition that a zero of σ occurs for $y=y_D\approx 10^{2.4}$. For values of $|\Delta_i^{(r)}|\approx 10^{-4}$, compatible with observed anisotropy of the CBR, we have the following estimated value: $|\Delta_i^{(s)}|\approx 10^{-8}$. This can be a constraint on entropy fluctuations in primordial quasiadiabatic perturbations.

baryon generated in the earlier, more primordial, inflationary era [4–6,10,34]. Primordial entropy fluctuations have been examined in connection with the creation of axions in an inflationary scenario [5,10]. However, these are isocurvature perturbations, and so might not be related to entropy fluctuations associated with $\Delta_i^{(s)}\neq 0$. The study of entropy fluctuations in inflationary models is still a highly speculative topic [34], and its connection with initial conditions in the models under consideration should be an exciting subject to examine in a future paper.

XI. SAHA'S EQUATION AND THE JEANS MASS

As mentioned earlier, it is important to compare τ and t_H with the timescale characteristic of the interaction rate of the photons and electrons for Compton and Thomson scattering (the dominant radiative processes in the temperature range of interest)

$$t_\gamma=\frac{1}{c\sigma_T n_e}, \quad t_c=\frac{m_e c^2}{k_B T}t_\gamma, \tag{45}$$

where $\sigma_T\approx 6.65\times 10^{-25}$ cm^2 is the Thomson scattering cross section, m_e is the electron mass, and n_e is the number density of free electrons, a quantity obtained from Saha's equation

$$\frac{X_e^2}{1-X_e}=\left[\frac{2\pi m_e k_B T}{h^2}\right]^{3/2}\frac{\exp(B_0/k_BT)}{n_B}, \tag{46}$$

$$X_e\equiv\frac{n_e}{n_B}, \quad n_B\approx n^{(m)},$$

where X_e is the fractional ionization, h is Planck's constant, n_B is the number density of baryons and $B_0\approx 13.6$ ev is the binding energy of the hydrogen atom. Combining Eqs. (45) and (46) we obtain

$$t_\gamma=\frac{1}{2c\sigma_B n^{(m)}}\left[1+\left(1+\frac{4h^3n^{(m)}\exp(B_0/k_BT)}{(2\pi m_e k_B T)^{3/2}}\right)^{1/2}\right]. \tag{47}$$

The recombination process is characterized by $X_e\approx 1/10$ in Eq. (46), so that most free electrons have combined with protons into neutral atoms, while the decoupling of matter and radiation strictly follows from the condition: $t_\gamma=t_H$, a condition analogous to $\tau=t_H$ and leading to the ''decoupling temperature'' from Eqs. (45) and (47) with the help of Eqs. (17b) and (22). Since t_γ must be smaller but qualitatively similar to τ (increasing as τ increases [29]), a comparison between τ and t_H for near decoupling temperatures should be qualitatively analogous to that between t_γ and t_H. The time scales t_γ and t_c can be easily obtained from Eqs. (45), (47), and (22) and compared with t_H,τ for a set of parameters characteristic of thermodynamically consistent models discussed in the previous section. As shown by Fig. 7, displaying the ratios of τ,t_γ,t_c to t_H for $\Delta_i^{(r)}=10^{-4}$ and $10^{-8}<\Delta_i^{(s)}<10^{-4}$. For $\Delta_i^{(s)}\approx 10^{-8}$ and for near decoupling temperatures ($10^3<$ T$<10^4$ K or equivalently $10^2<y<10^{2.6}$) we have t_γ smaller than τ (but of the order of magnitude) and overtaking t_H along a set of values of y that closely match the accepted value $T_D\approx 4\times 10^3$ K. This is consistent with the estimation $\Delta_i^{(s)}\approx 10^{-8}$ obtained in the previous section (see Fig. 6). On the other hand, for higher temperatures ($T\approx 10^6$ K or equivalently $y\approx 1$), the relaxation parameter τ is of the order of magnitud but greater than the Compton time scale, the dominant radiative process at these temperatures. It is also possible to show that $X_e\approx 1/10$ in Eq. (46) leads to the accepted values of the recombination temperature.

To end the discussion, we compute the Jeans mass associated to the initial conditions of the case $\Delta_i^{(s)}\neq 0$. This mass is given by [1,3–6]

$$M_J=\frac{4\pi}{3}mn^{(m)}\left[\frac{8\pi^2C_s^2}{\kappa(\rho+p)}\right]^{3/2}$$
$$=\frac{4\pi}{3}\frac{c^4\chi_i\Gamma^{1/2}}{\sqrt{\rho_i^{(r)}}}\left[\frac{\pi y^2\Psi}{3G\left(\Psi+\frac{3}{4}\chi_i y\right)^2}\right]^{3/2}, \tag{48}$$

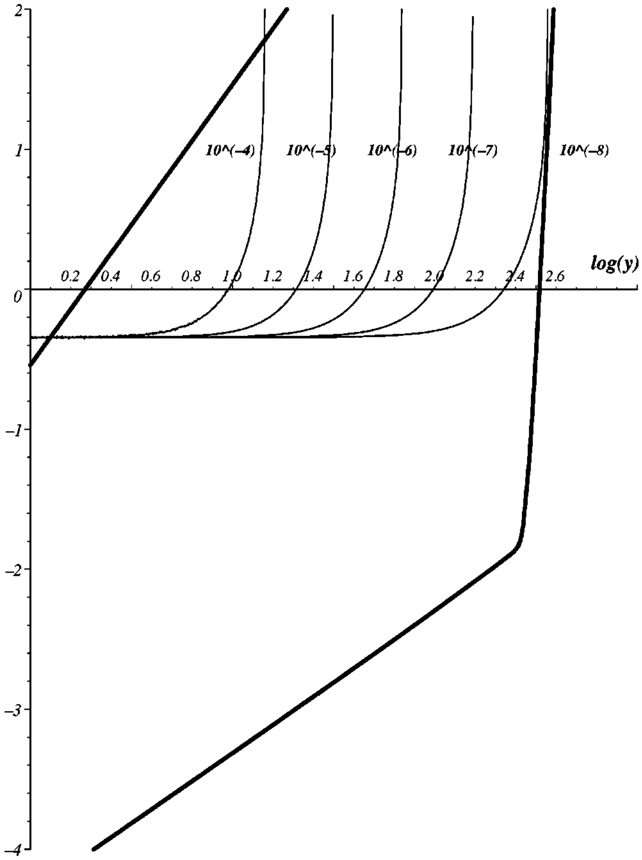

FIG. 7. Comparison between τ, the time scales of Thomson (t_γ) and Compton (t_c) scatterings, and t_H. The figure depicts the ratios $\log_{10}(\tau/t_H)$, $\log_{10}(t_c/t_H)$, and $\log_{10}(t_\gamma/t_H)$ along the range $0<\log_{10}(y)<3$ for $\Delta_i^{(m)}=(3/4)\Delta_i^{(r)}-\Delta_i^{(s)}$ with $\Delta_i^{(r)}=10^{-4}$ and $10^{-8}<\Delta_i^{(s)}<10^{-4}$. The curves that branch out correspond to $\log_{10}(\tau/t_H)$ for the displayed values of $\Delta_i^{(s)}$. The thick curves are $\log_{10}(t_c/t_H)$ (above) and $\log_{10}(t_\gamma/t_H)$ (below) for the same values of $\Delta_i^{(s)}$ and $\Delta_i^{(r)}$. Notice that τ is much more sensitive to changes in $\Delta_i^{(s)}$ than t_γ and t_c. It is clear from the figure that $\tau\approx t_\gamma$ for $\Delta_i^{(s)}\approx 10^{-8}$ for temperatures near the decoupling temperature, obtained from the condition $t_H=t_\gamma$ and closely matching $T_D\approx 4\times 10^3$ K [marked by $\log_{10}(y)\approx 2.4$]. The figure also reveals that τ and t_c are of the same order of magnitude for higher temperatures closer to $t=t_i$. This is consistent with the fact that Compton scattering is the dominant radiative process in this temperature range.

where $\rho, p, n^{(m)}$ are given by Eqs. (2), (7), (13), and (14), $\chi_i=\rho_i^m/\rho_i^r$, Ψ, and Γ follow from Eq. (19) and C_s is the speed of sound, which for the equation of state (2), has the form

$$C_s^2=\frac{c^2}{3}\left[1+\frac{3\rho^{(m)}}{4\rho^{(r)}}\right]^{-1}, \quad \rho^{(m)}=mc^2 n^{(m)},$$

$$\rho^{(r)}=3n^{(r)}k_B T.$$

Evaluating Eq. (48) for $y=y_D\approx 10^{2.4}$, $\epsilon\approx 1/\chi_i\approx 10^3$, and $\rho_i^{(r)}\approx a_B T_i^4\approx 7.5\times 10^9$ ergs/cm^3, yields $M_J\approx 10^{49}$ g m, or approximately 10^{16} solar masses. This value coincides with the Jeans mass obtained for baryon dominated perturbative models as decoupling is approached in the radiative era.

XII. CONCLUSIONS

We have derived a new class of exact solutions of Einstein's equations providing a physically plausible hydrodynamic description of a radiation-matter (photon-baryon) interacting mixture, evolving along adiabatic but irreversible thermodynamic processes. The conditions for these models to be consistent with the transport equation and entropy balance law of EIT (when shear viscosity is the only dissipative agent) have been provided explicitly, and their effect on initial conditions have been given in detail, briefly discussing the analogy of these conditions with gauge invariant initial perturbations in the isochronous gauge. As far as we are aware, and in spite of their limitations mentioned in the Introduction, we believe these models are the first example in the literature of a self-consistent hydrodynamical approach to matter-radiation mixtures that (a) is based on inhomogeneous exact solutions of Einstein's field equations and (b) is thermodynamically consistent. We believe these models can be a useful theoretical tool in the study of cosmological matter sources, providing a needed alternative and complement to the usual approaches based on perturbations or on numerical methods.

The solutions have an enormous potential as models in applications of astrophysical and cosmological interest. Consider, for example, the following possibilities

(a) *Structure formation in the acoustic phase*. There is a large body of literature on the study of acoustic perturbations in relation to the Jeans mass of surviving cosmological condensations. Equations of state analogous to Eq. (2) are often suggested in this context [1–4,6,10]. Since practically all work on this topic has been carried on with perturbations on a FLRW background, the exact solutions derived and presented here may be viewed as an alternative treatment for this problem.

(b) *Comparison with Perturbation Theory*. The models presented in this paper are based on exact solutions of Einstein's field equations, but their initial conditions and evolution can be adapted for a description of ''exact spherical perturbations'' on a FLRW background. It would be extremely interesting, not only to compare the results of this approach with those of a perturbative treatment, but to provide a physically plausible theoretical framework to examine carefully how much information is lost in the nonlinear regime that falls beyond the scope of linear perturbations. We have studied in this paper only the case $F=0$ in Eq. (11), thus restricting the evolution to the ''growing mode'' since all fluid layers expand monotonously. The study of the more general case, where $F(r)$ in Eq. (11) is an arbitrary function that could change sign, would allow a comparison with perturbations that include also a ''decaying mode'' related to condensation and collapse of cosmological inhomogeneities.

(c) *Inhomogeneity and irreversibility in primordial density perturbations*. The initial conditions of the models with $\Delta_i^{(s)}\neq 0$ are set for a hypersurface with temperature $T_i\approx 10^6$. These initial conditions can be considered the end product of processes characteristic of previous cosmological history, and so the estimated value $|\Delta_i^{(s)}|\approx 10^{-8}$, related to the spatial variation of photon entropy fluctuations, can be used as a constraint on the effects of inhomogeneity on primordial entropy fluctuations that might be predicted by inflationary models at an earlier cosmic time. Also, the deviation from equilibrium in the initial hypersurface (proportional to $[\Delta_i^{(r)}]^2\approx|\Delta_i^{(s)}|$) might be helpful to understand the irreversibility associated with the physical processes involved in the generation of primordial perturbations [34]. These and other applications are worth to be undertaken in future research efforts.

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