Exact spherically symmetric dust solution of the field equations in observational coordinates with cosmological data functions

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In this paper we summarize the status of what is known about the solution of the exact spherically symmetric Einstein field equations for dust in observational coordinates with the input of functions representing observer area distance and galaxy number counts data, and complete the integration in a practically applicable way. [S0556-2821(99)05720-3]

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I. INTRODUCTION

A key component of the observational cosmology program (Ellis *et al.* [1]) is the exact solution of the spherically symmetric Einstein equations for dust in observational coordinates, using cosmological data for observer area distances and galaxy number counts given as functions of redshift z. These data are given, not on a space-like surface of constant time, but rather on our past light cone $C^{-}(p_0)$, which is centered at our observational position p_0, "here and now." Thus, they can be very easily and naturally introduced into the field equations written in observational coordinates. For example, z is essentially g_{00}, and the observer area distance $C(w,y)$ is $g_{22}=g_{33}/(\sin^2\theta)$ evaluated on $C^{-}(p_0)$. To use that same data in the field equations expressed in the usual $3+1$ coordinates leads to difficulties, because the null geodesic equations which are needed to transform the parameters naturally defined on our past light cone to the $3+1$ coordinates cannot be solved exactly — only approximately — even for spherical symmetry.

By using observational coordinates, defined below, we can thus formulate Einstein's equations in a way which reflects both the geodesic flow of the cosmological fluid and the null geometry of $C^{-}(p_0)$, along which practically all of our information about the distant parts of our universe comes to us — via photons. In this formulation the field equations split naturally into two sets, as can be easily seen: a set of equations which can be solved on $C^{-}(p_0)$, that is on our past light cone, specified by $w=w_0$, where w is the observational time coordinate; and a second set which evolves these solutions off $C^{-}(p_0)$ to other light cones into the past or into the future. Solution to the first set is directly determined from the data, and those solutions constitute the initial conditions for the solution of the second set.

The foundations for this approach were inspired by the classic paper of Kristian and Sachs [2] and established by Ellis *et al.* [1]. In that paper they demonstrated that a complete observational set of analytic data on $C^{-}(p_0)$, consisting of galaxy redshifts, observer area distances, galaxy number counts, cosmological proper motions, and null shear measures down to some limiting redshift z^*, determine the existence of a unique solution to the field equations for dust on $C^{-}(p_0)$ down to a value v^* of the affine parameter corresponding to z^*, and for the past Cauchy development of that section of $C^{-}(p_0)$ between $v=0$ and $v=v^*$.

If the galactic observations are isotropic, i.e., if the proper motions and null shear vanish and the area distances and number counts are independent of direction, then this implies directly that the spacetime geometry is isotropic on $C^{-}(p_0)$ and in its past Cauchy development, as shown by Maartens [3,4]. This theorem constitutes an observational foundation for the spherically symmetric dust models. [If one further invokes the weak Copernican principle, i.e., that we do not occupy a privileged position in spacetime, then one arrives at an observational foundation for Friedmann-Lemaître-Robertson-Walker (FLRW) models.] We discuss this case thoroughly in this paper, and present the completion of the integration scheme initiated in Stoeger *et al.* [5], and significantly corrected and improved in Maartens *et al.* [6] and Humphreys [7]. The solution to these will, of course, give just the Lemaître-Tolman-Bondi (LTB) models (Lemaître [8], [9], Tolman [10], Bondi [11]; and cf. Humphreys [7] and references therein) in observational coordinates. We have already mentioned why it is not possible to obtain these observational solutions simply by transforming them from the usual $3+1$ coordinate representation to observational coordinates: Those transformations cannot be exactly specified in analytic form.

However, the spherically symmetric (SS) equations in cosmological coordinates are formidable and not easily solved. There are many subtleties involved. Stoeger *et al.* [5] presented a detailed integration scheme, but it is not correct — as it introduces an assumption which over-restricts the solutions, forcing them, as it turns out, to be FLRW. The basic system of SS field equations in that paper are, however, correct, as is the first step in the integration — the solution of the null Raychaudhuri equation on our past light cone to determine the null radial coordinate y as a function of the redshift z. Subsequently, Maartens *et al.* [6] demonstrated how the integration of the SS equations could be carried out and completed on $C^{-}(p_0)$, and then off into the past as a function of proper time. But they did not actually show how the solutions off the light cone can be determined as functions of the observational coordinate time w. Humphreys [7] has gone one step farther, deriving an evolution equation for the observer area distance C which can be integrated in two steps for obtaining $C(w,y)$. However, he only performed the first integration over y down $C^{-}(p_0)$ and did not explicitly show how the central conditions can be used at that point to recover the general w-dependence of the first integral or how the second integration over w is to be done.

In this paper we review the entire problem of determining the solution of the exact spherically symmetric Einstein equations for dust in observational coordinates, summarize the steps in integrating them, and complete the integration by explicitly carrying out the solution of the key $\dot{C}'$ evolution equation. From this all the other dependent metric variables are determined. Finally, we illustrate the procedure concretely by integrating these equations for FLRW data in the flat case to obtain the FLRW solution explicitly.

After defining observational coordinates and writing the general spherically symmetric metric using them in Sec. II, we summarize the basic observational parameters we shall be using in Sec. III. In Sec. IV we present several key relationships among the metric variables, including the key momentum conservation equation. Section V presents the full set of field equations for the spherically symmetric case, with dust. In Sec. VI, we outline in considerable detail the procedure for integrating these equations with data functions representing observer area distance and galaxy number counts, emphasizing how the complete solution of the momentum conservation equation can be obtained, which has not been presented before. Section VII shows how this procedure is applied in the simplest case for FLRW data.

II. THE SS METRIC IN OBSERVATIONAL COORDINATES

We begin by giving the SS spacetime in observational coordinates (which were first suggested by Temple in 1938 [12]). As described by Ellis *et al.* [1] these coordinates $x^i = \{w,y,\theta,\phi\}$ are centered on the observer's world line C and defined in the following way:

(i) w is constant on each past light cone along C, with $u^a\partial_a w > 0$ along C, where u^a is the 4-velocity of matter ($u^a u_a = -1$). In other words, each $w=$constant specifies a past light cone along C. Our past light cone is designated as $w=w_0$.

(ii) y is the null radial coordinate. It measures distance down the null geodesics — with affine parameter ν — generating each past light cone centered on C. $y=0$ on C and $dy/d\nu > 0$ on each null cone, so that y increases as one moves down a past light cone away from C.

(iii) θ and ϕ are the latitude and longitude of observation, respectively — spherical coordinates based on a parallelly propagated orthonormal tetrad along C, and defined away from C by $k^a\partial_a\theta = k^a\partial_a\phi = 0$, where k^a is the past-directed wave vector of photons ($k^a k_a = 0$).

There are certain freedoms in the specification of these observational coordinates. In w there is the remaining freedom to specify w along our world line C. Once specified there it is fixed for all other world lines. There is considerable freedom in the choice of y — there is a large variety of possible choices for this coordinate — the affine parameter, z, the area distance $C(w,y)$ itself. We normally choose y to be comoving with the fluid, that is $u^a\partial_a y = 0$. Once we have made this choice, there is still a little bit of freedom left in y, which we shall use below. The remaining freedom in the θ and ϕ coordinates is a rigid rotation at *one* point on C.

In observational coordinates the SS metric takes the general form

$$ds^2 = -A(w,y)^2 dw^2 + 2A(w,y)B(w,y)dw\,dy + C(w,y)^2 d\Omega^2 \tag{1}$$

where we assume that y is comoving with the fluid, so that the fluid 4-velocity is $u^a = A^{-1}\delta^a_w$.

The remaining coordinate freedom which preserves the observational form of the metric is a scaling of w and of y:

$$w \to \tilde{w} = \tilde{w}(w), \quad y \to \tilde{y} = \tilde{y}(y) \quad \left(\frac{d\tilde{w}}{dw} \neq 0 \neq \frac{d\tilde{y}}{dy}\right). \tag{2}$$

The first, as we mentioned above, corresponds to a freedom to choose w as any time parameter we wish along C, along our world line at $y=0$. This is usually effected by choosing $A(w,0)$. The second corresponds to the freedom to choose y as any null distance parameter on an initial light cone — typically our light cone at $w=w_0$. Then that choice is effectively dragged onto other light cones by the fluid flow — y is comoving with the fluid 4-velocity, as we have already indicated. We shall use this freedom to choose y by setting:

$$A(w_0,y) = B(w_0,y). \tag{3}$$

In general, these freedoms in w and y imply the metric scalings:

$$A \to \tilde{A} = \frac{dw}{d\tilde{w}}A, \quad B \to \tilde{B} = \frac{dy}{d\tilde{y}}B. \tag{4}$$

It is important to specify the central conditions for the metric variables $A(w,y)$, $B(w,y)$ and $C(w,y)$ in Eq. (1) — that is, their proper behavior as they approach $y=0$. These are

$$\begin{aligned} \text{as} \quad y \to 0: \quad & A(w,y) \to A(w,0) \neq 0, \\ & B(w,y) \to B(w,0) \neq 0, \\ & C(w,y) \to B(w,0)y = 0, \\ & C_y(w,y) \to B(w,0). \end{aligned} \tag{5}$$

III. THE BASIC OBSERVATIONAL QUANTITIES

The basic observable quantities on C are the following:

(i) Redshift. The redshift z at time w_0 on C for a comoving source a null radial distance y down $C^{-}(p_0)$ is given by

$$1+z=\frac{A(w_0,0)}{A(w_0,y)}. \tag{6}$$

This is just the observed redshift, which is directly determined by source spectra, once they are corrected for the Doppler shift due to local motions.

(ii) Observer area distance. The observer area distance, often written as r_0, measured at time w_0 on C for a source at a null radial distance y is simply given by

$$r_0=C(w_0,y), \tag{7}$$

provided the central condition (5), determining the relation between $C(w,y)$ and $B(w,y)$ for small values of y, holds. This quantity is also measurable as the luminosity distance because of the reciprocity relation [13].

(iii) Galaxy number counts. The number of galaxies counted by a central observer out to a null radial distance y is given by

$$N(y)=4\pi\int_0^y \mu(w_0,\tilde{y})m^{-1}B(w_0,\tilde{y})C(w_0,\tilde{y})^2 d\tilde{y} \tag{8}$$

where μ is the mass-energy density and m is the average galaxy mass. Then the total energy density can be written as

$$\mu(w_0,y)=m\,n(w_0,y)=M_0(z)\frac{dz}{dy}\frac{1}{B(w_0,y)} \tag{9}$$

where $n(w_0,y)$ is the number density of sources at (w_0,y), and where

$$M_0\equiv\frac{m}{J}\frac{1}{d\Omega}\frac{1}{r_0^2}\frac{dN}{dz}. \tag{10}$$

Here $d\Omega$ is the solid angle over which sources are counted, and J is the completeness of the galaxy count, that is, the fraction of sources in the volume that are counted is J. The effects of dark matter in biasing the galactic distribution may be incorporated via J. In particular, strong biasing is needed if the number counts have a fractal behavior on local scales [14].

It can be seen from the above characterization of these observational quantities how closely they fit into the metric as given in observational coordinates.

IV. OTHER KEY RELATIONSHIPS

There are a number of other important quantities which we catalog here for completeness and for later reference.

First there are the two fundamental four-vectors in the problem, the fluid four-velocity u^a and the null vector k^a, which points down the generators of past light cones. These are given in terms of the metric variables as

$$u^a=A^{-1}\delta^a_w, \quad k^a=(AB)^{-1}\delta^a_y. \tag{11}$$

The rate of expansion of the dust fluid is $3H=\nabla_a u^a$, so that, from the metric (1) we have

$$H=\frac{1}{3A}\left(\frac{\dot{B}}{B}+2\frac{\dot{C}}{C}\right) \tag{12}$$

where a ''dot'' indicates $\partial/\partial w$ and a ''prime'' indicates $\partial/\partial y$, which will be used later. For the central observer H is precisely the Hubble expansion rate. In the homogeneous (FLRW) case, H is constant at each instant of time t. But in the general inhomogeneous case, H varies with radial distance from $y=0$ on $t=t_0$. From our central conditions above Eq. (3), we find that the central behavior of H is given by

$$\text{as} \quad y\to 0: \quad H(w,y)\to\frac{1}{A(w,0)}\frac{\dot{B}(w,0)}{B(w,0)}=H(w,0). \tag{13}$$

At any given instant $w=w_0$ along $y=0$, this expression is just the Hubble constant $H_0=H(w_0,0)=A_0^{-1}B_0^{-1}(\dot{B})_0$ as measured by the central observer.

Finally, from the normalization condition for the fluid four-velocity, we can immediately see that it can be given (in covariant vector form) as the gradient of the proper time t along the matter world lines: $u^a=-t_{,a}$. It is also given by Eqs. (1) and (11) as

$$u_a=g_{ab}u^b=-Aw_{,a}+By_{,a}. \tag{14}$$

Comparing these two forms implies

$$dt=Adw-Bdy \quad \Leftrightarrow \quad A=t_w, \quad B=-t_y \tag{15}$$

which shows that the surfaces of simultaneity for the observer are given in observational coordinates by $Adw=Bdy$. The integrability condition of Eq. (15) is simply then

$$A'+\dot{B}=0. \tag{16}$$

This turns out precisely to be the momentum conservation equation, which is the key equation in the system and essential to finding a solution.

V. THE SS FIELD EQUATIONS IN OBSERVATIONAL COORDINATES

Stoeger *et al.* [5], using the fluid-ray (FR) tetrad formulation of the Einstein's equations developed by Maartens [3] and Stoeger *et al.* [15], give the detailed derivation of the SS field equations in observational coordinates. Besides the very important momentum conservation equation (16), they are as follows.

A set of two very simple fluid-ray tetrad time-derivative equations

$$\Delta\omega=-3b\omega \tag{17}$$

$$\Delta\mu=2\mu(n-b) \tag{18}$$

where $\Delta=(A(w,y))^{-1}\partial/\partial w$ is a tetrad derivative, $b\equiv\dot{C}/AC$ and $n\equiv C'/2AB$ are FR spin coefficients. Using Eq. (16) these equations can be quickly integrated to give

$$\mu(w,y) = \mu_0(y) B^{-1}(w,y) C^{-2}(w,y) \quad (19)$$

$$\omega(w,y) = \frac{\omega_0(y)}{C^3(w,y)} = -\frac{1}{2C^2} + \frac{\dot{C}}{AC}\frac{C'}{BC} + \frac{1}{2}\left(\frac{C'}{BC}\right)^2 \quad (20)$$

where the last equality in Eq. (20) comes from the fact that ω is defined in terms of the FR spin coefficients as $\omega \equiv r + bf + f^2/2$, $f \equiv C'/BC$, $r = -1/2C$, μ again is the relativistic mass-energy density of the dust, and $\omega_0 \equiv \omega(w_0, y)$ is a quantity closely related to $\mu_0 \equiv \mu(w_0, y)$ [see Eq. (27) below]. Both ω_0 and μ_0 are specified by data on our past light cone, as we shall show.

The fluid-ray tetrad radial equations are

$$\frac{C''}{C} = \frac{C'}{C}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{2}B^2\mu \quad (21)$$

$$\omega_0{}' = -\frac{1}{2}\mu_0\left(\frac{\dot{C}}{A} + \frac{C'}{B}\right) \quad (22)$$

$$\frac{\dot{C}'}{C} = \frac{\dot{B}}{B}\frac{C'}{C} - \omega A B. \quad (23)$$

The remaining ''independent'' time-derivative equations given by the fluid-ray tetrad formulation are

$$\frac{\ddot{C}}{C} = \frac{\dot{C}}{C}\frac{\dot{A}}{A} + \omega A^2 \quad (24)$$

$$\frac{\ddot{B}}{B} = \frac{\dot{B}}{B}\frac{\dot{A}}{A} - 2\omega A^2 - \frac{1}{2}\mu A^2. \quad (25)$$

From Eq. (22) we see that there is a naturally defined ''potential'' (see Stoeger *et al.* [5])

$$F(y) \equiv \frac{N'_\star}{N'} = \frac{\dot{C}}{A} + \frac{C'}{B}, \quad (26)$$

where $N_\star(y)$ is an arbitrary function, whose central behavior is the same as that of the number counts (Stoeger *et al.* [5]). Thus, from Eqs. (26) and (22) it follows that

$$\omega_0(y) = -\frac{1}{2}\int \mu_0(y) F(y)\, dy. \quad (27)$$

Connected with this relationship is Eq. (20), which we rewrite as

$$\frac{\dot{C}}{C}\frac{C'}{C} + \frac{A}{2B}\frac{C'^2}{C^2} - \frac{AB}{2C^2} = \frac{\omega_0}{C^3}AB. \quad (28)$$

Stoeger *et al.* [5] and Maartens *et al.* [6] have shown that Eqs. (26) and (28) can be transformed into equations for A and B, thus reducing the problem to determining C:

$$A = \frac{\dot{C}}{[F^2 - 1 - 2\omega_0/C]^{1/2}} \quad (29)$$

$$B = \frac{C'}{F - [F^2 - 1 - 2\omega_0/C]^{1/2}}. \quad (30)$$

The LTB exact solution is obtained by integration of Eq. (29) along the matter flow $y = \text{constant}$ using Eq. (15)

$$t - T(y) = \int \frac{dC}{[F^2 - 1 - 2\omega_0/C]^{1/2}} \quad (31)$$

where $T(y)$ is arbitrary, provided we identify

$$F^2 = 1 - kf^2, \quad k = 0, \pm 1. \quad (32)$$

Here $f = f(y)$ is a function commonly used in describing LTB models in the $3+1$ coordinates (Bonnor [16]).

VI. INTEGRATION PROCEDURE

In order to streamline our discussion, we shall discuss the flat case $(k=0)$ first. Then, from Eq. (32) $F(y) = 1$. Now, since we have specified $F(y)$, the observer area distance data $r_0(z) = C(w_0, z)$ and the galaxy number counts data given by $\mu(w_0, z)$ are *not* independent. So, we can choose one or the other, and then find the second from the relationship given in Eq. (30), temporarily choosing $y = z$, which is always legitimate. For instance, if we know $C(w_0, z)$, then $\omega_0(z)$ is given by

$$\omega_0 = \frac{CF^2}{2}\left\{1 - \frac{1}{F^2} - \left[1 - \frac{(1+z)C'}{FA_0}\right]^2\right\} \quad (33)$$

with $y = z$, where $A_0 \equiv A(w_0, 0)$ and we have used Eqs. (3) and (6) to write

$$B(w_0, y) = A(w_0, 0)/(1+z). \quad (34)$$

The next step is to solve the null Raychaudhuri equation (21) on our past light cone using $C(w_0, z)$ and $\mu(w_0, z)$, or equivalently $\omega_0(z)$, in order to determine the null radial coordinate y as a function of the redshift z. This step is described in detail by Stoeger *et al.* [5]. In principle, we could maintain the choice $y = z$, but this leads to analytically impossible integrations later. We now can write our data functions as functions of y — thus we have $C(w_0, y)$ and $\mu(w_0, y)$, and therefore $\omega_0(y)$.

We now substitute Eqs. (29) and (30) into the momentum conservation equation (16) and solve it algebraically for $\dot{C}'/\dot{C}$ to obtain

$$\frac{\dot{C}'}{\dot{C}}=\frac{(FF'-\omega_0'C^{-1})(2F^2-2F\sqrt{G}-1-2C^{-1}\omega_0)+\omega_0C^{-2}C'F(F-2\sqrt{G})}{GF(F-\sqrt{G})} \tag{35}$$

where

$$G\equiv F^2-1-2\omega_0/C. \tag{36}$$

This equation was originally derived by Humphreys [7] in a different way — essentially by evaluating the integral in Eq. (31), differentiating it down the light rays, substituting from Eq. (30) and differentiating by w. Equation (35) can also be obtained from the null Raychaudhuri equation (21) by writing it completely in terms of $C(w,y)$, using, once again, Eqs. (29) and (30). Finally, it can be shown with some extensive calculation that this key equation (35) is the first integral of Eq. (25), which, of course, can be written as a third-order partial differential equation in $C(w,y)$. The other two basic equations, Eqs. (23) and (24) reduce to algebraic identities we already know, when we try to turn them into equations just for $C(w,y)$. Thus they are of no further use to us. We can see, then, from this brief inventory of the equations constituting this formulation of Einstein's spherically symmetric equations for dust, that is Eqs. (17), (18), and Eqs. (21) to (25), that we will have either used or satisfied all the equations available, once we have solved Eq. (35).

Now, how do we go about integrating this $\dot{C}'$ equation (35)? As Humphreys [7] first noticed, and we can see from our calculations, it can be solved as an ordinary differential equation for $\dot{C}$, with w held constant. That is, since w and y are independent coordinates we fix $w=w_0$ and integrate the above equation with respect to y to obtain $\dot{C}(w_0,y)$. This is a crucial stage in the integration procedure. In particular, since we know $C(w_0,y)$ and $C'(w_0,y)$ we can carry out this integration on our past light cone, $w=w_0$. In determining this first integral we also need to include an integration constant which implements the proper behavior of $\dot{C}(w_0,y)$ at $y=0$, that is, on our central world line [Eq. (5)]: $\dot{C}(w_0,y)\to 0$.

This same procedure can be followed in obtaining a first integral in the general SS case, in which we do not impose spatial flatness (see below). But now the $\dot{C}'$ equation is a bit more complicated: $F(y)$ no longer equals 1, but must be found in a previous step using the procedures indicated in Maartens *et al.* [6] (see especially the erratum). In this case, of course, $C(w_0,y)$ and $\omega_0(y)$ are no longer interdependent, but constitute independent data parameters. Except for these nontrivial, but inessential changes, obtaining the first integral of Eq. (35) on our past light cone is the same as in the flat case.

Before we can proceed to integrate this resulting first-order $\dot{C}$ equation to obtain $C(w,y)$ we have to examine the central behavior of $\dot{C}(w_0,y)$ in somewhat more detail to determine the explicit dependence $\dot{C}$ on the variable w, which was set to w_0 in the previous integration. We made this choice because we know all the relevant quantities in Eq. (35) as functions of y on $w=w_0$. That is, there is hidden in our expression for $\dot{C}(w_0,y)$ an implicit dependence on w_0. We need to extract that dependence and make it explicit, so that we can then determine the general dependence of $\dot{C}$ on w. Knowing $\dot{C}(w_0,y)$ explicitly in terms of w_0 and y enables us to write $\dot{C}(w,y)$, if we can determine which part of the w_0 dependence of $\dot{C}(w_0,y)$ translates into w dependence (see below) when we let $\dot{C}$ move from light cone to light cone. This is true because w and y are independent coordinates.

The main complication is that, besides the w_0 dependence arising from the $w=w_0$ choice we made just before finding the first integral of Eq. (35), there is another part of the w_0 dependence which derives from previous integration constants and remains through the entire problem. This w_0 dependence *does not* translate into w dependence when we free $\dot{C}$ to move from light cone to light cone. These two w_0 dependences must be disentangled. How exactly is this done?

We do this by employing the full description of the central behavior of $\dot{C}$, that is [Eq. (5)]

$$\dot{C}(w,y)\to\dot{B}(w,0)y=0,y\to 0. \tag{37}$$

(In order to follow this brief description of the procedure, it may be a help to see each step exemplified in the FLRW case detailed in Sec. VII below.) Applying this to our $\dot{C}$ expression, we isolate the ''$\dot{B}(w_0,y)$'' part, which is trivially done—just divide out y. Then, setting $y=0$, we find $\dot{B}(w_0,0)$. This will give a constant. The two parts of the w_0-dependence are hidden in this constant—that is, the free w part which was set to w_0 in order to do the y-integration, and the w_0 part which was constant from the beginning. We can then choose to relate the two parts of this w_0-dependence in any way we wish. This choice of w is essentially the choice of the w parameter along the central world line, which is the one coordinate freedom we have not yet used. It is often, as indicated above, expressed as the freedom to choose the dependent metric variable $A(w,y)$ on our world line — that is, the freedom to choose $A(w,0)$. But making the choice in this way here effectively sets $A(w,0)$: Knowing $\dot{B}(w,0)$ determines $A'(w,0)$ through the conservation equation $A'+\dot{B}=0$. And from $A'(w,0)$, $A(w,0)$ itself is determined.

In this exact spherically symmetric dust case our integration scheme is valid for moving into the future, as well as for moving into the past. This is not true in general, nor even for perturbations away from spherical symmetry, for which schemes analogous to the one presented here provide solutions only in the past Cauchy development of that part of our past light cone $C^-(p_0)$ on which we have data [17,1]. In the exact spherically symmetric dust case this restriction does not hold, because there can be no gravitational waves coming in along future past light cones (spherical symmetry). Furthermore, we have effectively eliminated photons and sound waves, because we have restricted the problem to pressure free matter. The only characteristics in the problem are timelike world lines of the matter itself.

Having obtained $\dot{C}(w,y)$ by this procedure, we can then do the second integral, with respect to w, to obtain $C(w,y)$, being careful to use the value of $C(w_0,y)$ we already have as an initial-value condition. From this we can now easily determine $A(w,y)$ and $B(w,y)$ from Eqs. (29) and (30).

VII. INTEGRATION WITH FLAT FLRW DATA

Here we briefly illustrate the integration procedure described above by beginning with FLRW data to find the FLRW solution in observational coordinates. For simplicity we shall again restrict ourselves to the flat case, for which $F(y)=1$. Because of this restriction we need only the observer area distance, or the galaxy number counts — not both. $F=1$ establishes a relation between these data functions. In the flat case, the FLRW data have the form (Stoeger *et al.* [5])

$$r_0(z)=2H_0^{-1}(1+z)^{-2}\{z+1-(z+1)^{1/2}\} \tag{38}$$

and

$$M_0(z)=3H_0(1+z)^{1/2}, \tag{39}$$

where H_0 is the Hubble parameter measured at $w=w_0$ and $y=0$.

Solving the null Raychaudhuri equation with this data (see Stoeger *et al.* [5]) yields the following relation between redshift and the null coordinate y

$$(1+z)=\frac{1}{(1+\alpha y)^2} \tag{40}$$

$$\alpha\equiv\frac{H_0A_0}{2}. \tag{41}$$

The observer area distance $C(w_0,y)$ and ω_0 as functions of y are then given by

$$C(w_0,y)=2H_0^{-1}(1-\alpha y)^2\alpha y \tag{42}$$

$$\omega_0=-4H_0^{-1}(\alpha y)^3. \tag{43}$$

Substituting Eqs. (42), (43) and $F=1$ into Eq. (35) gives

$$\frac{\dot{C}'}{\dot{C}}=\frac{2\alpha y-1}{y(\alpha y-1)}. \tag{44}$$

Integrating the above equation yields

$$\dot{C}(w_0,y)=y(\alpha y-1). \tag{45}$$

This already obeys the central condition that $\dot{C}\to 0$ as $y\to 0$.

From the central conditions we also have that

$$\lim_{y\to 0}\dot{C}(w_0,y)=\lim_{y\to 0}y\dot{B}(w_0,0). \tag{46}$$

Therefore,

$$\lim_{y\to 0}\alpha y-1=\dot{B}(w_0,0)\Rightarrow\dot{B}(w_0,0)=-1. \tag{47}$$

Somehow the two w_0 dependences we discussed above are hidden in this constant. We are free (the remaining coordinate freedom in w) to make any choice which is consistent with this central behavior. The obvious ones are $\dot{B}(w,0)=w$, with $w_0=1$, $\dot{B}(w,0)=w_0/w$, or $\dot{B}(w,0)=w/w_0$. We choose the last. With this, together with our knowledge that for the flat case $\alpha=1/w_0$, we then have

$$\dot{C}(w,y)=y(y-w)/w_0. \tag{48}$$

Now keeping y constant we can integrate this to obtain

$$C(w,y)=y(y-w)^2/2w_0, \tag{49}$$

which is the correct expression for FLRW in observational coordinates. Using this expression we can obtain A and B from Eqs. (29) and (30).

VIII. CONCLUSION

In this paper we have summarized the spherically symmetric field equations for dust in observational coordinates and their integration with data on our past light cone $C^-(p_0)$, focusing on completing that integration by indicating an analytic procedure for solving the momentum conservation equation [Eq. (35)] for $C(w,y)$. This involves finding the first integral on $C^-(p_0)$ — that is for $w=w_0$ — and then recovering the dependence of $\dot{C}$ on w by using the central conditions. Once this is accomplished the second integration to determine $C(w,y)$ itself is straightforward. Determination of the other two remaining metric variables $A(w,y)$ and $B(w,y)$ then follow from Eqs. (29) and (30), respectively. We have also fully characterized the relationships among the original equations, showing that all of the information that is contained in them has been used. Finally, we have illustrated this procedure by beginning with FLRW observational data in the flat case, and integrating to find the FLRW metric in observational coordinates.

In a subsequent paper, we shall discuss the perturbed spherically symmetric case in detail.

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