Comment on entropy bounds and the generalized second law

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(Received 13 January 1999; published 15 October 1999)

In a gedanken experiment in which a box initially containing energy E and entropy S is lowered toward a black hole and then dropped in, it was shown by Unruh and Wald that the generalized second law of black hole thermodynamics holds, without the need to assume any bounds on S other than the bound that arises from the fact that entropy at a given energy and volume is bounded by that of unconstrained thermal matter. The original analysis by Unruh and Wald made the approximation that the box was ''thin,'' but they later generalized their analysis to thick boxes (in the context of a slightly different process). Nevertheless, Bekenstein has argued that, for a certain class of thick boxes, the buoyancy force of the ''thermal atmosphere'' of the black hole is negligible, and that his previously postulated bound on S/E is necessary for the validity of the generalized second law. In arguing for these conclusions, Bekenstein made some assumptions about the nature of unconstrained thermal matter and the location of the ''floating point'' of the box. We show here that under these assumptions, Bekenstein's bound on S/E follows automatically from the fact that S is bounded by the entropy of unconstrained thermal matter. Thus, a box of matter which violates Bekenstein's bound would violate the assumptions made in his analysis, rather than violate the generalized second law. Indeed, we prove here that no universal entropy bound need be hypothesized in order to ensure the validity of the generalized second law in this process. [S0556-2821(99)01320-X]

PACS number(s): 04.70.Dy, 04.62.+v

I. INTRODUCTION

A cornerstone of black hole thermodynamics is the *generalized second law* (GSL), which asserts that in any process, the *generalized entropy*

$$S' = S + S_{\rm bh} \tag{1}$$

never decreases, where S denotes the entropy of matter outside of black holes and $S_{\rm bh} = \mathcal{A}/4$, where $\mathcal{A}$ denotes the total surface area of the black hole horizons. (Here and throughout this paper we use units where $G = c = \hbar = k = 1$.) The validity of the GSL is essential for the consistency of black hole thermodynamics and for the interpretation of $\mathcal{A}/4$ as representing the physical entropy of a black hole.

It was already recognized at the time the GSL was first postulated that a potential difficulty arises when one lowers a box initially containing energy E and entropy S toward a black hole [1]. Classically, a violation of the GSL can be achieved if one lowers the box sufficiently close to the horizon. A resolution of this difficulty was proposed by Bekenstein by postulating that the entropy to energy ratio of any matter put into a box must be subject to the universal bound [2]

$$S/E \leq 2\pi R \tag{2}$$

where E denotes the energy in the box, and R denotes some suitable measure of the size of the box. Naively, at least, such a bound would rescue the GSL by preventing one from lowering a box close enough to a black hole to violate it.

However, an alternative resolution of the apparent difficulty with the GSL was given in [3]. There it was noted that there is a quantum ''thermal atmosphere'' surrounding a black hole, which produces a large ''buoyancy force'' on a box when it is slowly lowered very close to the horizon. When this buoyancy force is taken into account, the optimal place to drop such a box into a black hole no longer is at the horizon of the black hole but rather at the ''floating point'' of the box, which lies at a finite distance from the horizon. When the effects of the buoyancy force on the energy balance are properly taken into account, it was found that the GSL always holds in this process [3].

The analysis of [3] assumed, for simplicity, that the box was ''thin'' in the sense that its proper height, b, is small compared with the scale of variation of the redshift factor, χ, i.e., $b \ll \chi (d\chi/dl)^{-1}$, where l denotes proper distance from the horizon. This analysis was then generalized to the case of ''thick boxes'' in [4], although this generalization was done in the context of a slightly different process, wherein, rather than dropping the box into the black hole, the contents of the box are allowed to ''leak out'' as the box is raised (see also [5]). Nevertheless, several years ago Bekenstein argued [6] that for boxes with b at least as large as $A^{1/2}$ (where A denotes the horizontal cross-sectional area of the box), the buoyancy effects of the thermal atmosphere are negligible. He then showed that the bound (2) must hold for such boxes in order that the GSL be valid, in apparent contradiction with the conclusions of the analysis of [3,4].

The purpose of this paper is to resolve this apparent contradiction. In the course of his analysis, Bekenstein [6] made some assumptions concerning the nature of unconstrained thermal matter and the location of the floating point of the box. Under these assumptions, it is indeed necessary for the validity of the GSL that the bound (2) hold, as Bekenstein found. However, we shall show that Eq. (2) holds *automatically* as a consequence of the same assumptions used to show that it is necessary for the validity of the GSL. In other words, if one had matter which violated Eq. (2), then Bekenstein's assumption about the nature of unconstrained thermal matter and/or his assumption about the location of the floating point of the box could not be correct.

In the next section, we show that—whether or not Eq. (2) is satisfied—the GSL holds in any process where a (possibly ''thick'') box, initially containing energy E and entropy S, is lowered toward a black hole and then dropped in. Bekenstein's arguments are then analyzed in Sec. III.

II. VALIDITY OF THE GSL FOR ''THICK'' BOXES

It was shown in [3] that the bound (2) is not needed for the validity of the GSL for the case of a ''thin'' box. The analysis of [3] was generalized to ''thick'' boxes in the Appendix of [4]. However, in [4] a slightly different process was considered (in response to criticisms of [7]), in which the contents of the box are allowed to slowly leak out as the box is raised. Consequently, the formulas of [4] are not immediately applicable to the present situation where the box is dropped into the black hole. Thus, in this section, we shall extend the analysis and arguments given in the Appendix of [4] to the present case. For simplicity we restrict attention here to the case of a static (as opposed to stationary) black hole.

To begin, in a given region of space outside of the black hole, *unconstrained thermal matter* is *defined* to be the state of matter that maximizes entropy at a fixed energy (as measured at infinity).[1] It should be noted that the properties of unconstrained thermal matter may depend upon location; i.e., for unconstrained thermal matter the functional dependence of the entropy density, s, on energy density, e (measured locally by a static observer), may vary with position outside of the black hole. We make two assumptions about unconstrained thermal matter: (i) We assume that unconstrained thermal matter is (locally) homogeneous, so that the integrated Gibbs-Duhem relation holds [3]:

$$e+P-Ts=0 \tag{3}$$

where T is the temperature of the unconstrained thermal matter, and P is its pressure. (ii) We assume that the ''thermal atmosphere'' of a black hole is described by unconstrained thermal matter, with the locally measured temperature given by $T=T_{\rm bh}/\chi$, where $T=T_{\rm bh}=\kappa/2\pi$ is the Hawking temperature of the black hole. Both of these assumptions were also made in Bekenstein's analysis [6].

Following [4] and [6], we now compute the change in generalized entropy occurring when a thick box containing matter is slowly lowered toward a black hole and then dropped in. Consider a box of cross-sectional area A and height b, containing energy density ρ and total entropy S. (Here ρ includes any energy density that may be in box walls.) As the box is lowered toward the black hole, the energy density will depend on both the proper distance, l, of the center of the box from the horizon and the proper height, y, above the center of the box. Following [6], we adopt the abbreviation

$$\int f(y)dV \equiv A\int_{-b/2}^{b/2} f(y)dy. \tag{4}$$

The energy of the box as measured at infinity is

$$E_\infty(l)=\int \rho(l,y)\chi(l+y)dV \tag{5}$$

where χ is the redshift factor. The weight of the box at infinity is [3]

$$w(l)=\int \rho(l,y)\frac{\partial\chi(l+y)}{\partial l}dV. \tag{6}$$

The condition that no extra energy be fed into or taken out of the box as it is lowered is[2]

$$0=\frac{dE_\infty}{dl}-w=\int \frac{\partial\rho(l,y)}{\partial l}\chi(l+y)dV. \tag{7}$$

Thus the work done by the weight of the box on the agent lowering it is

$$W_g(l)=-\int_\infty^l w(l')dl'=E_i-\int \rho(l,y)\chi(l+y)dV \tag{8}$$

where E_i is the initial energy of the box.

Meanwhile, the thermal radiation exerts a buoyancy force on the box equal to

$$f_b(l)=A[(P\chi)_{l-b/2}-(P\chi)_{l+b/2}] \tag{9}$$

where P is the radiation pressure of the unconstrained thermal matter. The work done by the buoyancy force on the agent at infinity is then

$$W_b(l)=-\int_\infty^l f_b(l')dl'=-\int P(l+y)\chi(l+y)dV. \tag{10}$$

If the box is dropped into the black hole from position l, the increase in black hole entropy will be

$$\Delta S_{\rm bh}=\frac{1}{T_{\rm bh}}(E_i-W_g-W_b)$$
$$=\frac{1}{T_{\rm bh}}\int [\rho(l,y)+P(l+y)]\chi(l+y)dV. \tag{11}$$

[1]By contrast, the terminology ''thermal matter'' would be used to denote matter which is in thermal equilibrium but which may have additional ''constraints'' resulting, e.g., from the presence of box walls (which may exclude some modes of excitation of the matter) or restrictions on the species of particles that are present.

[2]If the box is filled with matter in thermal equilibrium, then the temperature in the box will follow the Tolman law $T\propto 1/\chi$. Using $d\rho=Tds$ (and, hence, $\partial\rho/\partial l=T\partial s/\partial l$), we see that Eq. (7) is equivalent to requiring that the entropy of the box remain constant as it is lowered.

Using Eq. (3) together with $T = T_{bh}/\chi$, we obtain

$$\Delta S_{bh} = \frac{1}{T_{bh}} \int [\rho(l,y) - e(l+y)]\chi(l+y)dV + S_{th} \quad (12)$$

where S_{th} is the entropy of the thermal radiation displaced by the box. Equation (12) is equivalent to Eq. (20) of [6] and it corresponds directly to Eq. (A12) of [4] for the process considered in that reference.

It also follows from Eq. (3) together with $T = T_{bh}/\chi$ that

$$d(P\chi) = -e d\chi. \quad (13)$$

Minimizing ΔS_{bh} with respect to l, and using Eqs. (7) and (13), we obtain

$$\int [\rho(l_0,y) - e(l_0+y)] \frac{\partial \chi(l+y)}{\partial l} dV = 0. \quad (14)$$

Thus, the entropy increase of the black hole is minimal when the contents are dropped in from the ''floating point,'' i.e. when the weight of the box is equal to the weight of the displaced thermal radiation. Equation (14) is identical to Eq. (14) of [6] and Eq. (A13) of [4].

To proceed further, we first consider an idealized situation in which we imagine that the box is filled with unconstrained thermal matter and the energy in the box walls is negligible.[3] Let T_0 denote the temperature of the matter in the box at the start of the process. Then, when lowered to position l, the matter in the box will have a temperature distribution $T = T_\infty(l)/\chi$, where $T_\infty(l)$ is determined by T_0 and Eq. (7). According to our analysis above, the optimal place (in the sense of minimizing ΔS_{bh}) to drop such a box is at its ''floating point,'' which is easily seen to be the position, l_0, at which $T_\infty(l_0) = T_{bh}$, since at this position we have $\rho = e$. By Eq. (12), when the box is dropped into the black hole from its floating point, $l = l_0$, we have

$$\Delta S_{bh} = S_{th} = S \quad (15)$$

and there is no change in the generalized entropy. Consequently, if the box is dropped from *any* position, l, we have

$$\Delta S' \geqslant 0 \quad (16)$$

and the GSL holds in this idealized process.

Now consider the actual process in which the box contains some (arbitrary) distribution of matter, is lowered to an arbitrary position l (not necessarily the floating point of the box) and then is dropped into the black hole. Let us compare the change in generalized entropy in this process with the change in generalized entropy that would occur in the above idealized process where we choose T_0 so that at position l the energies as measured at infinity, E_∞, of the two boxes agree. Then, it follows immediately from Eqs. (5) and (12) that the change in black hole entropy, ΔS_{bh}, is the same for both processes. However, since the boxes have the same energy at infinity and occupy the same region of space, the entropy, S, contained in the box in the actual process cannot be larger than the entropy contained in the box in the idealized process. Consequently, the change in generalized entropy in the actual process cannot be smaller than the change in generalized entropy in the idealized process, which was shown above to be non-negative. This proves that the GSL cannot be violated in the actual process.

III. BEKENSTEIN'S ANALYSIS

In [6] Bekenstein purports to show that for thick boxes whose ''height,'' b, is not small compared with $A^{1/2}$ (where, as above, A denotes the horizontal cross-sectional area of the box), the contents of the box must satisfy the entropy bound (2) if the GSL is to hold. We now briefly review Bekenstein's assumptions and conclusions, and then reconcile them with the results of the previous section.

In his analysis, Bekenstein assumes that unconfined thermal matter can be modelled as an N-species mixture of noninteracting massless particles, so that

$$P = \frac{e}{3} = \frac{N\pi^2 T^4}{45}. \quad (17)$$

Bekenstein then makes the approximation[4] that

$$\chi(l) \approx \kappa l \quad (18)$$

where κ denotes the surface gravity of the black hole. Using this approximation, Bekenstein finds that the exact floating point condition (14) reduces to

$$\frac{(l_0^2 - b^2/4)^3}{3 l_0^2 b^4 + b^6/4} = \frac{NA}{720\pi^2 E(l_0) b^3} \quad (19)$$

where $E(l_0) = \int \rho dV$ is the locally measured energy of the box at the floating point.

Bekenstein then argues that at the floating point, the quantity

$$\eta^3 \equiv \frac{NA}{720\pi^2 E(l_0) b^3} \quad (20)$$

must satify $\eta \ll 1$. In making this argument, Bekenstein makes two additional assumptions: (1) that $b \gg 1/E$ and (2) that N is of order unity. (It is easy to see that these assumptions together with $A \lesssim b^2$ imply $\eta \ll 1$.) However, these as-

[3]It should be emphasized that we are not assuming here that it is physically realistic to actually have a box filled with unconstrained thermal matter. The consideration of such a box is done here purely for mathematical purposes, to compare the generalized entropy change that would occur in this idealized process to that which occurs in the actual process (see below).

[4]Equation (18) is a good approximation sufficiently near the black hole. Bekenstein's justification for this approximation is somewhat circular in nature, but Eq. (18) is not the source of any difficulties in Bekenstein's analysis.

sumptions are not innocuous ones since, in conjunction with Eq. (17) they would imply that entropy bound (2) is already satisfied by a wide margin for a box in Minkowski spacetime. Namely, since the box must have lower entropy than unconstrained thermal matter at the same energy and volume, we have, for the model of unconstrained thermal matter assumed by Bekenstein,

$$\frac{S}{E} \leqslant \left(\frac{S}{E}\right)_{\rm th} \sim \frac{1}{T}. \tag{21}$$

Hence, given that $b \gg 1/E$, $N \sim 1$, and $A \lesssim b^2$, we have

$$E \sim AbT^4 \gg \frac{1}{b}, \tag{22}$$

from which it follows that

$$\frac{S}{E} \ll (Ab^2)^{1/4} \lesssim b = 2R. \tag{23}$$

Nevertheless, Bekenstein's arguments correctly show that—irrespective of the above two additional assumptions—if the floating point of the box is very close to the horizon [in which case, by Eqs. (19) and (20), we have $\eta \ll 1$], then buoyancy effects are negligible, and the bound (2) is needed for the validity of the GSL. However, we now show that if unconstrained thermal matter is described by Eq. (17), then any box that floats very close to the horizon must automatically satisfy Eq. (2). Once again, we use the fact that unconstrained thermal matter maximizes entropy at a fixed volume and energy at infinity,

$$S(E_\infty, l_0) \leqslant S_{\rm th}(E_\infty, l_0). \tag{24}$$

The unconstrained thermal matter is described by Eq. (17) with $T = T_\infty(l_0)/\chi$, where $T_\infty(l_0)$ is determined by imposing $\int e\chi dV = E_\infty$. Evaluating this integral using the approximation (18), we find

$$[T_\infty(l_0)]^4 = \frac{15(l_0^2 - b^2/4)^2 \kappa^3 E_\infty}{N\pi^2 Abl_0}. \tag{25}$$

The entropy density of the thermal radiation is $s = 4e/3T$, so

$$S_{\rm th}(E_\infty) = \frac{4E_\infty}{3T_\infty(l_0)}. \tag{26}$$

It is convenient to express E_∞ in terms of the position, $l_{\rm c.m.}$, of the center of mass of the box. Again applying the $\chi \propto l$ approximation, we obtain the simple relation

$$l_{\rm c.m.} \equiv \frac{\int (l+y)\rho dV}{E(l_0)} = \frac{E_\infty}{\kappa E(l_0)}. \tag{27}$$

By Eqs. (24), (26) and (27), we have

$$\frac{S}{E} \leqslant \frac{8\pi}{3}\left(\frac{T_{\rm bh}}{T_\infty(l_0)}\right) l_{\rm c.m.}, \tag{28}$$

and from Eq. (25) and the definition, Eq. (20), of η, we find

$$\left(\frac{T_{\rm bh}}{T_\infty(l_0)}\right)^4 = \frac{3\eta^3 b^4 l_0}{(l_0^2 - b^2/4)^2 l_{\rm c.m.}}. \tag{29}$$

Now, assuming $\eta \ll 1$, the floating point condition (19) yields $l_0^2 \approx (1/4 + \eta)b^2$. Consequently,

$$\frac{T_{\rm bh}}{T_\infty(l_0)} \approx \left(3\eta \frac{l_0}{l_{\rm c.m.}}\right)^{1/4}, \tag{30}$$

and, finally, to leading order in η,

$$\frac{S}{E} \leqslant \frac{8\pi}{3}(3\eta l_{\rm c.m.}^3 l_0)^{1/4} \leqslant \frac{8\pi}{3} b(3\eta)^{1/4} \ll b = 2R. \tag{31}$$

Thus, we see that if the box floats very near the horizon, it follows that the entropy bound (2) is already satisfied by a wide margin. Consequently, the bound (2) does not have to be postulated as an additional requirement.

ACKNOWLEDGMENTS

This research was supported in part by NSF grant PHY 95-14726 and ONR grant N00014-96-1-0127 to the University of Chicago.

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