G_1 spacetimes with gravitational and scalar waves

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(Received 21 December 1998; published 15 October 1999)

A new algorithm to generate families of inhomogeneous massless scalar field spacetimes is presented. New solutions to Einstein field equations, having a single isometry, are generated by breaking the homogeneity of massless scalar field G_2 models along one direction. As an illustration of the technique, spacetimes which in their late time limit represent perturbations in the form of gravitational and scalar waves propagating on a non-static inhomogeneous background are considered. Several features of the obtained metrics are discussed, such as their early and late time limits, structure of singularities and physical interpretation.
[S0556-2821(99)09518-1]

PACS number(s): 04.20.Jb, 04.30.−w, 98.80.Hw

I. INTRODUCTION

The high degree of isotropy observed in the Universe on large scales today is usually combined with the Copernican principle to justify the assumption of homogeneity on the same scales. However, there is no known reason to assume that the Universe was isotropic nor homogeneous at very early epochs. The puzzling question of how the Universe might have evolved from an initially irregular state into the current isotropic and apparently homogeneous state lacks a complete answer at present. To date, several regularization mechanisms have been put forward, such as Misner's chaotic cosmological program [1,2], the standard inflationary scenario [3–6] and, more recently, the alternative pre-big bang inflationary scenario [7]. However, none of these is completely satisfactory, and in general one cannot know for certain which range of initial conditions could have allowed the Universe to evolve into its present form. Such scenarios can only provide one with partial indications of what initial conditions would have led a generic universe into the one observed at present. One way to maximize the amount of information obtainable from any cosmological program, such as those mentioned above, relies on studying the evolution of models with as many degrees of freedom as possible. This idea has motivated the special attention paid to inhomogeneous cosmological models in the last decades (see Krasinski [8] for a review).

In general, attempts to obtain new inhomogeneous metrics involve some symmetry assumption, so that the full implications of the non-linearity of the theory are compromised to some extent. In addition, non-vacuum spacetimes are required to have a physically meaningful matter content, so that they portray realistic situations. Motivated by the possibility of the existence of non-trivial massless scalar fields in the early universe, I will concern myself here with non-static inhomogeneous solutions to the Einstein equations induced by such matter sources. In particular, I present a new algorithm to generate families of massless scalar field G_1 metrics, i.e. time-dependent spacetimes with a single isometry. These new sets of solutions will be generated starting from generalized vacuum Einstein-Rosen spacetimes, which admit an Abelian group of isometries G_2 acting transitively on spacelike surfaces. In the last decades there has been intensive study of G_2 vacuum and matter filled cosmological models and several major reviews on the subject have been written [9,10,8].

Given the large number of known G_2 cosmologies and the various available techniques to generate further ones, algorithms transforming such spacetimes into G_1 metrics represent powerful tools for generating new inhomogeneous solutions. Explicit examples of G_1 metrics are rare, and, therefore, new solutions of this kind are significant items to the collection of exact solutions of Einstein's field equations. The new algorithm displays the nice property of reducing the symmetry while keeping the type of matter source unaltered. Nonetheless, the input and output solutions may not admit the same physical interpretation, or even share some of their relevant features. For this reason, if one wishes to grasp the physical meaning of every new solution, an independent analysis of it will have to be carried out.

Herein, I construct and analyze G_1 models representing the propagation of gravitational and matter waves on a spatially curved non-static background. The study of primordial inhomogeneities in the form of waves is an active area of research. This is motivated by the fact that wave-like primordial perturbations originating from vacuum fluctuations during inflation may be responsible for structure formation. Unlike other types of inhomogeneities formed in the early universe, they would have remained nearly unaltered up to the present, therefore allowing the possibility of their detection.

The exact inhomogeneous G_1 spacetimes seeded by gravitational and matter waves studied here represent a generalization of more symmetric configurations considered by Charach and Malin [11], because in those models the background hosting the waves was homogeneous. Another important difference between the two sets of solutions is that the new models are manifestly degenerate at $x=0$, and for that reason, it seems inadequate to refer to them as cosmologies.

In the context of colliding plane waves, G_2 diagonal spacetimes with waves of scalar and gravitational nature have also been considered. Spacetimes such as those studied by Wu [12] or any solution generated by the methods of

Barrow [13] and Wainwright *et al.* [14] could be taken as starting points to construct new G_1 metrics modeling interactions of waves on a curved background.

Furthermore, interest in this generation procedure is not restricted to the relativistic framework; solutions to Einstein's equations with a massless scalar field (MSF) may be used to generate solutions to alternative theories of gravity, such as Brans-Dicke theory or string theory in its low energy limit. In the latter case, one could even take those spacetimes to generate new solutions with other massless modes in the characteristic spectrum of the theory as recently discussed by Clancy *et al.* [15].

The plot of the paper is as follows: First, the G_1 massless scalar field solution generating algorithm is introduced. Then, I construct new inhomogeneous metrics starting from a infinite dimensional family of solutions which in their WKB limit admit an interpretation in terms of waves. It will be shown that at early times these solutions behave like a Belinskii-Khalatnikov generalized model [16] with homogeneity broken along one spatial direction. The structure of spacelike singularities of the new solutions in the early time limit will be analyzed as well, and the special features due to the presence of the matter source will be indicated. Next, the solutions will be considered in their high frequency limit and it will be shown that these models can be thought of in the same physical terms as their G_2 counterparts. In particular they represent spacetimes with a spatially inhomogeneous background filled with a scalar field and a null fluid of ''gravitons'' and ''scalar particles.'' In this regime, the scalar field is made of the addition of a time dependent term and another one depending on a spatial coordinate. As time grows the null fluid's contribution to the energy-momentum tensor grows faster than the one associated with the homogeneous part of the scalar field generating the background geometry. However, that will not be necessarily the case with the contribution of the inhomogeneous part of the scalar field; initial conditions will determine whether at very late times the matter content corresponds to a null fluid, a scalar field, or a combination of both. Finally, the main conclusions will be outlined.

II. SOLUTION GENERATION ALGORITHM

Basically, the new generating technique is a prescription to break homogeneity along one direction in G_2 MSF cosmologies, ending up with new spacetimes possessing a single isometry, but having the same type of matter content. Remarkably, the pioneering investigations on matter filled universes of Einstein-Rosen type, carried out respectively by Tabenski and Taub [17] and Liang [18] considered models with noninteracting scalar fields, even though at the time there was no clear physical motivation.

According to a conjecture due to Belinskii, Lifshitz and Khalatnikov (BLK) [19–24], G_2 metrics seem to be specially relevant for the description of the early universe, as such solutions give the leading approximation to a general solution near the singularity at $t=0$. Their claim has recently found the support of numerical results [25–31]. In particular BLK considered approximate Einstein-Rosen solutions and performed an analysis of their local behavior in the early and late time regime. An interesting result reached in the course of those investigations, which is specially relevant for the present paper, was the prediction of a high frequency gravitational wave regime in the late epochs of the Universe.

Before going any further it is convenient to explain how G_2 spacetimes induced by a MSF can be generated starting from vacuum solutions to Einstein's equations with the same symmetry. For the sake of simplicity, the discussion will be restricted here to a particular case of a well-known general procedure [11,13,14,32–35]. The generic diagonal line-element with G_2 symmetry will be taken as a starting point:

$$ds_v^2=e^{f_v(t,z)}(-dt^2+dz^2)+G_v(t,z)(e^{p_v(t,z)}dx^2+e^{-p_v(t,z)}dy^2), \tag{1}$$

where the subscript v stands for vacuum. A new solution $g_{\mu\nu}$ of the Einstein equations with a massless scalar field φ as a source, and line element:

$$ds^2=e^{f(t,z)}(-dt^2+dz^2)+G(t,z)(e^{p(t,z)}dx^2+e^{-p(t,z)}dy^2), \tag{2}$$

can be obtained by the following transformations:

$$G=G_v, \tag{3a}$$

$$p=Bpv_v+C\log G_v, \tag{3b}$$

$$f=f_v+Ep_v+F\log G_v, \tag{3c}$$

$$\varphi=Ap_v+D\log G_v; \tag{3d}$$

provided that the constants *A*, *B*, *C*, *D*, *E* and *F* are subject to the constraints:

$$BC+2AD=E, \tag{4a}$$

$$C^2+2D^2=2F, \tag{4b}$$

$$B^2+2A^2=1. \tag{4c}$$

Conditions (4) arise by demanding the following equations to be satisfied:

$$R_{\mu\nu}=\varphi_{,\mu}\varphi_{,\nu}, \tag{5a}$$

$$\nabla^\gamma\nabla_\gamma\varphi=0. \tag{5b}$$

In principle, a large number of new MSF G_2 cosmologies can be obtained by simply applying the procedure sketched above to any of the representatives of the populated family of vacuum Einstein-Rosen spacetimes. However, generating MSF solutions with a lower degree of symmetry is a more cumbersome task. At this point, attention should be drawn to a method given by Feinstein *et al.* [36], which allows one to generate families of solutions with a two-dimensional degree of inhomogeneity and a self-interaction term for the massless field of the form

$$V = V_0(\lambda) e^{-\lambda\varphi}. \tag{6}$$

With this procedure new metrics are obtained by introducing an x-dependent conformal factor on an input G_2 metric, and where in general the potential term only vanishes for $|\lambda| = 6$. This difficulty in canceling the self-interaction term for the scalar field can be traced back to the highly symmetric x-dependence of the models.

In order to find a prescription for introducing an additional degree of symmetry in MSF G_2 metrics without switching on a potential in the process, I have considered the possibility of allowing the metric to depend on x in a more general way. In particular, I have sought metrics of the form:

$$d\tilde{s}^2 = \Omega(x) e^{f(t,z)}(-dt^2 + dz^2) + G(t,z)[e^{p(t,z)}dx^2 + \Xi(x) e^{-p(t,z)} dy^2], \tag{7}$$

and made the following ansatz for the scalar field:

$$\tilde{\varphi}(t,z,x) = \varphi(t,z) + \Lambda(x). \tag{8}$$

Note that the massless scalar field case included in the solutions of Feinstein *et al.* is also a particular case of the models here. Since the requirement here is that no potential should arise in the transformations, the equations that must hold are

$$\tilde{R}_{\mu\nu} = \tilde{\varphi}_{,\mu}\tilde{\varphi}_{,\nu}, \tag{9a}$$

$$\tilde{\nabla}^{\gamma}\tilde{\nabla}_{\gamma}\tilde{\varphi} = 0. \tag{9b}$$

Explicitly, Eq. (9a) is equivalent to the set of equations:

$$\tilde{R}_{00} = R_{00} + e^{f-p}\frac{\Xi_{,x}\Omega_{,x} + 2\Xi\Omega_{,xx}}{4G\Xi} = \tilde{\varphi}_{,t}^2, \tag{10a}$$

$$\tilde{R}_{11} = R_{11} - e^{f-p}\frac{\Xi_{,x}\Omega_{,x} + 2\Xi\Omega_{,xx}}{4G\Xi} = \tilde{\varphi}_{,z}^2, \tag{10b}$$

$$\tilde{R}_{01} = R_{01} = \tilde{\varphi}_{,t}\tilde{\varphi}_{,z}, \tag{10c}$$

$$\tilde{R}_{22} = R_{22} + \frac{\Xi_{,x}^2 - 2\Xi\Xi_{,xx}}{4\Xi^2} + \frac{\Omega_{,x}^2 - 2\Omega\Omega_{,xx}}{2\Omega^2} = \tilde{\varphi}_{,x}^2, \tag{10d}$$

$$\tilde{R}_{12} = \frac{G_{,z}\Omega_{,x}}{2G\Omega} + \frac{\Xi_{,x}p_{,z}}{2\Xi} = \tilde{\varphi}_{,z}\tilde{\varphi}_{,x}, \tag{10e}$$

$$\tilde{R}_{02} = \frac{G_{,t}\Omega_{,x}}{2G\Omega} + \frac{\Xi_{,x}p_{,t}}{2\Xi} = \tilde{\varphi}_{,t}\tilde{\varphi}_{,x}, \tag{10f}$$

$$\tilde{R}_{33} = R_{33} + e^{-2p}\frac{\Xi_{,x}^2 - 2\Xi\Xi_{,xx}}{4\Xi} - e^{-2p}\frac{\Xi_{,x}\Omega_{,x}}{2\Omega} = 0; \tag{10g}$$

whereas Eq. (9b) in explicit form reads

$$\frac{g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi}{\Omega} + e^{-p}\frac{(\Omega\Xi^{1/2}\tilde{\varphi}_{,x})_{,x}}{\Xi^{1/2}} = 0. \tag{11}$$

Inspection of Eqs. (10) indicates that a solution to Eqs. (10), (11) is given by

$$\Omega(x) = x^k, \tag{12a}$$

$$\Xi(x) = x^n, \tag{12b}$$

$$\Lambda(x) = m \log|x|; \tag{12c}$$

subject to the following constraints on the parameters k, n and m:

$$k(2k+n-2) = 0, \tag{13a}$$

$$n(2k+n-2) = 0, \tag{13b}$$

$$k + nC = 2Dm, \tag{13c}$$

$$nB = 2Am, \tag{13d}$$

$$2m^2 = 4k - 3k^2. \tag{13e}$$

In this case, Eq. (11) reduces to

$$m(2k+n-2) = 0, \tag{14}$$

which is automatically satisfied provided that (13a),(13b), (13e) hold. Note that the parameter k must be non-negative and not larger than 4/3. Consistency of the solutions is reflected by the fact that for the choice

$$m = n = k = 0, \tag{15}$$

the set of equations (13),(14) is satisfied for any value of A, B, C and D, and the input G_2 MSF is recovered. On the one hand, for $m \neq 0$ and $k \neq 1$, if k and C are taken as free parameters, one can parametrize the solution's constants in the form:

$$n = 2 - 2k, \tag{16a}$$

$$m = \mathrm{sgn}(m)\frac{\sqrt{4k-3k^2}}{\sqrt{2}}, \tag{16b}$$

$$A = \sqrt{2}\,\mathrm{sgn}(A)\left|\frac{1-k}{2-k}\right|, \tag{16c}$$

$$B = \mathrm{sgn}(Am)\left|\frac{1-k}{2-k}\right|\frac{\sqrt{4k-3k^2}}{1-k}, \tag{16d}$$

$$D = \mathrm{sgn}(m)\frac{k + (2-2k)C}{\sqrt{8k-6k^2}}, \tag{16e}$$

$$E = \mathrm{sgn}(Am)\left|\frac{2-2k}{2-k}\right|\frac{C(2-k)^2 + (2-2k)k}{\sqrt{4k-3k^2}}, \tag{16f}$$

$$F = \frac{C^2}{2} + \frac{[k + (2-2k)C]^2}{8k-6k^2}. \tag{16g}$$

Four different subcases can be distinguished, depending on the choice of the sign of A and m. On the other hand, in the particular case $m\neq 0$, $k=1$, the constants take the values:

$$\sqrt{2}|m|=\sqrt{2}|D|=|B| \tag{17a}$$

$$A=0 \tag{17b}$$

$$C^2+1=2F \tag{17c}$$

$$E=\mathrm{sgn}(B). \tag{17d}$$

Whatever the values of the parameters, the metric $\tilde{g}_{\mu\nu}$ will only admit one Killing vector, namely $\xi_{kv}=\partial/\partial y$.

An important feature of the new models is the manifest degeneracy of their line element at $x=0$. In order to prove that this singularity is essential, and not a mere effect of the coordinate system chosen, one can compute the curvature scalars for a generic exact solution belonging to this class. Due to the inhomogeneous character of the metric it is possible in principle to have a conspiracy between the parameters of the solutions, so that on certain hypersurfaces some of those invariants are identically null, and therefore do not reveal the presence of a singularity in the spacetime. In the case here, however, either the Kretchmann scalar $K=R_{\gamma\lambda\sigma\delta}R^{\gamma\lambda\sigma\delta}$ or the Ricci scalar $R_{\sigma\delta}R^{\sigma\delta}$ are non-negative everywhere and they can be used to spot singularities. For instance, in the case of a generic solution one has

$$R_{\sigma\delta}R^{\sigma\delta}=\frac{1}{x^{2k}}\left[\frac{(\phi_t^4+\phi_z^4)}{e^{2f}}+\frac{m^2}{x^n e^{2p}}\right]. \tag{18}$$

Clearly, curvature is infinite at $x=0$. Moreover, some components of the energy-momentum tensor will become unbounded on this hypersurface as well. This suggest the presence of trapped energy on the hyperplane $x=0$, a situation that resembles the one occurring in topological defects, in which, however, the associated singularity can be regularized in some circumstances (see for e.g. [37]).

Bearing in mind the similarity between the new algorithm and the one of Feinstein *et al.*, one might wonder whether geometries like (7) can be also seeded by a scalar field with an exponential potential. Such a situation, however, will only be possible if $n=k$, which is nothing but the case already found in Ref. [36]. In order to prove this, let us consider the case where the generic geometry (7), under the constraint (12), is induced by an exponential potential $\tilde{V}(\tilde{\varphi})=V_0(\lambda)e^{-\lambda\tilde{\varphi}}$. In this case the field equations are

$$\tilde{R}_{\mu\nu}=\tilde{\varphi}_{,\mu}\tilde{\varphi}_{,\nu}+\tilde{g}_{\mu\nu}V_0e^{-\lambda\tilde{\varphi}}, \tag{19a}$$

$$\tilde{\nabla}^{\gamma}\tilde{\nabla}_{\gamma}\tilde{\varphi}=-\frac{\partial\tilde{V}(\tilde{\varphi})}{\tilde{\partial}\tilde{\varphi}}. \tag{19b}$$

It is only necessary to look at the equations for $\tilde{R}_{00}$ and $\tilde{R}_{33}$ to realize the following constraint must hold:

$$k(2-2k-n)=n(2-2k-n)=4V_0\neq 0. \tag{20}$$

Compatibility of the latter set of equations in the case of a non-vanishing potential requires $n=k$, or in other words, that the x-dependence of the metric is given by a global conformal factor, as in the case already known.

It is important to note here that the generation technique does not restrict the character of the gradient of metric function $G(t,z)$ of the input metric. This vector field determines the local behavior of the spacetime, depending on whether it is globally timelike, spacelike, null, or varies from point to point. Although in what follows I am focusing on a case with a timelike character of the gradient of $G(t,z)$, a window is left open for the study of other physically appealing cases.

Moreover, even though our generating prescription has been used to break homogeneity of an input MSF solution with a single degree of inhomogeneity, it is also possible to construct an equivalent algorithm to transform MSF static spacetimes into non-static ones. One should start with a MSF model with two commuting Killing vectors, one of them being timelike, and then generalize this solution by introducing time dependent factors in the metric and scalar field as it has been done here.

III. SPACETIMES FILLED WITH GRAVITATIONAL AND SCALAR WAVES

After having outlined our method to generate uniparametric families of G_1 spacetimes, I shall illustrate the technique by taking as a starting point the vacuum Gowdy-Berger-Misner models [38–42], defined by:

$$G_v=t, \tag{21a}$$

$$p_v=\beta\log t+\sum_{j=1}^{\infty}\cos[j(z-z_n)]\alpha_jJ_0(jt), \tag{21b}$$

$$\begin{aligned}f_v=&\frac{\beta^2-1}{2}+\beta\sum_{j=1}^{\infty}\alpha_j\cos[\alpha_j(z-z_j)]J_0(jt)\\&+\frac{t^2}{4}\sum_{j=1}^{\infty}j^2\{[\alpha_jJ_0(jt)]^2+[\alpha_jJ_1(jt)]^2\}\\&-\frac{t}{2}\sum_{j=1}^{\infty}j\alpha_j^2\cos^2[\alpha_j(z-z_j)]J_0(jt)J_1(jt)\\&+\frac{t}{2}\sum_{l=1}^{\infty}\sum_{j=1}^{\infty}\frac{lj}{l^2-j^2}\{\sin[l(z-z_l)]\sin[j(z-z_j)]\\&\times[l\alpha_l\alpha_jJ_1(lt)J_0(jt)-j\alpha_l\alpha_jJ_0(lt)J_1(jt)]\\&+\cos[l(z-z_l)]\cos[j(z-z_j)][j\alpha_l\alpha_jJ_1(lt)J_0(jt)\\&-l\alpha_l\alpha_jJ_0(lt)J_1(jt)]\}.\end{aligned} \tag{21c}$$

In this paper, however, the G_1 derived from the latter vacuum metrics solutions will only be considered in the asymptotic limits $t\sim 0$, and $jt\gg 1$ for any value of j.

Charach and Malin [11] studied MSF G_2 counterparts of the same vacuum models. These can be thought of as inhomogeneous sinusoidal perturbations of the Belinskii-

Khalatnikov homogeneous solution [16], which also has a MSF as a seed. Charach and Malin's cosmologies represent propagation of gravitational and scalar waves on an anisotropically expanding flat background. The gravitational and scalar degrees of freedom of those spacetimes satisfy linear wave equations, with the form of cylindrically symmetric waves propagating on Minkowski spacetime, for which the spatial and temporal coordinates have been interchanged. Since the solutions have a standing wave form they are fully compatible with the $S^1 \otimes S^1 \otimes S^1$ topology (three-torus). Nevertheless, the G_1 counterparts of Gowdy-Berger-Misner models cannot have the same topology as them. In the homogeneity breaking process a term proportional to $\log|x|$ is introduced in the scalar field $\tilde{\varphi}$, and the coordinate x cannot be cyclic any longer.

A. Early time behavior and singularities

From the analysis of the new G_1 solution's early time behavior it can be determined whether spacelike singularities arise at the time origin $t=0$. A look at expressions (21) shows that in the case here, the periodic inhomogeneities can be neglected in the very first stages of that spacetime's evolution. Our metrics will be then G_2 inhomogeneous spacetimes generalizing the cosmological models of Doroshkevich-Zeldovich-Novikov (DZN) [43], which have three commuting Killing vectors. In this limit, the metric and scalar field read

$$\tilde{g}_{\mu\nu} \sim \mathrm{diag}(-x^k \epsilon_1(t), x^k \epsilon_1(t), \epsilon_2(t), x^n \epsilon_3(t)) \tag{22a}$$

$$\tilde{\varphi} = \varphi_0 \log t + m \log|x| \tag{22b}$$

where

$$\epsilon_1 = t^{f_0}, \tag{23a}$$

$$\epsilon_2 = t^{1+p_0}, \tag{23b}$$

$$\epsilon_3 = t^{1-p_0}, \tag{23c}$$

and

$$p_0 = B\beta + C, \tag{24a}$$

$$f_0 = \frac{\beta^2 - 1}{2} + E\beta + F, \tag{24b}$$

$$\varphi_0 = A\beta + D. \tag{24c}$$

In addition, the following relation holds:

$$f_0 = \frac{p_0^2 - 1}{2} + \varphi_0^2. \tag{25}$$

Following Charach and Malin, the metric can be rewritten using a synchronous set of coordinates in which the new time coordinate τ is defined

$$d\tau = \sqrt{\epsilon_1(t)}\, dt; \tag{26}$$

this way the metric can be recognized as a simple inhomogeneous generalization of a Belinskii-Khalatnikov [16] solution. Their structure is very similar to that of Bianchi I (Kasner) vacuum models.

In broad terms, breakdowns of the coordinate systems, and in particular a spacelike singularity at the beginning of time, will be reflected in the behavior of the curvature invariants. Again, we choose one of the two such invariants which are non-negative everywhere, in particular the Kretchmann scalar $K = R_{\mu\nu\sigma\delta} R^{\mu\nu\sigma\delta}$, and address the question of whether the spacetime's curvature becomes unbounded at $t=0$. In the case of the models under discussion it explicitly reads:

$$K = \frac{3 f_0^2}{8 t^{6 f_0 + 4} x^{6k}} [(1+p_0^2)^2 + 2(f_0+1)^2] + \frac{3}{16 t^{2 f_0 + 4} x^{2k}} \times [(1-p_0)^2 (1+f+p)^2 + (1+p)^2 (1+f-p)^2]. \tag{27}$$

Because of (25), K is singular at $t=0$ for any value of the three free parameters of the solution, indicating thus the generic presence of a spacelike singularity at the time origin.

It is also possible to study the spacelike singularity structure of the solutions in a more refined way, in particular by investigating the expansion along each spatial axis. In general, the behavior will strongly depend on the values of the parameters k, C and β. It can be shown analytically that for large enough β there will necessarily be contraction along the z axis; moreover, if $k > 1$ that will be the case regardless the value of β and C. Another fact one can easily check is the impossibility of having simultaneous contraction along axes x and y. Three main types of singular behavior exist:

(a) Point-like singularities (Quasi-Friedmann behavior)
All three spatial directions shrink as the initial time $t=0$ is approached; or explicitly $\lim_{t\to 0} \epsilon_i = 0 \,\forall i$. Depending on how many directions have the same expansion rate, the behavior will be completely anisotropic, axially symmetric or isotropic.

(b) Finite lines
This type of singular behavior occurs when in the vicinity of $t=0$ one of the spatial directions neither expands nor contracts with time. In other words, it is said that the direction i is a finite line if $\lim_{t\to 0} \epsilon_i = 1$. The subcases can be classified according to which direction behaves in that way. In general, there will be a single finite line, though it is possible to have particular cases for which a second finite line exists.

(c) Infinite lines (Quasi Kasner regime)
An infinite line along the i direction exists when $\lim_{t\to 0} \epsilon_i = \infty$. Again, three cases can be seen to occur, depending on which is the axis displaying that feature. For some particular values of the parameters the maximum allowed number of two infinite lines can be reached.

It is interesting to compare the main features of the new models with those of Kasner spacetimes, to which they are closely related. Kasner models are genuinely anisotropic, in

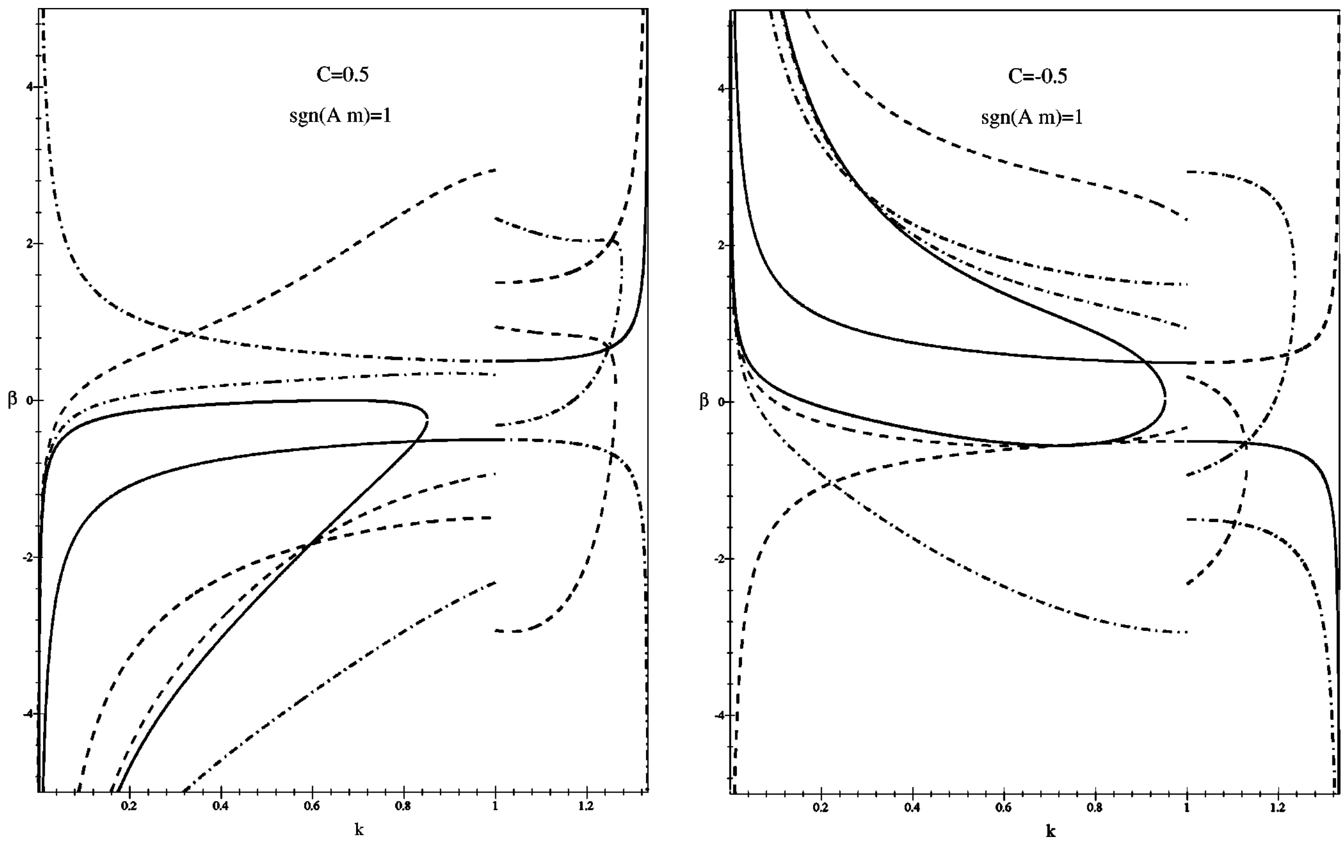

FIG. 1. Structure of singularities of the inhomogeneous generalization of Belinskii-Khalatnikov's model for $C=0.5$ (l.h.s.) and $C=-0.5$ (r.h.s.) with $\mathrm{sgn}(Am)=1$. The black lines indicate the value of β as a function of k for which a given direction behaves as a finite line. The black continuous, dashed and dashed-dotted lines correspond to a finite line along direction *z, y* or *x* respectively. The gray lines indicate the value of β as a function of k for which two spatial directions have the same expansion rate. The gray continuous, dashed and dashed-dotted lines correspond respectively to $\epsilon_2=\epsilon_3$, $\epsilon_1=\epsilon_3$ and $\epsilon_1=\epsilon_2$.

the sense that particular cases with isotropic expansion are excluded. It is worth mentioning as well that in Kasner models finite lines always come in pairs, and the fact there cannot be more than one infinite line.

Since the metric functions of the worked-out examples depends on the three free parameters in a rather cumbersome way, graphic methods will be the most convenient to provide further insight in the structure of spacelike singularities. For simplicity, the parameter C will be kept fixed. In Fig. 1 two different sets of lines can be distinguished. On the one hand, the black lines in each plot correspond to the curves along which a spatial direction becomes a finite line. In the region delimited by the black continuous line and the axes, a infinite line type of singularity arises along direction z. Here one can see how for $C=0.5$ it is possible to have simultaneously the same behavior along directions y and z. For $k>1$, in the region above the black dashed line, there is contraction along direction y, though this behavior gets reversed for $k<1$. Similarly, for β values less than those along the dashed-dotted line contraction takes place, and the contrary happens for $k>1$. The points where two black lines intersect correspond to having two finite lines. On the other hand, the gray lines represent the curves along which two spatial directions display the same expansion rate. The fact there is one point at which the three gray lines intersect, reflects the possibility of having isotropic expansion, and for the two C values chosen that point lies in the $k>1$ region.

B. WKB limit

With regard to the physical interpretation of the new solutions constructed here, it is their high-frequency limit which turns out to be most interesting. This limit, also called the WKB regime, corresponds to the epoch when the time elapsed since the beginning of the Universe is much larger than the period of any perturbation mode. By taking $jt\gg1$, for every value of j, in the normal mode expansions of the scalar field and metric functions, Charach and Malin were able to show that the relativistic solutions taken here as input, represent scalar and gravitational waves propagating on an spatially flat background. In this limit, such universes are causally connected because the particle horizon is larger than the wavelength of any of the modes of the independent degrees of freedom, namely the transverse part of the gravitational field p and the scalar field $\tilde{\varphi}$.

In this vein, it will be proved here that the G_1 counterparts to Charach and Malin's cosmological models can also be thought of in terms of non-static spacetimes containing wave-like perturbations. Thus, the physical interpretation is not spoiled in the process of homogeneity breaking. A direct consequence of the additional degree of inhomogeneity present in the G_1 solutions, as compared to their more symmetric counterparts, is that the background on which the waves live is not spatially flat. This can be checked by direct computing of the curvature of the $t = constant$ three dimensional hypersurfaces. Non-static backgrounds perturbed by waves have also been considered in a series of paper by Centrella and Matzner [44–46] who studied collisions of plane gravitational waves in Kasner cosmologies. Exact solutions describing propagation of waves in MSF Friedman-Robertson-Walker (FRW) universes have also been thoroughly studied [47–52].

Let us consider now the late time expressions for the metric functions and the scalar field of the G_1 spacetimes obtained by applying the new technique to Gowdy-Berger-Misner cosmological models (21):

$$p \sim p_0 \log t + B\overline{p}, \tag{28a}$$

$$f \sim f_0 \log t + \left(\frac{t}{2\pi}\right) \sum_{j=1}^{\infty} j\alpha_j^2, \tag{28b}$$

$$\tilde{\varphi} \sim \varphi_0 \log t + A\overline{p} + m \log|x|, \tag{28c}$$

$$\overline{p} = \sum_{j=1}^{\infty} \sqrt{\frac{2}{j\pi t}} \alpha_j \cos\left(jt - \frac{\pi}{4}\right) \times \cos[j(z - z_j)]. \tag{28d}$$

Since in this regime $\overline{p} \ll 1$, the metric $\tilde{g}_{\mu\nu}$ can be split into a background $\tilde{\eta}_{\mu\nu}$ plus a perturbation metric $\tilde{h}_{\mu\nu}$, that is

$$\tilde{g}_{\mu\nu} \sim \tilde{\eta}_{\mu\nu} + \tilde{h}_{\mu\nu}, \tag{29a}$$

$$\tilde{\eta}_{\mu\nu} = \text{diag}(-x^k t^{f_0} e^{f_1 t}, x^k t^{f_0} e^{f_1 t}, t^{1+p_0}, x^n t^{1-p_0}), \tag{29b}$$

$$\tilde{h}_{\mu\nu} = \text{diag}(0,0,t^{1+p_0} B\overline{p}, -x^n t^{1-p_0} B\overline{p}). \tag{29c}$$

Here, in addition to (24), I have made the following definition:[1]

$$f_1 = \sum_{j=1}^{\infty} \frac{j\alpha_j^2}{2\pi}(B^2 + 2A^2). \tag{30}$$

A peculiarity regarding the perturbations on the scalar and gravitational degrees of freedom is that for $0 < k < 1$ they will be on phase, whereas for $1 < k < 4/3$ they will be phase-shifted by π. Nonetheless, it will be seen later that whatever the value of k the scalar and gravitational perturbations contribute constructively to the energy momentum tensor.

Let us now proceed to analyze the $\tilde{h}_{\mu\nu}$ tensor and thereby show that it represents tensor perturbations of a background spacetime $\tilde{\eta}_{\mu\nu}$. The wave equation $\tilde{\nabla}^{\gamma}\tilde{\nabla}_{\gamma}\tilde{h}_{\mu\nu} = 0$ is satisfied because $\tilde{h}_{\mu\nu}$ is proportional to the transversal gravitational degree of freedom $\tilde{p}$, which is in turn a solution to this equation. One has to make sure however, that $\tilde{h}_{\mu\nu}$ is free of trace and divergence, thus not containing pieces which transform as scalars or vectors [53–55], i.e.:

$$\tilde{h}_{\gamma}^{\gamma} = 0, \tag{31}$$

$$\tilde{\nabla}^{\gamma}\tilde{h}_{\gamma\lambda} = 0, \tag{32}$$

with γ, $\lambda = 2,3$, and where the D'Alambertian and the covariant derivative must be calculated using the corresponding background metric $\tilde{\eta}_{\mu\nu}$.

Since it is straightforward to see that the trace-free condition (31) is satisfied by construction, the problem reduces to checking the fulfilment of the divergence-free condition (32), which in this case reads:

$$\tilde{h}_{22,x} - 2\tilde{\Gamma}_{22}^{2}\tilde{h}_{22} = 0, \tag{33a}$$

$$\tilde{h}_{33,x} - 2\tilde{\Gamma}_{32}^{3}\tilde{h}_{33} = 0. \tag{33b}$$

On the one hand, expression (33a) is identically null because neither $\tilde{\eta}_{22}$ nor $\tilde{h}_{22}$ are x-dependent. On the other hand, it can be seen that (33b) is also satisfied by just having in mind that

$$\tilde{h}_{33} = B\tilde{\eta}_{33}, \tag{34a}$$

$$\tilde{\Gamma}_{32}^{3} = (\log\sqrt{\tilde{\eta}_{33}})_{,x}. \tag{34b}$$

Then, $\tilde{h}_{\mu\nu}$ describes metric tensor perturbations in the form of gravitational waves, and as such, they are gauge invariant [53–55].

In order to give additional arguments in favor of the interpretation of the solution in terms of waves propagating on a nonflat background, I shall follow Charach and Malin and analyze the energy-momentum tensor of the background metric $\tilde{\eta}_{\mu\nu}$. It will be shown that the stress-energy tensor is naturally manifested in terms of two components. One of these corresponds to a null fluid, supporting thus the interpretation suggested above; while the other term corresponds to an inhomogeneous massless scalar field with no z-dependence. In particular,

$$^{(\eta)}\tilde{T}_{\mu}^{\nu} = {}^{(1)}\tilde{T}_{\mu}^{\nu} + {}^{(2)}\tilde{T}_{\mu}^{\nu}, \tag{35}$$

where ${}^{(1)}\tilde{T}_{\nu}^{\nu} \neq 0$ and ${}^{(2)}\tilde{T}_{\nu}^{\nu} = 0$. Explicitly

$$^{(1)}\tilde{T}_{0}^{0} = -t^{-(1+p_0)}\frac{m^2}{2x^2} - t^{-(2+f_0)}\frac{1+2f_0 - p_0^2}{4e^{f_1 t}x^k}, \tag{36a}$$

$$^{(1)}\tilde{T}_{1}^{1} = -t^{-(1+p_0)}\frac{m^2}{2x^2} + t^{-(2+f_0)}\frac{1+2f_0 - p_0^2}{4e^{f_1 t}x^k}, \tag{36b}$$

[1]Though the factor $B^2 + 2A^2$ equates to unity in the simple case I am dealing with, it has been deliberately introduced in the definition of f_1; so that the trail of the separate contributions to the energy momentum tensor of the graviton and scalar field pair can be followed. Had I considered the general case of the procedure to generate a G_2 massless scalar field solution, then $f_1 \neq \sum_{j=1}^{\infty} j\alpha_j^2/(2\pi)$.

$$ {}^{(1)}\tilde{T}_2^0 = t^{-(1+f_0)} \frac{2p_0(k-1)-k}{2e^{f_1 t} x^{1+k}}, \tag{36c} $$

$$ {}^{(1)}\tilde{T}_2^2 = t^{-(1+p_0)} \frac{m^2}{2x^2} + t^{-(2+f_0)} \frac{1+2f_0-p_0^2}{4e^{f_1 t} x^k}, \tag{36d} $$

$$ {}^{(1)}\tilde{T}_3^3 = -t^{-(1+p_0)} \frac{m^2}{2x^2} + t^{-(2+f_0)} \frac{1+2f_0-p_0^2}{4e^{f_1 t} x^k}, \tag{36e} $$

$$ {}^{(2)}\tilde{T}_0^0 = -t^{-(1+f_0)} \frac{f_1}{2e^{f_1 t} x^k}, \tag{36f} $$

$$ {}^{(2)}\tilde{T}_1^1 = t^{-(1+f_0)} \frac{f_1}{2e^{f_1 t} x^k}. \tag{36g} $$

I will now proceed to give an interpretation for the ${}^{(1)}\tilde{T}_\mu^\nu$ term. The Klein-Gordon equation for a scalar field $\tilde{\psi}$, calculated using the metric $\tilde{\eta}_{\mu\nu}$, takes the form

$$ \frac{(x\tilde{\psi}_{,x})_{,x}}{x^{1-k}} - \frac{(t\tilde{\psi}_{,t})_{,t}}{e^{f_1 t} t^{f_0-p_0}} = 0, \tag{37} $$

a solution of which is

$$ \tilde{\psi} = \varphi_0 \log t + m \log|x|. \tag{38} $$

The energy momentum tensor for the field $\tilde{\psi}$ propagating on the spacetime $\tilde{\eta}_{\mu\nu}$ yields ${}^{(1)}\tilde{T}_\mu^\nu$ exactly. This being so, it follows that the exact solution obtained after applying the generating technique to (21) asymptotically evolves into a solution with a single degree of inhomogeneity, and the sinusoidal inhomogeneities along the z-axis vanish with time.

On the other hand, the traceless term ${}^{(2)}\tilde{T}_\mu^\nu$ can be shown to account for waves. It can also be separated into two parts, namely:

$$ {}^{(2)}\tilde{T}_{\mu\nu} = {}^{(GW)}\tilde{T}_{\mu\nu} + {}^{(SW)}\tilde{T}_{\mu\nu}, \tag{39a} $$

$$ {}^{(GW)}\tilde{T}_{\mu\nu} = \sum_{j=-\infty}^{\infty} \frac{B^2\alpha_j^2}{4|j|} \kappa_{\mu j}\kappa_{\nu j}, \tag{39b} $$

$$ {}^{(SW)}\tilde{T}_{\mu\nu} = \sum_{j=-\infty}^{\infty} \frac{A^2\alpha_j^2}{2|j|} \kappa_{\mu j}\kappa_{\nu j}. \tag{39c} $$

The null vector κ_μ is defined by

$$ \kappa_{\mu j} = \frac{1}{\sqrt{\pi t}} (|j|, j, 0, 0). \tag{40} $$

It is clear that ${}^{(2)}\tilde{T}_{\mu\nu}$ corresponds to a null fluid describing a collisionless flow of "gravitons" and "scalar particles." It was to be expected, however, that the interpretation of the solutions' matter content should be preserved despite the homogeneity breaking process, because no additional term was introduced in the traceless part of the energy momentum tensor. So, like the cases studied by Charach and Malin, the new solutions represent in their WKB limit, waves propagating on a non-static spacetime, the difference being the inhomogeneous character of the background.

There is an alternative approach to show how the matter source behaves asymptotically. Let us calculate the energy-momentum tensor corresponding to the scalar field $\tilde{\varphi}$ in the late time limit and then just retain terms up to the order t^{-1}. Under this restriction the only non-null terms of the energy momentum tensor are $\tilde{T}_{00}$ and $\tilde{T}_{11}$, which are given by:

$$ \begin{aligned} \tilde{T}_{00} = -\tilde{T}_{11} = \left(\frac{A^2}{\pi t}\right) \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} \alpha_l \alpha_j \sqrt{lj} \Big\{ & \sin[l(z-z_l)] \\ & \times \sin[j(z-z_j)] \cos\left[lt - \frac{\pi}{4}\right] \cos\left[jt - \frac{\pi}{4}\right] \\ & - \cos[l(z-z_l)] \cos[j(z-z_j)] \sin\left[lt - \frac{\pi}{4}\right] \\ & \times \sin\left[jt - \frac{\pi}{4}\right] \Big\} + \mathcal{O}(t^{-2}). \end{aligned} \tag{41} $$

Averaging $\tilde{T}_{00}$ over a region $0 \le t \le 2\pi$, $0 \le z \le 2\pi$ on the (t,z)-plane one obtains:

$$ \begin{aligned} \langle \tilde{T}_{00} \rangle = -\langle \tilde{T}_{11} \rangle = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \langle \tilde{T}_{00} \rangle dz dt = \frac{1}{2\pi t} \sum_{j=1}^{\infty} jA^2 \\ = {}^{(SW)}\tilde{T}_{00}. \end{aligned} \tag{42} $$

So, essentially some terms of the stress-energy tensor evolve into those of a null fluid. One must be careful, however, when dealing with the rest of terms. Since in this approximation

$$ \tilde{T}_2^2 = -\tilde{T}_3^3 = -\tilde{T} = t^{-(1+p_0)} \frac{m^2}{2x^2}, \tag{43} $$

these three terms will only be negligible with respect to $\langle \tilde{T}_0^0 \rangle$ provided $1+p_0 > 1$ holds. Summarizing, the null fluid's contribution to the energy-momentum tensor dominates eventually the one due to the homogeneous part of the scalar field. Depending on the initial conditions, another term corresponding to an x-dependent scalar field may give a non-negligible contribution to the energy-momentum tensor. Even more, in the case $p_0 > 1$ this other term will represent the leading contribution to $T_{\mu\nu}$. Thus, as far as the evolution of the model is concerned, the presence of the additional degree of inhomogeneity in the model plays a crucial role. The scalar curvature of the background at $t \gg 1$ will be given by:

$$ {}^{(\eta)}R \sim \frac{m^2}{x^2 t^{1+p_0}}, \tag{44} $$

so, depending on the value of p_0, it will either vanish like in the G_2 counterpart models, or become unbounded at $t = \infty$. This is thus one more peculiarity arising in the singular behavior of the models which is entirely due to the presence of an additional degree of inhomogeneity.

The WKB regime of the solutions under discussion admits a reformulation in terms of the density of particles contributing to the modes of two fields. I shall strictly follow here an approach which consists of performing a quasi-classical treatment based on the geometrical optics energy-momentum tensor [56–58]. A family of Lorentz local frames is introduced so that the density of particles in each normal mode can be defined through

$$\lambda^{(a)}=\sqrt{|\tilde{\eta}_{\nu\nu}|}\,\delta^{(a)}_{\nu} \quad (\text{no summation over } \nu), \tag{45}$$

where $\tilde{\eta}_{\nu\nu}$ represents the components of the inhomogeneous generalization of the DZN metric. A set of observers corresponding to this tetrad are characterized by the 4-velocity:

$$u^{\nu}\equiv\lambda^{\nu}_{(0)}=(\sqrt{\tilde{\eta}^{00}},0,0,0). \tag{46}$$

Let us consider now Eqs. (39b),(39c), which give the WKB stress-energy tensors of the model with gravitational and scalar wave perturbations, namely:

$$^{(GW)}\tilde{T}_{\mu\nu j}=\frac{B^2\alpha_j^2}{4|j|}\,\kappa_{\mu j}\kappa_{\nu j} \tag{47a}$$

$$^{(SW)}\tilde{T}_{\mu\nu j}=\frac{A^2\alpha_j^2}{2|j|}\,\kappa_{\mu j}\kappa_{\nu j}\,. \tag{47b}$$

The density of scalar and gravitational particles in the n-th mode is given by

$$\rho_j^S=\frac{\tilde{T}^{(0)SW}_{(0)j}}{h\kappa^{(0)}_{\mu j}}=\frac{A^2\alpha_j^2}{2t\sqrt{x^k e^f}} \tag{48a}$$

$$\rho_j^G=\frac{\tilde{T}^{(0)GW}_{(0)j}}{h\kappa^{(0)}_{\mu j}}=\frac{B^2\alpha_j^2}{4t\sqrt{x^k e^f}}, \tag{48b}$$

where $\kappa_{\mu\nu}$ is a null vector with dimensions of length. Besides, since the description here is based on units $c=G=1$, the Planck constant has dimensions of $(\text{length})^2$, where $h\sim10^{-66}\,\text{cm}^2$. The density does not depend on the direction along which the particles propagate, that is

$$\rho_j^S=\rho_{-j}^S \tag{49a}$$

$$\rho_j^G=\rho_{-j}^G\,. \tag{49b}$$

In the G_1 models considered here, the volume of the spatial sections $t=constant$ is not finite; for that reason the total number of particles in each mode of the two degrees of freedom has no upper bound.

In light of this reformulation it was suggested that one should regard the evolution of G_2 models filled with waves as describing a process of transforming the initial inhomogeneities along z into quanta of various fields. Clearly, this interpretation's validity is extendible to models with just one isometry, like the ones here.

IV. CONCLUSIONS

Before I finish, I will summarize the main results. I have presented the first method to generate uniparametric families of general relativistic spacetimes having a two-dimensional inhomogeneity and a MSF as a source. In the context of either General Relativity or alternative theories of gravity, one can obtain a large number of new inhomogeneous metrics using this algorithm, where moreover the only input needed is any of the many known vacuum relativistic cosmologies with two commuting Killing vectors.

It has also been shown that this technique allows one to construct families of spacetimes which represent waves propagating on a spatially curved cosmological background. The spacelike singularity structure of these solutions has been studied, and several peculiarities due to the matter content have been elucidated.

ACKNOWLEDGMENTS

I am much indebted to A. Feinstein, M. A. Vázquez-Mozo and D. Clancy for valuable suggestions and enlightening discussions. Useful comments from A. Achúcarro and J. A. Valiente are also acknowledged. This work has been carried out thanks to the financial support of the Basque Government under fellowship number BFI98.79.

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