Expanding, axisymmetric pure-radiation gravitational fields with a simple twist

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New expanding, axisymmetric pure-radiation solutions are found, exploiting the analogy with the Euler-Darboux equation for aligned colliding plane waves. [S0556-2821(99)05520-4]

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I. INTRODUCTION

There exist many papers dealing with algebraically special, expanding and twisting pure-radiation solutions of the Einstein equations [1–7]. The standard form of the metric is [1]

$$ds^2=\frac{2d\zeta d\bar{\zeta}}{\rho\bar{\rho}P^2}-2\Omega(dr+Wd\zeta+\bar{W}d\bar{\zeta}+H\Omega), \tag{1}$$

$$\Omega=du+Ld\zeta+\bar{L}d\bar{\zeta}. \tag{2}$$

Here r is the coordinate along the null congruence of geodesics, u is the retarded time, and the complex coordinates ζ, $\bar{\zeta}$ span a two-dimensional surface. The metric components are determined by the r-independent real functions P, m, M and the complex function L:

$$2i\Sigma=P^2(\bar{\partial}L-\partial\bar{L}), \tag{3}$$

$$\rho=-\frac{1}{r+i\Sigma}, \tag{4}$$

$$W=\rho^{-1}L_u+i\partial\Sigma, \tag{5}$$

$$H=-r(\ln P)_u-(mr+M\Sigma)\rho\bar{\rho}+\frac{K}{2}, \tag{6}$$

$$K=2P^2\,\mathrm{Re}[\partial(\bar{\partial}\ln P-\bar{L}_u)], \tag{7}$$

where $\partial=\partial_\zeta-L\partial_u$ and Σ is the twist. The basic functions P, L, m, M satisfy the system

$$(\partial-3L_u)(m+iM)=0, \tag{8}$$

$$P^{-3}M=\mathrm{Im}\,\partial\partial\bar{\partial}\bar{\partial}V, \tag{9}$$

$$n^2=-2P^3[P^{-3}(m+iM)]_u+2P^3(\partial\partial\bar{\partial}\bar{\partial}V)_u-2P^2(\partial\partial V)_u(\overline{\partial\partial V})_u, \tag{10}$$

where $V_u=P$, n is the energy density of pure radiation and the Newton constant is set to 1. Equations (8), (9), and (10) are in fact Eqs. (26.32) and (26.33) from Ref. [1].

It has been noticed in different contexts that the condition $M=0$ [vanishing Newman-Unti-Tamburino (NUT) parameter] simplifies the equations [4–6,8,9]. Twisting gravitational fields with $M=0$ generalize the classes of Robinson-Trautman [10] and Kerr-Schild fields [11] which are physically realistic, their simplest representatives being the Schwarzschild, Kerr, and Vaidya solutions.

In the present paper we explore this condition applying the method of Stephani [2]. We discuss axisymmetric fields with the simplest possible twist. In Sec. II Eqs. (8)–(10) are reformulated in terms of an invariant potential which leads to the $L_u=0$ gauge. In Sec. III the main equation (9) for simplest twist is shown to be equivalent to the Euler-Darboux equation, which is central in the theory of aligned colliding plane waves (CPW). We use the known solutions and techniques to find solutions for our problem. In Sec. IV a closing discussion is presented.

II. FIELD EQUATIONS IN THE $L_u=0$ GAUGE

Following [2] we introduce the invariant complex potential ϕ which solves Eq. (8):

$$m+iM=\phi_u^3, \tag{11}$$

$$L=\frac{\phi_\zeta}{\phi_u}. \tag{12}$$

When $M=0$ we can apply the gauge transformation

$$u'=f(u,\sigma), \tag{13}$$

$$(m+iM)'=f_u^{-3}(m+iM) \tag{14}$$

to make the mass parameter m a positive or negative constant m_0 so that

$$\phi=m_0^{1/3}[u+iq(\sigma)], \tag{15}$$

$$L=i\bar{\zeta}q(\sigma)_\sigma, \tag{16}$$

where q is real and due to the axial symmetry depends only on $\sigma=\zeta\bar{\zeta}$. The complex coordinate ζ is related to the angular coordinates θ, φ on the distorted spheres ($r=r_0$, $u=u_0$) according to

$$\zeta=\sqrt{2}\tan(\theta/2)e^{i\varphi}. \tag{17}$$

Obviously $L_u=0$. This gauge differs from the usual Kerr's gauge [12] $P_u=0$, but is very suitable when the NUT parameter vanishes. Equations (9) and (10) simplify

$$\partial\partial\overline{\partial\partial}V=\overline{\partial\partial}\partial\partial V, \tag{18}$$

$$n^2=6m_0P^{-1}P_u+2P^3\partial\partial\overline{\partial\partial}P-2P^2\partial\partial P\overline{\partial\partial}P. \tag{19}$$

The second equation is in fact an inequality. When $P_u\neq 0$, n^2 can be made positive by the choice of m_0 at least for some region of spacetime [1,4,8]. The expressions for the metric components simplify too, e.g., the gauge invariants Σ and K read

$$\Sigma=P^2Q, \tag{20}$$

$$K=P^2(\bar{\partial}\partial+\partial\bar{\partial})\ln P, \tag{21}$$

where $Q=q_{\zeta\bar{\zeta}}$.

When $m+iM=0$ (Petrov types III and N) Eq. (8) is an identity but still a potential ϕ may be introduced with the property $\partial\phi=0$ and the subclass of solutions satisfying Eqs. (15) and (16) (with $m_0=1$) can be studied. This results in setting $m_0=0$ in all other equations.

In both cases the main equation (18), which is of fourth order with respect to V, becomes a linear second order equation for P. Let us choose the simplest possible twist $q=\sigma$, $Q=1$, $L=i\bar{\zeta}$. Then Eqs. (18) and (19) read

$$(\bar{\partial}\partial+\partial\bar{\partial})P=0, \tag{22}$$

$$n^2=6m_0P^{-1}P_u-6P^3P_{uu}-2\sigma^2P^2\times\left[\left(2P_{uu}+\frac{1}{\sigma}P_\sigma\right)^2+4P_{u\sigma}^2\right]. \tag{23}$$

The last term in Eq. (23) is obviously negative, so necessarily the first must be positive for a type II solution and the second must be positive for a type III or N solution.

III. REDUCTION TO THE EULER-DARBOUX EQUATION

Let us introduce the complex variable $z=\frac{1}{2}(\sigma+iu)$. Then Eq. (22) becomes

$$2(z+\bar{z})P_{z\bar{z}}+P_z+P_{\bar{z}}=0. \tag{24}$$

When z and $\bar{z}$ are two real variables this is the Euler-Darboux equation, the main equation in the theory of aligned CPW [13]. We can adapt the numerous solutions for our problem. We must ensure that P is real and investigate the regions where $n^2>0$.

In the original variables Eq. (24) may be written as

$$P_{uu}+P_{\sigma\sigma}+\frac{1}{\sigma}P_\sigma=0, \tag{25}$$

which is the analogue of the equation for vacuum Gowdy cosmologies [14]. Equation (25) has been derived from another viewpoint in Ref. [8] and discussed there. Solutions with separated variables behave as $(u+c)\ln\sigma$ or $e^{au}J_0(b\sigma)$ where a, b, c are constants and J_0 is a Bessel function. The first class includes the simplest CPW solution $P=-\ln(z+\bar{z})$.

A simple solution is obtained by separating the variables in Eq. (24) [13]:

$$P=A[(a+z)(a-\bar{z})]^{-1/2}, \tag{26}$$

where A, a are constants. A is ignorable, while a must vanish for a real P. Hence

$$P=B^{-1/2}, \tag{27}$$

$$B=\sigma^2+u^2. \tag{28}$$

The energy density becomes

$$n^2=-6m_0uB^{-1}-12B^{-3}. \tag{29}$$

We shall discuss only positive retarded times $u>u_0>0$. The energy density is regular in u and σ. If $m_0<0$ and $-m_0>2/u_0^5$ then $n^2>0$. Type III solutions, however, always have negative energy and are not realistic. The gauge invariants are also regular and vanish for big u:

$$\Sigma=B^{-1}, \tag{30}$$

$$K=-2\sigma B^{-2}. \tag{31}$$

The Weyl scalars [1,15] have the following leading terms:

$$\Psi_2=m_0\rho^3, \tag{32}$$

$$\Psi_3=-\rho^2P^3\partial I+O(\rho^3), \tag{33}$$

$$\Psi_4=\rho P^2I_u+O(\rho^2), \tag{34}$$

where $I=P^{-1}\overline{\partial\partial}P$. Plugging Eq. (27) into Eqs. (33) and (34) we obtain

$$\Psi_3=-6\rho^2\zeta(\sigma+iu)^2B^{-7/2}+O(\rho^3), \tag{35}$$

$$\Psi_4=6i\rho\zeta^2(\sigma+iu)^3B^{-4}+O(\rho^2). \tag{36}$$

It can be shown, using the full expressions for Ψ_3 and Ψ_4 [15], that the Weyl scalars are regular in σ and u. When $u\to\infty$, Ψ_3, and Ψ_4 vanish, while $\Psi_2\to -m_0/r^3$.

There is an analogue of the $\cosh^{-1}$ solution for CPW. Suppose that $P=P(y)$ where

$$y=\frac{i(\bar{z}-z)}{z+\bar{z}}=\frac{u}{\sigma}. \tag{37}$$

Then Eq. (24) yields the solution

$$P=\sinh^{-1}y. \tag{38}$$

The energy density is given by the expression

$$n^2=\frac{6m_0}{uP(1+y^{-2})^{1/2}}+\frac{6P^3y^3}{u^2(y^2+1)^{3/2}}-\frac{2P^2y^2(y^2+4)}{u^2(y^2+1)}. \tag{39}$$

The first term is regular in y and positive when $m_0>0$. When $y\to\infty$ the energy density becomes negative and $n\sim -y^2\ln^2 y$ no matter how big m_0 may be.

Equation (25) is the starting point for the procedure leading to the first Yurtsever solution [13,16]. It comprises the expressions

$$P=B^{l/2}P_l(x), \tag{40}$$

$$P=B^{l/2}Q_l(x), \tag{41}$$

where P_l and Q_l are Legendre functions of the first and second kinds and $x=uB^{-1/2}$. Solutions with P_l grow infinitely when $\sigma\to\infty$, $u\to\infty$ and change sign. A promising solution is

$$P=2Q_0(x)=\ln\frac{1+x}{1-x}. \tag{42}$$

Its energy density is

$$n^2=\frac{12m_0}{u}\left(\frac{P}{x}\right)^{-1}+\frac{12}{u^2}(xP)^3-8P^2\frac{4\sigma^2+u^2}{B\sigma^2}. \tag{43}$$

The range of x is $0\leqslant x<1$ and the first two terms are always positive when $m_0>0$. The last term has a negative pole for $\sigma\to 0$, $(x\to 1)$ of the type $\sigma^{-2}\ln^2\sigma$ and it can not be compensated by the positive singularity of the second term which is $\sim\ln^3\sigma$. Hence $n^2<0$ for $\sigma\to 0$.

In Ref. [13] a general method is presented for obtaining solutions of the Euler-Darboux equation. Angular coordinates are introduced, which is possible for CPW because the variables are bounded. This method does not have an analogue for expanding and twisting solutions. At last, it should be mentioned that any linear combination of the solutions derived above is also a solution of Eq. (24).

IV. DISCUSSION

We have shown that when the NUT parameter vanishes and the gauge $L_u=0$ is used, the main equation (22) for axisymmetric expanding pure radiation fields with the simplest twist becomes the second order, linear, Euler-Darboux equation for P. This is the central equation in the theory of aligned colliding plane waves. We have found analogues of some of the numerous known solutions adapted to our problem and studied the region of spacetime where the energy density of pure radiation is positive.

It is interesting that the interaction region of two colliding aligned plane waves is mathematically equivalent to an expanding distorted spherical wave with a simple twist. However, this is not the first example of such kind. It was shown in Ref. [17] that the field equations for nonaligned colliding electrogravitational plane waves coincide with the Ernst equations [18] for stationary axisymmetric Einstein-Maxwell fields. The Bell-Szekeres interacting solution is equivalent to the twistless conformally flat Bertotti-Robinson solution [13]. Colliding gravitational waves are algebraically general due to the presence of both Ψ_1 and Ψ_4. When an electromagnetic wave collides with a gravitational one sometimes the solution is of type II [19] without twist and falls in a class of solutions generalizing the Robinson-Trautman solutions (distorted spherical twistless waves) [20]. The structure of time-dependent expanding and twisting algebraically special pure-radiation solutions obviously is very rich, because in the case of simplest possible twist they coincide mathematically with the large variety of aligned colliding plane waves.

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