Natural vacua in hyperbolic Friedmann-Robertson-Walker spacetimes

Ian H. Redmount*

Department of Physics, Parks College of Engineering and Aviation, Saint Louis University,
3450 Lindell Blvd., St. Louis, Missouri 63103
(Received 22 January 1999; published 14 October 1999)

Recent evidence indicates that the Universe is open, i.e., spatially hyperbolic, long-standing theoretical preferences to the contrary notwithstanding. This makes it possible to select a vacuum state, Fock space, and particle definition for a quantized field by requiring concordance with ordinary flat-spacetime theory at late times. The particle-number basis states thus identified span the physical state space of the field at all times. This construction is demonstrated here explicitly for a massive, minimally coupled, linear scalar field in an open, radiation-dominated Friedmann-Robertson-Walker spacetime. [S0556-2821(99)05118-8]

PACS number(s): 04.62.Cv, 98.80.Cq, 98.80.Hw

I. INTRODUCTION

For more than 20 years it has been well known that the definitions of a vacuum state and ''particles'' for a quantized field in curved spacetime are problematical. Not that a Fock-space basis of particle-number eigenstates cannot be defined for such a field; rather, these can be defined in an infinity of inequivalent ways. It is not even necessary to consider gravitation to find this problem, which arises even in flat spacetime.

For a similar length of time it had been widely held that the Universe, as approximated by a homogeneous, isotropic Friedmann-Robertson-Walker spacetime geometry, must be spatially flat. (It was not always so; for many years a model with closed, positively curved spatial sections was believed to be the only natural choice [1].) The preference for a spatially flat model was upheld either as a consequence of inflationary cosmology or a motivation for it, or on the basis of various considerations of ''naturalness.'' Spatially flat or positive-curvature models require a cosmological mass density at or above critical density; this underlies the enormous body of theoretical and observational work on the cosmological dark-matter problem. But the weight of evidence, greatly strengthened by recent observations, indicates that the Universe is not spatially flat or positively curved—it is best described by an open model with hyperbolic spatial sections, theoretical preferences to the contrary notwithstanding [2,3].

Remarkably, these two conundra may be related. It proves easy to select a particularly simple vacuum state for a quantized field in a hyperbolic Friedmann-Robertson-Walker (FRW) spacetime; the corresponding particle-number eigenstates span the state space of the field. Conversely, hyperbolic (open) FRW models are distinguished from spatially flat and positive-curvature models in admitting such a ''natural'' definition of particles.

The choice of this ''natural'' vacuum state arises from the asymptotic behavior of the hyperbolic FRW spacetime. In the absence of a cosmological constant, any such spacetime approaches the empty Milne spacetime geometry [4,5] at late

times, as the expansion of the model drives its mass-energy density to zero. But Milne spacetime is a portion of flat (Minkowski) spacetime. A vacuum choice can be made for a quantized field in Milne spacetime which is equivalent to the familiar vacuum state of flat-spacetime theory, with all of its desirable physical properties [6]. Hence a vacuum state is selected on the original spacetime: that which approaches this ''physical'' Milne vacuum at late times, thus inheriting all of its features. The choice of a vacuum determines a Fock space of states for the field; basis states for this space define particles in the spacetime.

Of course, this defines an ''out'' vacuum and ''out'' particles, notions long familiar in curved-spacetime quantum field theory. But in an open universe (with ordinary matter-energy content), the asymptotically Minkowskian ''out'' region is physical. The behavior of a quantized field *must* accord with familiar Minkowski-spacetime field theory there. The state of the field must lie in the Fock space spanned by the usual Minkowski vacuum and many-particle states in the late-time limit, hence, by this ''natural'' choice of basis states. Since both the actual field state and the basis states are exact solutions of the field theory, the actual state must be the same superposition of basis states at all times. That is, this choice of basis states spans the physical state space of the field, at all times.

This basis-state construction is illustrated here by means of an example: a massive, minimally coupled, linear scalar field in radiation-dominated FRW spacetime. (The present Universe is matter dominated, but this example is simpler.) The relevant spacetime geometries and quantum field theory are described in Sec. II. The choice of a vacuum state for the field in Milne spacetime, equivalent to the familiar Minkowski-spacetime vacuum, is shown in Sec. III. The selection of a vacuum state for the field in the radiation-dominated spacetime, which matches this Milne vacuum at late times, is demonstrated in Sec. IV. The results are discussed in Sec. V. I use units here with $\hbar = c = 1$, and sign conventions and general notation as in Misner, Thorne, and Wheeler [7].

II. FUNDAMENTALS

A. Spacetime geometries

The spacetimes of interest here are open Friedman-Robertson-Walker geometries, with hyperbolic spatial sec-

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*Email address: redmount@hypatia.slu.edu

tions and curvature parameter $k = -1$. They have metrics of the form

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\,d\phi^2)], \quad (2.1)$$

with $t \in [0, +\infty)$, $\chi \in [0, +\infty)$, $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi)$. The scale factor $a(t)$ completely characterizes each spacetime.

The Milne universe [4,5] is identified by the scale factor

$$a_M(t) = t. \quad (2.2)$$

The resulting geometry corresponds, via the Einstein field equations, to an empty universe. It is simply a portion of Minkowski spacetime, in coordinates tied to observers moving radially with all possible velocities. In coordinates $T = t\cosh\chi$, $R = t\sinh\chi$, θ, and ϕ, metric (2.1) with scale factor a_M takes on the familiar flat-spacetime form.

The radiation-dominated FRW model has a scale factor given parametrically by

$$a_R(\eta) = a_*\sinh\eta, \quad (2.3a)$$

$$t(\eta) = a_*(\cosh\eta - 1), \quad (2.3b)$$

with conformal-time parameter $\eta \in [0, +\infty)$ and a_* a constant. According to the Einstein field equations, the resulting geometry corresponds to a density which decreases with time as a_R^{-4} and a pressure one-third of this density, hence "radiation dominated." For $\eta \gg 1$ the density and pressure rapidly approach zero, and a_R approaches the Milne form a_M.

B. Quantized scalar field

A suitable example is a real scalar field φ interacting only with the spacetime geometry, with mass μ and minimal coupling. The corresponding field equation is

$$(\Box - \mu^2)\varphi = 0, \quad (2.4)$$

with $\Box$ the covariant d'Alembertian corresponding to metric (2.1). The quantized field is expanded in terms of normal-mode solutions of this equation and operator coefficients:

$$\varphi = \sum_{l,m} \int_0^\infty d\kappa \left\{ b_{\kappa lm} u_\kappa(t) \left[\prod_{j=0}^l (j^2 + \kappa^2)^{1/2} \right] \right.$$

$$\left. \times (\sinh\chi)^{-1/2} P_{-1/2+i\kappa}^{-(l+1/2)}(\cosh\chi) Y_{lm}(\theta,\phi) + \text{H.c.} \right\}. \quad (2.5)$$

Here Y_{lm} is a spherical harmonic and $P_{-1/2+i\kappa}^{-(l+1/2)}$ is an associated Legendre function; the parameter κ represents radial momentum in the hyperbolic spatial sections of the spacetime. The time dependence $u_\kappa(t)$ is a solution of the equation

$$\frac{d^2 u_\kappa}{dt^2} + \frac{3}{a}\frac{da}{dt}\frac{du_\kappa}{dt} + \left(\mu^2 + \frac{1+\kappa^2}{a^2}\right)u_\kappa = 0, \quad (2.6)$$

satisfying the "positive-frequency" normalization condition

$$ia^3\left(u_\kappa^* \frac{\overleftrightarrow{\partial}}{\partial t} u_\kappa\right) = 1. \quad (2.7)$$

Consequently, the canonical commutation relations for the field φ impose on the operators $b_{\kappa lm}$ and their Hermitian conjugates commutation relations appropriate to annihilation and creation operators, respectively. The vacuum state of the field is defined by the condition $b_{\kappa lm}|0\rangle = 0$ for all modes κlm; a Fock-space basis of particle states is obtained by applying the creation operators $b_{\kappa lm}^\dagger$ to this vacuum state.

The positive-frequency functions u_κ are not uniquely determined by Eqs. (2.6) and (2.7). In general, there is a one-complex-parameter family of such functions for each mode. Each choice of u_κ corresponds to a different operator $b_{\kappa lm}$, which defines a different vacuum and particle states. The Fock spaces spanned by these various bases can be unitarily inequivalent.

III. "PHYSICAL" VACUUM IN MILNE SPACETIME

For the Milne geometry, Eq. (2.6) assumes a familiar form; a general solution can be written

$$u_\kappa(t) = \frac{1}{t}[c_\kappa^{(1)} H_{i\kappa}^{(1)}(\mu t) + c_\kappa^{(2)} H_{i\kappa}^{(2)}(\mu t)], \quad (3.1)$$

where $c_\kappa^{(1,2)}$ are constants and $H_{i\kappa}^{(1,2)}$ are Hankel functions. Condition (2.7) imposes the constraint

$$\frac{4}{\pi}(e^{-\pi\kappa}|c_\kappa^{(2)}|^2 - e^{\pi\kappa}|c_\kappa^{(1)}|^2) = 1 \quad (3.2)$$

on the constants. Each choice of $c_\kappa^{(1)}$, then, identifies a different positive-frequency mode function, giving rise to a different definition of vacuum and particle states.

There is a unique choice of coefficients $c_\kappa^{(1,2)}$, however, for which the resulting mode functions consist of superpositions entirely of positive-frequency Minkowski-spacetime mode functions. That choice is

$$c_\kappa^{(1)} = 0 \quad \text{and} \quad c_\kappa^{(2)} = \frac{\sqrt{\pi}}{2}e^{-\pi\kappa/2}. \quad (3.3)$$

A standard integral representation of the Hankel functions can be used to show

$$\frac{1}{t}H_{i\kappa}^{(2)}(\mu t)(\sinh\chi)^{-1/2}P_{-1/2+i\kappa}^{-(l+1/2)}(\cosh\chi)$$

$$= \int_0^\infty \alpha_{\kappa l}(\beta)e^{-i\omega T}j_l(kR)dk, \quad (3.4a)$$

with $k = \mu\sinh\beta$, $\omega = \mu\cosh\beta$,

$$\alpha_{\kappa l}(\beta) = \frac{-i}{\kappa^2 + l^2}\left[\frac{\partial}{\partial\beta} + \tanh\beta - (l+1)\coth\beta\right]\alpha_{\kappa,l-1}(\beta), \quad (3.4b)$$

and

$$\alpha_{\kappa 0}(\beta) = -\left(\frac{2}{\pi}\right)^{1/2}\frac{e^{-\pi\kappa/2}}{\pi\kappa/2}\sin(\kappa\beta)\tanh\beta. \quad (3.4c)$$

In Eq. (3.4a), j_l is a spherical Bessel function; the integrand is the time and radial dependence of a positive-frequency normal-mode solution of Eq. (2.4) in Minkowski coordinates (T, R, θ, ϕ), with angular momentum l. Consequently, the Bogoliubov transformation between the Milne-spacetime field decomposition with mode choice (3.3) and the usual Minkowski-spacetime analysis is ''trivial,'' in the sense that it does not mix positive- and negative-frequency mode functions or creation and annihilation operators. The vacuum state associated with mode choice (3.3) is the same state as the usual Minkowski-spacetime vacuum, with all its symmetry, regularity, and minimal-energy properties. Milne-spacetime particle states built on this vacuum span the familiar Fock space of the Minkowski-spacetime field. Thus mode choice (3.3) may be distinguished as providing the physical vacuum and particle definition [6] for the field φ in Milne spacetime.

IV. ''NATURAL'' VACUUM IN RADIATION-DOMINATED FRW SPACETIME

Field-mode functions for the radiation-dominated FRW geometry can be found as functions of conformal time η.

With $u_\kappa(\eta) = v_\kappa(\eta)/a_R(\eta)$, Eq. (2.6) takes the form

$$\frac{d^2 v_\kappa}{d\eta^2} - [(\tfrac{1}{2}\mu^2 a_*^2 - \kappa^2) - \tfrac{1}{2}\mu^2 a_*^2 \cosh(2\eta)]v_\kappa = 0. \quad (4.1)$$

This is the modified form of Mathieu's equation. Its solutions can be expressed in many different ways; for those which approach the Milne-spacetime functions $H_{i\kappa}^{(2)}(\mu t)$, the form

$$v_\kappa(\eta) = \sum_{n=-\infty}^{+\infty} c_n H_{n+\nu}^{(2)}(\mu a_* \cosh \eta) \quad (4.2)$$

is most appropriate. In consequence of Eq. (4.1), the coefficients c_n must satisfy the system of equations

$$\tfrac{1}{4}\mu^2 a_*^2(c_{n+2} + c_{n-2}) + [\tfrac{1}{2}\mu^2 a_*^2 - \kappa^2 - (n+\nu)^2]c_n = 0 \quad (4.3)$$

for all n, with overall normalization determined by Eq. (2.7). The necessary and sufficient condition for system (4.3) to have a solution is that the infinite determinant

$$\Delta(\nu) \equiv \begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & \dfrac{q}{A-(3+\nu)^2} & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 0 & \dfrac{q}{A-(2+\nu)^2} & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & \dfrac{q}{A-(1+\nu)^2} & 0 & \cdots \\ \cdots & \dfrac{q}{A-\nu^2} & 0 & 1 & 0 & \dfrac{q}{A-\nu^2} & \cdots \\ \cdots & 0 & \dfrac{q}{A-(-1+\nu)^2} & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & \dfrac{q}{A-(-2+\nu)^2} & 0 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & \dfrac{q}{A-(-3+\nu)^2} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}, \quad (4.4)$$

with $q \equiv \tfrac{1}{4}\mu^2 a_*^2$ and $A \equiv \tfrac{1}{2}\mu^2 a_*^2 - \kappa^2$, equal zero. Standard methods exist for the treatment of such determinants [8]. The condition $\Delta(\nu) = 0$ determines the parameter ν, viz.,

$$\nu = \frac{1}{\pi}\arcsin[\sqrt{\Delta(0)}\sin(\pi\sqrt{A})]. \quad (4.5)$$

The coefficients c_n can then be found, establishing the solution (4.2).

Every term in series (4.2) behaves asymptotically as

$$(\mu a_* \cosh\eta)^{-1/2}e^{-i\mu a_* \cosh\eta} \sim (\mu t)^{-1/2}e^{-i\mu t} \quad (4.6)$$

for $\eta \gg 1$. Hence the resulting solution matches the physical positive-frequency mode function for the Milne-spacetime field determined above, in the late-time limit. The vacuum state and Fock space based on this mode choice will exhibit the same regularity (lack of extraneous singularities [6]), analyticity, and other properties as standard Minkowski-spacetime field theory in that limit—as indeed does quantum field theory in the actual Universe at the present time. In this

sense, then, this choice may be termed the ''natural'' vacuum and particle definition in open, radiation-dominated FRW spacetime.

V. CONCLUSIONS

This example illustrates how a possible ''natural'' choice of positive-frequency normal modes, vacuum, and particle states for a quantized field can be made, specifically in an *open* or hyperbolic FRW spacetime geometry. That choice is based on correspondence with standard Minkowski-spacetime theory at late times, in accordance with all of experimental particle physics.

Conversely, hyperbolic or open cosmological models can be viewed as ''natural,'' in that they admit such a definition of vacuum and particle states. The construction illustrated here can be applied to any hyperbolic FRW spacetime without a cosmological constant, i.e., any spacetime which approaches the Milne geometry at late times. Of course, this includes the matter-dominated model to which the present Universe more closely corresponds, though the forms of Eq. (2.6) and its solutions are then more unwieldy and less familiar than in the radiation-dominated example.

Current evidence also admits the possibility that the cosmological constant may be nonzero [2,3]. If it is positive, then the Universe will ultimately approach a de Sitter geometry, as its expansion suppresses all other density contributions. In that case a similar ''natural'' vacuum can also be chosen, on physical grounds akin to those in the zero-cosmological-constant case: The ''natural'' vacuum is that which approaches the Euclidean (Chernikov-Tagirov [9] or Bunch-Davies [10]) vacuum in de Sitter space at late times. An explicit example would be more involved than that shown above and will not be attempted here.

Though identified by their behavior at late times, the vacuum and particle states chosen as described here can serve to describe the physical state of the field at all times. The mode functions used are exact, not approximate, solutions of the field equation—equivalently, the vacuum and particle states are exact solutions of the functional Schrödinger equation for the field. Consequently, the actual state of the field is the same superposition of these basis states at early as at late times (though physical properties of the states may be very different). This choice of vacuum state could even serve as the ''preferred natural, geometrical vacuum state'' in the normal-ordering program described by Brown and Ottewill [11].

The dependence of this construction on the particular behavior of the spacetime geometry suggests the possibility that the formulation of physics might be more intimately related to the actual structure of the Universe than is often supposed. It may be noted, however, that no ''teleology'' is involved in the notion that the vacuum and particle states—hence, the state space of the field—are determined at all times by particular late-time behavior. Since both the spacetime geometry and the mode functions of the field are solutions of second-order differential equations, the late-time behavior of both is implicit in their initial conditions.

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