Constructing exact perturbations of the standard cosmological models

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In this paper we show a procedure to construct cosmological models which, according to a covariant criterion, can be seen as exact (nonlinear) perturbations of the standard Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models. The special properties of this procedure will allow us to select some of the characteristics of the models and also to study in depth their main geometrical and physical features. In particular, the models are conformally stationary, which means that they are compatible with the existence of isotropic radiation, and the observers that would measure this isotropy are rotating. Moreover, these models have two arbitrary functions (one of them is a complex function) which control their main properties, and in general they do not have any isometry. We study two examples, focusing on the case when the underlying FLRW models are flat dust models. In these examples we compare our results with those of the linearized theory of perturbations about a FLRW background. [S0556-2821(99)08320-4]

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I. INTRODUCTION

The standard picture of the universe is based on the well-known *cosmological principle* (see [1,2] for recent accounts), which states that the universe is homogeneous and isotropic. This principle implies that the geometry of spacetime is of the Robertson-Walker type and therefore the models that describe the dynamics are Friedmann-Lemaître-Robertson-Walker (FLRW) solutions of Einstein's equations. However, as astronomical and astrophysical observations indicate, the cosmological principle should be considered as an approximation valid only to a certain extent. In order to get a precise description of the universe we need to consider that there are inhomogeneities as well as anisotropies in the distribution of the energy-matter content of the universe. Taking into account these considerations, the study of perturbations of FLRW models has been an important subject of research for a long time (see [3] for a review), and even today important developments on this issue are appearing in the literature.

The aim of this paper is to contribute to the study of cosmological models that are *close* to the FLRW standard models, but instead of using approximate methods we will use exact techniques, dealing in this way with models that are exact solutions of the Einstein (nonlinear) equations. This kind of approach has two obvious advantages: first, we can study gravitational nonlinear effects in a cosmological scenario, and second, the fact that the models are exact solutions allows us to compute and study any physical quantity. In contrast, in the linearized theory of perturbations, in order to avoid spurious gauge mode solutions it is necessary to fix the gauge freedom in the map between the background and perturbed spacetimes (see [4,5]), but for some physical quantities, as for instance the spatial variation of the energy density, the results are difficult to interpret (see [5] for a extensive discussion). An alternative is to use a gauge-invariant approach [6,5], but not all the physical quantities are gauge invariant, only those which in the background spacetime vanish, or are constant scalars, or linear combinations (with constant coefficients) of Kronecker δ's (see [4]). Then, other related gauge-invariant quantities must be used.

The particular method we employ in this work is a combination of different transformations of spacetimes: we will start from the static FLRW models, then we will make a generalized Kerr-Schild transformation, and finally, we will apply to the result a conformal transformation. This procedure provides the necessary tools to control some properties of the final spacetimes and facilitates the study of the physical properties. As we will see, choosing adequately the parameters of these transformations we will find stationary cosmological models, that is, we find cosmological models that allow the existence of isotropic radiation (see [7,8]), and this radiation can represent the cosmic microwave background radiation (CMBR). The models have not, in general, any Killing symmetry and they have two arbitrary functions, one of them corresponds to the conformal factor, which plays the role of the scale factor for the world lines of the observers that would observe the CMBR isotropic (the motion of these observers will be shear free, but in general it will be rotating and nongeodesic). The other function is a complex function that will control other properties, as for instance, the rotation of the preferred observers and the Petrov type.

In order to give a meaning to the term *close to* FLRW we are going to use a covariant criterion. This criterion is a generalization of that given in [9,10], where the characterization of when a perfect-fluid spacetime is close to a FLRW model is introduced. Here, we give the generalization for arbitrary cosmological models, since our models will not be in general of the perfect-fluid type. This criterion is formulated in terms of the well-known hydrodynamical formalism introduced by Ehlers [11] and popularized by Ellis and collaborators (see, e.g., [12–14]). We do not enter here in details about this formalism, they can be consulted in the references just given.

Then, the criterion we are going to use to say that a given cosmological model is close to a FLRW model is the following: ''A cosmological model is said to be *close to FLRW* in some open domain of the spacetime if and only if there are observers, moving with a unit velocity $\boldsymbol{u}$ $(u^a u_a = -1)$, such that for some suitable small positive constant ϵ the following inequalities hold in that domain:[1]

Kinematical variables. Expansion θ, shear σ_{ab}, rotation ω_{ab} and acceleration a^a:

$$\frac{|\sigma_{ab}|}{\theta}<\epsilon, \quad \frac{|\omega_{ab}|}{\theta}<\epsilon, \quad \frac{|\mathrm{a}^a|}{\theta}<\epsilon, \quad \frac{|\mathcal{D}_a\theta|}{\theta^2}<\epsilon.$$

Matter variables. Energy density ϱ, isotropic pressure p, heat flow q^a and anisotropic pressure Π_{ab}:

$$\frac{|\mathcal{D}_a\varrho|}{\theta\varrho}<\epsilon, \quad \frac{|\mathcal{D}_a p|}{\theta\varrho}<\epsilon, \quad \frac{|q^a|}{\varrho}<\epsilon, \quad \frac{|\Pi_{ab}|}{\varrho}<\epsilon.$$

Weyl tensor variables. Electric E_{ab} and magnetic H_{ab} parts:

$$\frac{|E_{ab}|}{\theta^2}<\epsilon, \quad \frac{|H_{ab}|}{\theta^2}<\epsilon.,,$$

Where for any spatial (orthogonal to $\boldsymbol{u}$) tensor $A^{a\cdots}{}_{b\ldots}$ we have defined the following scalar:

$$|A^{a\cdots}{}_{b\ldots}|\equiv(h_{ac}\cdots h^{bd}\cdots A^{a\cdots}{}_{b\ldots}A^{c\cdots}{}_{d\ldots})^{1/2},$$

which vanishes if and only if $A^{a\cdots}{}_{b\ldots}$ vanishes (where $h_{ab}\equiv g_{ab}+u_a u_b$ denotes the orthogonal projector to $\boldsymbol{u}$). Moreover, we have used the derivative $\mathcal{D}$, defined as follows $\mathcal{D}_a A^{b\cdots}{}_{c\ldots}\equiv h^b{}_e\cdots h_c{}^f\cdots h_a{}^d\nabla_d A^{b\cdots}{}_{c\ldots}$. Obviously, we are assuming in this criterion that the model is neither a vacuum spacetime nor the observers under consideration are expansion free. Note that all the quantities compared with ϵ are dimensionless quantities, they have been formed by dividing by the expansion θ and the energy density ϱ, which we have assumed to be nonvanishing. The form of the quantities compared with ϵ is such that all of them can be taken as first-order terms in the theory of linear perturbations of FLRW cosmological models, because they have a vanishing background value that makes them gauge invariant. In that case, ϵ would correspond with the parameter or function controlling the strength of the perturbation. In the models that we are going to present in this work, we will be able to control the value of the quantities compared with ϵ by means of a function, in such a way that when this function is equal to zero we recover the FLRW models. This is also what happens with much of the cosmological models known in the literature (see [15] for a wide review of models containing FLRW cosmologies).

Note also that the conditions given in this criterion are sufficient but not necessary, in some circumstances some of them can be a consequence of the rest of the conditions: For instance, when there are equations of state relating the matter variables, or in the perfect-fluid case, in which the inequality for the acceleration follows from the conservation equations for the energy-momentum tensor T_{ab} (see [11–13]) and the inequalities for the pressure. However, this is not true in the general case, since although q^a and Π_{ab} are assumed to be small, their derivatives could not be small. A remaining open issue is finding the minimum set of inequalities that characterizes when a cosmological model is *closed to* a FLRW model.

The plan of this paper is the following. In Sec. II, we describe the techniques used to construct the models and give the expression of the line element, studying the main geometrical properties that follow from the procedure and the spacetimes used in the process. In Sec. III we focus on the main physical properties of these models. As we have said before, the models are conformally stationary, then we consider the unit timelike vector field proportional to the conformal Killing vector field, which correspond to the unit velocity of the observers that would measure isotropic radiation. We study the kinematics of these observers and their consequences on the properties of our cosmological models. Afterwards, we decompose the energy-momentum tensor with respect to these preferred observers and study the different quantities. Moreover, we study two particular examples of interest focusing on the case of a dust FLRW background. The comparison with the results of the linearized theory is also made. Finally, in Sec. IV we discuss the results obtained and the possible extensions of this work. In the Appendix we give some useful formulas for the basis associated with the different metrics that appear in this paper.

II. CONSTRUCTION AND GEOMETRICAL PROPERTIES OF THE MODELS

In this section we are going to give the explicit form of the line element of our models and also to study some of their geometrical properties. The procedure we are going to follow is based on the use of the generalized Kerr-Schild (GKS) and conformal transformations. Broadly speaking, given a spacetime (V_4,g), a GKS transformation is any transformation of the form $\boldsymbol{g}\rightarrow\boldsymbol{g}'=\boldsymbol{g}+2H\boldsymbol{l}\otimes\boldsymbol{l}$, such that H is a scalar and $\boldsymbol{l}$ is a null vector field (general properties of this transformation can be found, for instance, in [16–18]). On the other hand, as is well known, a conformal transformation is any transformation of the form $\boldsymbol{g}\rightarrow\boldsymbol{g}'=\Phi^2\boldsymbol{g}$, where Φ is any scalar (see, e.g., [19,20]).

The manner in which these transformations will be applied in this paper and the kind of spacetimes that we obtain are represented in Fig. 1. We start from the static FLRW models (V_4,g), then a GKS transformation is applied, in which we take $\boldsymbol{l}$ to be geodesic and shear free, such that the final spacetimes $(\tilde{V}_4,\tilde{g})$ are perfect-fluid stationary spacetimes. Finally, we make a conformal transformation, obtaining in this way conformally stationary spacetimes $(\hat{V}_4,\hat{g})$. In fact, the global transformation, from (V_4,g) to $(\hat{V}_4,\hat{g})$, can be seen as a GKS transformation applied to conformally flat spacetimes.

There are several reasons for considering the GKS transformation as the tool to find the exact perturbations of the FLRW models. First, as it has been shown in the plentiful

[1] Throughout this paper we use units in which $8\pi G=c=1$.

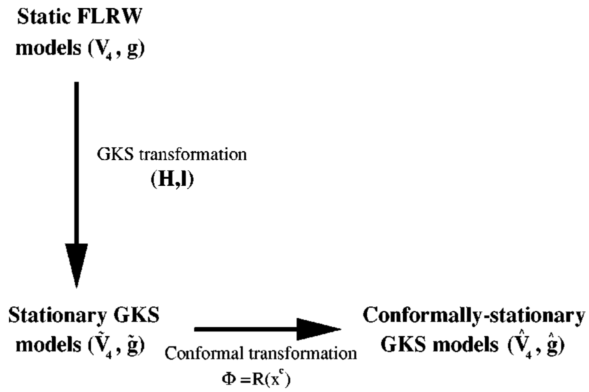

FIG. 1. Scheme followed in the construction of the cosmological models. Given the initial spacetime (in our case the static FLRW models), the GSK transformation is determined by the function H and the null vector field $\boldsymbol{l}$, and the conformal transformation by means of the conformal factor Φ.

literature on the KS and GKS transformations,[2] they allow one to integrate Einstein's equations in cases of low symmetry and to obtain large families of solutions. Moreover, in most cases the properties of this transformation provide simple ways of studying the final spacetimes. The second important reason is a theorem, shown first by Xanthopoulos [21] for the vacuum to vacuum GKS transformation, which states that given two spacetimes (V_4, g) and $(\hat{V}_4, \hat{g})$ related by a GKS transformation, and such that the null vector field of the transformation $\boldsymbol{l}$ is geodesic, the equations for the tensor $2H\boldsymbol{l}\otimes\boldsymbol{l}$, obtained from the Einstein equations for the metric $\hat{\boldsymbol{g}}$, coincide with the linearized Einstein equations for a traceless perturbation, the initial spacetime (V_4, g) being the background spacetime. In fact, it can be shown that the Einstein tensor of the final spacetimes $\hat{G}^a{}_b$ is linear in H and its derivatives, provided $\boldsymbol{l}$ is geodesic. As a consequence, the proximity of these spacetimes to the FLRW models, in the sense explained in the Introduction, is controlled through the function H. Finally, it is also important to point out that the GKS transformation preserves the volume four form, which is a useful property with which to study quantities defined through integration.

When we choose the conformal factor Φ to be a function of the proper time t of the fundamental observers in the static FLRW spacetimes (V_4, g), namely $R(t)$, the metric $R^2\boldsymbol{g}$ is the metric of the FLRW models (see, e.g., [22]). Then, in that particular case the final result is a metric of the form $\hat{\boldsymbol{g}} = \boldsymbol{g}_{\mathrm{FLRW}} + 2H\boldsymbol{L}\otimes\boldsymbol{L}$, where $\boldsymbol{L} = R\boldsymbol{l}$ is a geodesic and shear-free null vector field, and therefore, the global procedure can be seen as a GKS transformation of the FLRW models (that is what is done for some particular vector field $\boldsymbol{l}$ in Refs. [17,23]). In this case, the term $2H\boldsymbol{L}\otimes\boldsymbol{L}$ satisfies the equations for a traceless perturbation in a FLRW background.

However, in order to get more general models containing the FLRW spacetimes, we can consider the only restriction on the conformal factor, that in the limit $H\to 0$ becomes a function of t only. On one hand, this allows us to choose widely the conformal factor, and on the other hand, this restriction ensures that we can always recover the FLRW spacetimes by taking the limit $H\to 0$.

Now, let us carry out the procedure just outlined (see the scheme in Fig. 1). The starting point is the static FLRW models, that is, those FLRW spacetimes without expansion $(\theta = 0)$. The line element can be written in the following way:

$$ds^2 = -dt^2 + a^2(d\chi^2 + \Sigma_k^2 d\mathcal{S}^2), \tag{1}$$

where a is an arbitrary constant and k is the curvature parameter of the hypersurfaces $\{t = \text{constant}\}$ $(k = 0,1,-1$ for flat, closed and open models, respectively), $\Sigma_k = \Sigma_k(\chi)$ satisfies the differential equation $\Sigma_{k,\chi}^2 + k\Sigma_k^2 = 1$ $({}_{,\chi} \equiv \partial/\partial\chi)$, or alternatively it can be given by

$$\Sigma_k = \begin{cases} \sin\chi & \text{for } k = 1, \\ \chi & \text{for } k = 0, \\ \sinh\chi & \text{for } k = -1, \end{cases}$$

and $d\mathcal{S}^2$ is the line element of the unit two-dimensional sphere, which, using complex stereographic coordinates $\{\xi, \bar{\xi}\}$, can be written as follows:

$$d\mathcal{S}^2 = \frac{4 d\xi d\bar{\xi}}{(1 + \xi\bar{\xi})^2}.$$

These spacetimes have a perfect-fluid energy-momentum tensor, $T_{ab} = (\varrho + p)u_a u_b + p g_{ab}$, where the fluid velocity $\boldsymbol{u}$ $(u^a u_a = -1)$, energy density ϱ, and pressure p are given by

$$\boldsymbol{u} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} t}, \quad \varrho = \frac{3k}{a^2} = -3p. \tag{2}$$

For $k = 1$, the line element (1) represents the Einstein static universe, for $k = 0$ the Minkowski flat spacetime, and for $k = -1$ a negative curvature model that we will call here the *anti-Einstein* universe. As we can see from Eq. (2), the energy-momentum content of the anti-Einstein universe does not satisfy the usual energy conditions [19]. It is important to remark that applying a conformal transformation to these spacetimes we obtain all the FLRW models, the conformal factor being the usual scale factor, that is to say, a function of t only (see, e.g., [22]).

The first step of our procedure (see Fig. 1) is to perform the most general GKS transformation

$$g_{ab} \to \tilde{g}_{ab} = g_{ab} + 2H l_a l_b, \tag{3}$$

which leads from the initial spacetimes (1) to stationary perfect-fluid spacetimes. Hereafter we will use a tilde to denote objects associated with the spacetimes $(\tilde{V}_4, \tilde{g})$. In this transformation $\boldsymbol{l}$ is taken to be the most general geodesic and shear-free null vector field for the metrics (1), and it can be

[2]The Kerr-Schild transformation is obviously a GKS transformation with the Minkowski spacetime as the initial one.

shown that it has the same properties with respect to the new metrics (3). The case with a nonrotating $\boldsymbol{l}$ was given in [24], whereas the rotating case was treated in [22] (see also [18]). Here we present an unified treatment for both cases.

The expression for $\boldsymbol{l}$ is the following:

$$\boldsymbol{l}=\frac{-a}{\sqrt{2}(1+\Omega\bar{\Omega})}\left[(1+\Omega\bar{\Omega})\frac{\boldsymbol{d}t}{a}-(1-\Omega\bar{\Omega})\boldsymbol{d}\chi+\frac{2\Sigma_k}{1+\xi\bar{\xi}}(\bar{\Omega}\boldsymbol{d}\xi+\Omega\boldsymbol{d}\bar{\xi})\right], \tag{4}$$

where Ω is a complex function of the coordinates $\{\chi,\xi,\bar{\xi}\}$ such that it satisfies the following two complex partial differential equations:

$$(1+\xi\bar{\xi})\Omega\Omega_{,\xi}-\bar{\xi}\Omega^2=(\Sigma_k\Omega)_{,\chi}, \tag{5}$$

$$(1+\xi\bar{\xi})\Omega_{,\bar{\xi}}+\xi\Omega=-\Sigma_k^2\Omega\left(\frac{\Omega}{\Sigma_k}\right)_{,\chi}. \tag{6}$$

This system of partial differential equations comes from the shear-free and geodesic character of $\boldsymbol{l}$, and the nondependence on the coordinate t is a consequence of imposing the stationary character on the GKS spacetimes (3). The multiplicative factor of $\boldsymbol{l}$ was chosen so that it is affinely parametrized ($l^b\nabla_b l^a=0$).

Then, from Eqs. (3),(1),(4) the line element of these stationary spacetimes is

$$d\tilde{s}^2=-dt^2+a^2\left(d\chi^2+\frac{4\Sigma_k^2 d\xi d\bar{\xi}}{(1+\xi\bar{\xi})^2}\right)+\frac{a^2H}{(1+\Omega\bar{\Omega})^2}\left[(1+\Omega\bar{\Omega})\frac{dt}{a}-(1-\Omega\bar{\Omega})d\chi+\frac{2\Sigma_k}{1+\xi\bar{\xi}}(\bar{\Omega}d\xi+\Omega d\bar{\xi})\right]^2. \tag{7}$$

On the other hand, from part of the Einstein equations we get partial differential equations for the function H (the rest of Einstein's equations determine completely the energy-momentum content). A remarkable feature of these equations is the fact that when we write them in an adequate Newman-Penrose (NP) basis (see the basis in the Appendix), they can be solved without introducing a coordinate system. The solution depends on whether $\boldsymbol{l}$ has rotation or not, equivalently, on whether the complex divergence ρ of $\boldsymbol{l}$ (the definition of this quantity can be found for, instance, in [20]) is real or not. We can write the expression found for H in the following form:

$$H=\begin{cases} 2m\rho+wH_2 & \text{when} \quad \rho=\bar{\rho}, \\ m(\rho+\bar{\rho}) & \text{when} \quad \rho\neq\bar{\rho}, \end{cases} \tag{8}$$

where m and w are arbitrary constants, and H_2 is given by

$$H_2=\begin{cases} 1+\sqrt{2}a\rho\tan^{-1}(\sqrt{2}a\rho) & \text{for } k=1, \\ 1/\rho^2 & \text{for } k=0, \\ 1-\sqrt{2}a\rho\tanh^{-1}(\sqrt{2}a\rho) & \text{for } k=-1. \end{cases} \tag{9}$$

As we have said before, the case $\rho=\bar{\rho}$ was solved in [24], where the result was given in a different form, and the case $\rho\neq\bar{\rho}$ was solved in [22]. As we can see from these expressions, the function H depends only on the expansion of $\boldsymbol{l}$ $[-1/2(\rho+\bar{\rho})=1/2\nabla_a l^a]$.

We can compute for both cases the expression of ρ, which is also a spin coefficient, using the NP basis constructed in the Appendix. Taking into account Eqs. (5),(6) we find the following result:

$$\rho=\begin{cases} \dfrac{1}{\sqrt{2}a}\dfrac{\Omega_{,\chi}}{\Omega} & \text{for } \Omega\neq 0 \\ \dfrac{1}{\sqrt{2}a}\dfrac{\Sigma_{k,\chi}}{\Sigma_k} & \text{for } \Omega=0. \end{cases} \tag{10}$$

Once we have found the possible functions Ω, expressions (8),(9),(10) determine the function H, and hence we have determined completely the line element (7). As is clear, in order to find Ω we need to solve the system of partial differential equations (5),(6). As it can be shown, (5),(6) is a quasilinear system that can be solved completely [22] (more details are given in [18]), in such a way that Ω is given implicitly by the following equation:

$$G\left(\frac{e^{-\sqrt{-k}\chi}\Omega-\xi}{1+\bar{\xi}e^{-\sqrt{-k}\chi}\Omega},\frac{(1-\xi\bar{\xi})\Sigma_{k,\chi}\Omega-\xi+\bar{\xi}\Omega^2}{(1+\xi\bar{\xi})\Sigma_k\Omega}\right)=0, \tag{11}$$

where $G(z_1,z_2)$ is any analytic complex function of two complex variables $\{z_1,z_2\}$. In the cases $k=0,1$ we must impose $\partial_{z_2}G\neq 0$, otherwise we would obtain a vanishing function H, which means that the transformation (3) is the identity. On the other hand, it turns out that in the nonrotating case ($\rho=\bar{\rho}$) Eqs. (5),(6) can be solved explicitly. The expression for Ω can be written as follows:

$$\Omega=(\mathcal{B}+\sqrt{1+\mathcal{B}^2})\mathcal{U}, \quad \mathcal{U}=\sqrt{\frac{\mathcal{Z}}{\bar{\mathcal{Z}}}} \tag{12}$$

where B and Z are real and complex functions, respectively, whose expressions are

$$\mathcal{B}=\frac{1}{\sqrt{\mathcal{Z}\bar{\mathcal{Z}}}}\{c(1+\xi\bar{\xi})\Sigma_k-[(1-\xi\bar{\xi})\cos\phi+(\xi+\bar{\xi})\sin\phi]\Sigma_{k,\chi}\},$$

$$\mathcal{Z}=2\xi\cos\phi-(1-\xi^2)\sin\phi,$$

where c and ϕ are arbitrary constants.

As we have already said, we have performed the GKS transformation in such a way that the final metrics $\tilde{g}_{ab}$ are of the perfect-fluid type. Then, the energy-momentum tensor has the form $\tilde{T}_{ab}=(\tilde{\varrho}+\tilde{p})\tilde{u}_a\tilde{u}_a+\tilde{p}\tilde{g}_{ab}$, where

$$\tilde{\boldsymbol{u}}=\frac{1}{\sqrt{1-H}}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}t}, \quad \tilde{\varrho}=\frac{3k}{a^2}(1-H)=-3\tilde{p}. \tag{13}$$

As we can see, $\tilde{\boldsymbol{u}}$ $(\tilde{u}^a\tilde{u}_a=-1)$ can only be defined when $1-H>0$. As is obvious, this condition depends on the choice of the function Ω [more precisely, it depends on the choice of the function G in Eq. (11)] and the constants m,w. It is possible that in some cases this condition is automatically fulfilled, but in some other cases it would imply that we can only define this vector field in the region determined by the inequality $1-H>0$ (see [22] for more details).

The nonvanishing kinematical quantities associated with $\tilde{\boldsymbol{u}}$ are the acceleration $\tilde{a}^a$ and the rotation $\tilde{\omega}_{ab}$. Using the NP basis constructed in the Appendix (and the associate differential operators), their expressions are

$$\tilde{\omega}_{ab}=-\left(\frac{\bar{\delta}H}{1-H}\tilde{v}_{[a}\tilde{m}_{b]}+\text{c.c.}\right)-\frac{\sqrt{2}(\rho-\bar{\rho})H}{\sqrt{1-H}}\tilde{m}_{[a}\bar{\tilde{m}}_{b]}, \tag{14}$$

$$\tilde{a}^a=-\frac{DH}{\sqrt{2(1-H)}}\tilde{v}^a-\left(\frac{\bar{\delta}H}{1-H}\tilde{m}^a+\text{c.c.}\right), \tag{15}$$

where c.c. stands for complex conjugation. It is interesting to note that from Eq. (14) we can deduce that the rotation $\tilde{\omega}_{ab}$ of $\tilde{\boldsymbol{u}}$ vanishes if and only if so does the rotation $\rho-\bar{\rho}$ of l^a.

On the other hand, the vector field $\boldsymbol{\partial}/\boldsymbol{\partial}t$ is the timelike Killing vector field. In general there are no more Killing vectors (see [22]), although there are special cases (special forms of the complex function Ω) in which there is an additional Killing vector. With regard to the algebraic structure of these spacetimes, they are in general Petrov type-II unless the following condition holds:

$$\Omega(k\Omega^2+\Omega_{,\chi}^2)\Omega_{,\chi\chi\chi}=[3(\Omega\Omega_{,\chi\chi}-\Omega_{,\chi}^2+k\Omega^2)\Omega_{,\chi\chi} -k\Omega(4\Omega_{,\chi}^2+k\Omega^2)]\Omega_{,\chi}. \tag{16}$$

In that case the Petrov type is D and the fluid velocity does not lie in the preferred two-space spanned by the two multiple null principal directions of the Weyl tensor. As it can be checked, the function Ω given in Eq. (12) satisfies this equation and, therefore, all the spacetimes in this case $(\rho=\bar{\rho})$ are Petrov type-D.

Now, we are going to carry out the second step of the procedure (see Fig. 1), that is, we are going to make a conformal transformation $\tilde{\boldsymbol{g}}\rightarrow\hat{\boldsymbol{g}}=\Phi^2\tilde{\boldsymbol{g}}$. To that end, in what follows we consider Φ to be any arbitrary function $R(x^c)$ such that in the limit $H\rightarrow 0$ [which can be done by taking in Eq. (8) the limit $m,w\rightarrow 0$] becomes a function of t only. Then, the complete transformation from the seed spacetimes (1) to the *perturbed* FLRW ones looks as follows:

$$\hat{g}_{ab}=R^2\tilde{g}_{ab}=R^2g_{ab}+2H(Rl_a)(Rl_b), \tag{17}$$

where we use a hat to denote objects associated with the final spacetimes $(\hat{V}_4,\hat{g})$. Then, we can see Eq. (17) as a GKS transformation in which the seed spacetimes are conformal to the metrics (1), which include the FLRW spacetimes, and the null vector field of the transformation is $R(x^a)\boldsymbol{l}$. This vector field is also geodesic and shear free in the final spacetimes, for any conformal transformation preserves these properties.

The line element of these spacetimes can be obtained by substituting expressions (1),(4) or Eq. (7) in Eq. (17), which yields to the following result:

$$ds^2=R^2\Bigg\{-dt^2+d\chi^2+\frac{4\Sigma_k^2 d\xi d\bar{\xi}}{(1+\xi\bar{\xi})^2}+\frac{H}{(1+\Omega\bar{\Omega})^2} \times\Bigg[(1+\Omega\bar{\Omega})\,dt-(1-\Omega\bar{\Omega})d\chi +\frac{2\Sigma_k}{1+\xi\bar{\xi}}(\bar{\Omega}d\xi+\Omega d\bar{\xi})\Bigg]^2\Bigg\}, \tag{18}$$

where we have eliminated the constant a by absorbing it into the conformal factor R and rescaling t.

Now, we are going to study the main geometrical properties of the final spacetimes $(\hat{V}_4,\hat{g})$ that follow from this construction and the properties of the spacetimes involved (see Fig. 1). In the next section we will discuss the main physical consequences of these properties.

First of all, these models have no symmetry in general, that is, there are in general no Killing vector fields. As we have pointed out above, in general the only Killing vector field in the $(\tilde{V}_4,\tilde{g})$ spacetimes is $\boldsymbol{\partial}/\boldsymbol{\partial}t$, which is now a conformal Killing vector field with a timelike character when $1-H>0$ (this condition depends on Ω and m,w). Therefore, the spacetime is conformally stationary in the region $1-H>0$, or in the whole spacetime in the case that this condition holds everywhere.

With regard to the algebraic structure of the spacetimes $(\hat{V}_4,\hat{g})$, we have to take into consideration the fact that the Weyl tensor is conformally invariant, and therefore, the final spacetimes will have the same algebraic structure, that is, they will be Petrov type-II in general, and type-D when condition (16) is fulfilled. Again, the case $\rho=\bar{\rho}$ imply type-D. Moreover, the null vector field $\hat{\boldsymbol{l}}=R\tilde{\boldsymbol{l}}=R\boldsymbol{l}$ is the null multiple eigenvector of the Weyl tensor.

Depending on the form we take for the conformal factor, some special features can appear (see Sec. III). The most simple case is when we take the conformal factor Φ to be a function of the coordinate t, namely $R(t)$. In that case, Eq. (17) is a GKS transformation in which the seed spacetimes are now the FLRW spacetimes, $R(t)\boldsymbol{l}$ being the null, geodesic, and shear-free vector field of the transformation. Within this special case, the particular subcase defined by $k=1$ and the choice in Eq. (11) of $G(z_1,z_2)=z_2+i/(\sqrt{2}c)$, c being an arbitrary real constant, was already given in [23,17], where the models were interpreted as the Kerr metrics in a cosmological background (it is important to note that these models are special particular cases of our models). Moreover, although the techniques used in these papers allowed one to find models containing only the closed $(k=1)$ FLRW models, the extension to the other cases $(k=0,-1)$ can be done by reparametrizations or limiting processes (see [15], and references therein). The extension to the Kerr-Newman case is given in [25] (see [15] for more information). Finally, it is important to remark that all the solutions of the type (17) previously known belong to this particular case in which $\Phi=R(t)$.

III. PHYSICAL PROPERTIES OF THE MODELS

In this section we are going to study the main physical characteristics and properties of the spacetimes $(\hat{V}_4,\hat{g})$, constructed in the previous section. To that end, we are going to consider the preferred unit timelike vector field of these spacetimes, $\hat{\boldsymbol{u}}$ $(\hat{g}_{ab}\hat{u}^a\hat{u}^b=-1)$, which is given by

$$\hat{u}^a \equiv R^{-1}\tilde{u}^a = \frac{1}{R\sqrt{1-H}}u^a. \tag{19}$$

As we can see, it is privileged because it is the unit timelike vector field proportional to the conformal Killing vector field $\boldsymbol{\partial}/\boldsymbol{\partial} t$. In the same way as it happened with the vector field $\tilde{\boldsymbol{u}}$ (13), $\hat{\boldsymbol{u}}$ can be defined only in the region $1-H>0$.

From the results obtained in the previous section, we deduce that the nonzero kinematical quantities of $\hat{\boldsymbol{u}}$ are the expansion $\hat{\theta}$, the rotation $\hat{\omega}_{ab}$, and the acceleration $\hat{a}^a$, that is to say, $\hat{\boldsymbol{u}}$ is shear free. The expression for these kinematical quantities is given by

$$\hat{\theta}=3R^{-1}\hat{u}^a\partial_a R \equiv 3R^{-1}\dot{R}, \tag{20}$$

$$\hat{\omega}_{ab}=R\tilde{\omega}_{ab}, \tag{21}$$

$$\hat{a}^a=R^{-2}\tilde{a}^a+R^{-1}\hat{\mathcal{D}}^a R, \tag{22}$$

respectively. The expressions for $\tilde{a}^a$ and $\tilde{\omega}_{ab}$ are given in Eqs. (14),(15).

From these expressions we can deduce some interesting properties of the cosmological models (18). The first important physical feature of these models is a consequence of the existence of the conformal timelike vector field $\boldsymbol{\partial}/\boldsymbol{\partial} t$. Studies of kinetic theory in general relativity [7,8] led to the conclusion that the spacetimes allowing the existence of isotropic radiation must be conformally stationary and the distribution function describing the radiation (gas of photons) must depend only on the first integral of the null geodesic equation, which is defined by the conformal Killing vector field (see also [26]). This result led to the establishment of the well-known Ehlers-Geren-Sachs theorem [8]. Recently, an extension of this result for *almost*-FLRW models has been given in [27].

Therefore, the models here obtained can be considered as inhomogeneous exact perturbations of the FLRW cosmological models, compatible with the existence of isotropic radiation (see [28] for a different study of the rotation-free case). In our case, the observers who see the radiation isotropic are those moving along the world lines of the unit velocity field $\hat{\boldsymbol{u}}$. Then, we can use these models to study a universe containing an isotropic CMBR. As is well known, in this case the changes in the redshift measured by the preferred observers are due only to the expansion (isotropic contribution) and to the acceleration (gravitational redshift)

$$\lambda^{-1}d\lambda=(\tfrac{1}{3}\hat{\theta}+\hat{a}_a\hat{e}^a)d\hat{l},$$

where $\hat{e}^a$ is a unit spacelike vector representing an orthogonal direction to $\hat{\boldsymbol{u}}$ and $d\hat{l}$ is a proper distance element relative to $\hat{\boldsymbol{u}}$ (see [11,12,14], and references therein). Since the magnitude of the acceleration $|\hat{a}^a|$ depends on the function H, we can control the term corresponding to the gravitational redshift.

From the expressions for the nonzero kinematical quantities (20)–(22), we can deduce other properties of these models. Looking at the expansion (20) of $\hat{\boldsymbol{u}}$, it follows that the preferred congruence of world lines is in general expanding (or contracting), and the conformal factor R is a scale factor,[3] which reduces to the scale factor of the FLRW models in the limit $H\to 0$. Moreover, from Eq. (21) we can see that these models have, in general, rotation. The rotation vanishes only when the rotation (14) of the stationary spacetimes (7) vanishes, that is, only when the rotation of $\boldsymbol{l}$ vanishes $(\rho=\overline{\rho})$. This kind of cosmological model, in which there is a shear-free and irrotational timelike congruence, has been studied recently in connection with the study of the Newtonian limit of general relativity in a cosmological context [29] (see also [30]). Following the naming of this work, the spacetimes $(\hat{V}_4,\hat{g})$ in which $\hat{\omega}_{ab}$ vanishes would be *quasi-Newtonian cosmologies* and the preferred congruence would be a *Newtonian-like* timelike congruence. Later, we will study the effect of the rotation on these cosmological models in two examples. Finally, the expression for the acceleration (22) tells us that in general the world lines of the preferred observers are not geodesics.

Now, we are going to study the energy-momentum content of these cosmological models. To that end we need to compute the energy-momentum tensor associated with the metrics (18). This calculation can be done by computing the Einstein tensor of the metrics (18) and using Einstein's equations $\hat{G}_{ab}=\hat{T}_{ab}$. A simple way to carry out this calculation is to use the well-known formulas for a conformal transformation (see, e.g., [19,20]), which allow us to compute all the quantities for the final spacetimes $(\hat{V}_4,\hat{g})$ in terms of the

[3]The scale factor is always fixed up to a general first integral of the velocity field, since given the expansion θ, the scale factor is defined by $3R^{-1}u^a\partial_a R=\theta$.

initial ones $(\tilde{V}_4,\tilde{g})$ and the conformal factor Φ. On the other hand, in order to express the results of the calculations and to facilitate the study of the energy-momentum content, we are going to decompose the energy-momentum tensor $\hat{T}_{ab}$ with respect to the unit velocity $\hat{\boldsymbol{u}}$ (19) of the preferred observers. This decomposition is standard and is as follows:

$$\hat{T}_{ab}=\hat{\varrho}\hat{u}_a\hat{u}_b+\hat{p}\hat{h}_{ab}+2\hat{q}_{(a}\hat{u}_{b)}+\hat{\Pi}_{ab},$$

where $\hat{h}_{ab}\equiv\hat{g}_{ab}+\hat{u}_a\hat{u}_b$ is the orthogonal projector and $\hat{\varrho}$, $\hat{p}$, $\hat{q}^a$, $\hat{\Pi}_{ab}$ are the energy density, isotropic pressure, heat flow, and anisotropic pressures relative to $\hat{\boldsymbol{u}}$, respectively (see [11] for more details). For a generic conformal factor $\Phi=R(x^a)$ they are given by

$$\hat{\varrho}=\varrho_0-3kR^{-2}H+3R^{-2}(\hat{\mathcal{D}}^aR)(\hat{\mathcal{D}}_aR)-2R^{-1}\hat{\mathcal{D}}^a\hat{\mathcal{D}}_aR, \tag{23}$$

$$\hat{p}=p_0+kR^{-2}H-3R^{-2}(\hat{\mathcal{D}}^aR)(\hat{\mathcal{D}}_aR)+\tfrac{4}{3}R^{-1}\hat{\mathcal{D}}^a\hat{\mathcal{D}}_aR+2\hat{a}^a\hat{\mathcal{D}}_aR, \tag{24}$$

$$\hat{q}^a=\tfrac{2}{3}\hat{\mathcal{D}}^a\hat{\theta}+2R^{-1}\hat{\omega}^{ab}\hat{\mathcal{D}}_bR, \tag{25}$$

$$\hat{\Pi}_{ab}=-2R^{-1}\hat{\mathcal{D}}_{\langle a}\hat{\mathcal{D}}_{b\rangle}R, \tag{26}$$

where here the angled brackets on indices denote the symmetric and trace-free part. Moreover, ϱ_0 and p_0 are given by

$$\varrho_0=\frac{3}{R^2}(\dot{R}^2+k),\quad p_0=-\frac{1}{R^2}(\dot{R}^2+2R\ddot{R}+k). \tag{27}$$

As we can see, these terms of the density and pressure have the same dependence on the scale factor as the density and pressure of the FLRW models have, and in the limit $H\to 0$ they coincide.

With regard to the covariant criterion exposed in the introduction, we can check, by studying the dependence of the derivative $\hat{\mathcal{D}}_aR$ on the function H, that given a ϵ we can always choose the constants contained in the function H (m and w) so that the kinematical variables (20)–(22) and the matter variables (23)–(26) associated with the fluid velocity $\hat{\boldsymbol{u}}$ satisfy the corresponding inequalities.

On the other hand, as is well-known the energy-momentum tensor determines part of the Riemann curvature tensor, the Ricci tensor (through Einstein's equations). The other part of the curvature tensor is described by the Weyl tensor $\hat{C}_{abcd}$. Their components can be decomposed with respect to a given unit timelike vector field [we consider here the preferred timelike vector field $\hat{\boldsymbol{u}}$ given in Eq. (19)] into the electric $\hat{E}_{ab}$ and magnetic $\hat{H}_{ab}$ parts (see, e.g., [12,13]), which are spacelike symmetric and trace-free tensors. In our case, these tensors are in general nonzero, but when the rotation $\hat{\omega}_{ab}$ vanishes, we can show that the magnetic part vanishes,

$$\hat{\omega}_{ab}=0\Rightarrow\hat{H}_{ab}=0.$$

This is in fact a consequence of the Ricci identities for the velocity field $\hat{\boldsymbol{u}}$ (see [12,13] for details on the Ricci identities for a unit timelike vector field) and the vanishing of the shear. Furthermore, taking into account that when the rotation vanishes we have $\rho=\bar{\rho}$, and since in this case the Petrov type of our models is D, we deduce that the electric part $\hat{E}_{ab}$ becomes degenerate, that is, at least two of their eigenvalues are equal (see [20] for details on the Petrov classification in terms of the electric and magnetic parts of the Weyl tensor). Finally, we can also check that for a given ϵ, we can choose the constant m and w so that the electric and magnetic parts associated with $\hat{\boldsymbol{u}}$ satisfy the conditions of the criterion, showing that our models are closed to the FLRW cosmological models in the sense already explained.

Now, in order to go deeply into the study of the cosmological models obtained, let us consider some examples. Here, we will deal with the following two cases: (i) The case $R=R(t)$, and (ii) the case $R=R(\hat{\tau})$, where $\hat{\tau}$ is the proper time of $\hat{\boldsymbol{u}}$ (19).

The case in which $R=R(t)$, which is the most simple one, has been the only one studied in the literature, and as we have pointed out before, only in some very special cases. The simple form of the conformal factor allows us to write the metric in the GKS form with the FLRW models as the initial spacetimes, that is $\hat{\boldsymbol{g}}=\boldsymbol{g}_{\rm FLRW}+2H\boldsymbol{L}\otimes\boldsymbol{L}$ (see Sec. II for more details).

Taking into account this special form of the conformal factor we can expand the expressions (23)–(26). After some calculations we arrive at the following result:

$$\hat{\varrho}=\varrho_{\rm FLRW}+\frac{3}{R^2}(R_{,\tau}^2-k)H+\sqrt{2}\frac{R_{,\tau}}{R^2}\left\{\frac{2-H}{1-H}DH-2(\rho+\bar{\rho})H\right\}+\frac{2}{R^2}(R_{,\tau}^2-RR_{,\tau\tau})\frac{H^2}{1-H}, \tag{28}$$

$$\hat{p}=p_{\rm FLRW}-\frac{1}{R^2}(R_{,\tau}^2+2RR_{,\tau\tau}-k)H+\frac{\sqrt{2}}{3}\frac{R_{,\tau}}{R^2}\left\{\frac{5H-4}{1-H}DH+4(\rho+\bar{\rho})H\right\}+\frac{2}{3R^2}(R_{,\tau}^2-RR_{,\tau\tau})\frac{H^2}{1-H}, \tag{29}$$

$$\hat{q}_a=\frac{1}{R^2}\{2(R_{,\tau}^2-RR_{,\tau\tau})H+\sqrt{2}R_{,\tau}DH\}\frac{\hat{v}_a}{1-H}+\frac{R_{,\tau}}{R^2}\left\{\frac{\bar{\delta}H}{\sqrt{1-H}}\hat{m}_a+c.c.\right\}, \tag{30}$$

$$\hat{\Pi}_{ab}=\frac{4}{3R^2}\left\{\frac{R_{,\tau}}{\sqrt{2}}\left[\frac{2-H}{1-H}DH+(\rho+\overline{\rho})H\right]\right.$$
$$+(R^2_{,\tau}-RR_{,\tau\tau})\frac{H^2}{1-H}\Bigg\}(\hat{v}_a\hat{v}_b-\hat{m}_{(a}\hat{\bar{m}}_{b)})$$
$$+2\frac{R_{,\tau}}{R^2}\left\{\frac{\overline{\delta}H}{\sqrt{1-H}}\hat{v}_{(a}\hat{m}_{b)}+\text{c.c.}\right\}, \tag{31}$$

where τ is the proper time of the initial FLRW model, which is related to the conformal time t by $d\tau=R(t)dt$. On the other hand, the terms $\varrho_{\rm FLRW}$ and $p_{\rm FLRW}$ are the energy density and the pressure of the initial FLRW spacetimes:

$$\varrho_{\rm FLRW}=\frac{3}{R^2}(R^2_{,\tau}+k),\quad p_{\rm FLRW}=-\frac{1}{R^2}(R^2_{,\tau}+2RR_{,\tau\tau}+k),$$

which have the same form as (27) but changing $\hat{u}^a\nabla_a$ by $u^a\nabla_a$. Moreover, as is obvious $\varrho_{\rm FLRW}$ and $p_{\rm FLRW}$ do not depend on the function H.

Now, let us study some properties of these quantities. First of all, from their expressions (28)–(31), we can see explicitly how these models are close to the FLRW models in the sense exposed in the introduction. In particular, we can see how the function H (or the constant parameters m and w that it contains) controls the magnitude of the quantities in the left-hand side of the inequalities of our criterion.

On the other hand, in the static limit $(R_{,\tau}=0)$, we recover obviously the GKS models (7). We can also check that we cannot impose the energy-momentum tensor $\hat{T}_{ab}$ to be of the perfect-fluid type with respect to $\hat{\boldsymbol{u}}$, otherwise either we get the stationary models (7) or the acceleration $\hat{a}^a$ and rotation $\hat{\omega}_{ab}$ would vanish, which implies that our models are exactly FLRW models.

In what follows, and in order to study the evolution, we are going to consider only those models in which the initial FLRW spacetime is flat $(k=0)$ and has an equation of state $p=\gamma\varrho$ [we will consider $\gamma\in(-1,1)$ and $\varrho>0$ in order that they satisfy the dominant energy condition], which implies that their scalar factor grows as a power of the proper time τ, that is

$$R(\tau)=R_0\left(\frac{\tau}{\tau_0}\right)^{2/[3(1+\gamma)]}, \tag{32}$$

where R_0 and τ_0 are arbitrary constants. Now, it is important to notice that the proper time of the fundamental observers in the FLRW background and that of the preferred observers in the final spacetimes are related by (choosing the same origin)

$$\hat{\tau}=\tau\sqrt{1-H},$$

and, therefore, both parameters, $\hat{\tau}$ and τ, trace properly the evolution of the quantities in the final spacetimes (along the world lines of the preferred observers). Moreover, from Eq. (32), the expansion $\hat{\theta}$ in this case is

$$\hat{\theta}=\frac{2}{1+\gamma}\frac{1}{\hat{\tau}}, \tag{33}$$

and therefore, in this case it depends only on the proper time $\hat{\tau}$.

When we introduce Eq. (32) in expressions (28)–(31), we realize that these quantities evolve according to two different powers of $\hat{\tau}$. There are terms that evolve like the background terms $\varrho_{\rm FLRW}$ and $p_{\rm FLRW}$, that is, like $\hat{\tau}^{-2}$, and the other terms evolve like $\hat{\tau}^{-(5+3\gamma)/[3(1+\gamma)]}$. In the case of the energy-density parameter $\hat{\Omega}$, from expressions (33),(28), we obtain the following relationship:

$$\hat{\Omega}\equiv\frac{3\hat{\varrho}}{\hat{\theta}^2}=1+\gamma H^2+\frac{(1+\gamma)\hat{\tau}_0}{R_0\sqrt{2(1-H)}}[(2-H)DH$$
$$-2(\rho+\overline{\rho})(1-H)H]\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{(1+3\gamma)/[3(1+\gamma)]}, \tag{34}$$

where $\hat{\tau}_0\equiv\tau_0\sqrt{1-H}$. This expression shows how terms containing H will become important (for $3\gamma>-1$) as the models evolve. Furthermore, if we restrict ourselves to the case of a dust FLRW background $(\gamma=0)$, corresponding to the well-known Einstein–de Sitter universe [31], we can get from Eq. (34) the following equation:

$$\hat{\Omega}_{\hat{\tau}}-1=(\hat{\Omega}_{\hat{\tau}_0}-1)\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{1/3}, \tag{35}$$

which shows how the density parameter deviates from the unity in a matter-dominated stage of the universe. Here the subscripts denote the proper time at which the density parameter is evaluated.

To finish this example, we are going to see what is the behavior of the rotation and the acceleration of the preferred observers. This can be studied by using expressions (21),(22), from which we can get the following relations:

$$\frac{|\hat{\omega}_{ab}|}{\hat{\theta}}=\frac{(1+\gamma)\hat{\tau}_0}{2R_0}|\tilde{\omega}_{ab}|\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{(1+3\gamma)/[3(1+\gamma)]},$$

$$\frac{|\hat{a}^a|^2}{\hat{\theta}^2}=\frac{(1+\gamma)^2\hat{\tau}_0^2}{4R_0^2}|\tilde{a}^a|^2\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{2(1+3\gamma)/[3(1+\gamma)]}+\frac{1}{9}H^2$$
$$+\frac{(1+\gamma)\hat{\tau}_0}{3\sqrt{2}R_0}\frac{HDH}{\sqrt{1-H}}\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{(1+3\gamma)/[3(1+\gamma)]}. \tag{36}$$

As we can see, both kinematical quantities will become important as the models evolve. Moreover, in the case of the rotation we find the following relation:

$$|\hat{\omega}_{ab}|_{\hat{\tau}}=|\hat{\omega}_{ab}|_{\hat{\tau}_0}\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{-2/[3(1+\gamma)]}, \tag{37}$$

which shows the fact that the rotation decays slower (for $\gamma > -1/3$) than the expansion (33).

The second example we are going to consider here, the case when $R=R(\hat{\tau})$, is an example of a different choice of the conformal factor. It has the special feature that always the scale factor and expansion associated with the preferred observers only depend on the proper time $\hat{\tau}$, and the same happens with the *background* terms ϱ_0 and p_0 in the energy density and pressure (23),(24).

In this case the explicit calculation of the quantities (23)–(26) is more involved. Here, for the sake of brevity, we only give the expression for the energy density:

$$\hat{\varrho}=\varrho_o-\frac{3k}{R^2}H+2\frac{\dot{R}}{R^2}\left\{\sqrt{2(1-H)}[(2-3H)DH-(\rho+\bar{\rho})H]\right.$$
$$\left.+\tilde{\tau}\left[DDH-(\rho+\bar{\rho})DH-2(\rho-\bar{\rho})^2\frac{H}{1-H}+\frac{\delta H\bar{\delta}H}{(1-H)^2}\right]\right\}$$
$$-\frac{\dot{R}^2+2R\ddot{R}}{2(1-H)R^2}[(H\sqrt{2(1-H)}+\tilde{\tau}DH)^2+\tilde{\tau}\delta H\bar{\delta}H], \quad (38)$$

where $\tilde{\tau}$ is the proper time of the fluid velocity $\tilde{\boldsymbol{u}}$ of the stationary spacetimes $(\tilde{V}_4,\tilde{g})$. The relation between $\tilde{\tau}$ and $\hat{\tau}$ is $R^{-1}(\hat{\tau})d\hat{\tau}=d\tilde{\tau}$, that is, $\tilde{\tau}$ is a function of $\hat{\tau}$ only and vice versa. If we consider, like in the previous example, the particular case in which $k=0$ and $R(\hat{\tau})$ is a power of $\hat{\tau}$ [like in Eq. (32)], apart from the terms evolving like the powers $\hat{\tau}^{-2}$ (background terms) and $\hat{\tau}^{-(5+3\gamma)/[3(1+\gamma)]}$, which also appeared in this previous example, we find terms evolving like $\hat{\tau}^{-4/[3(1+\gamma)]}$. Therefore, for ordinary matter, terms evolving with this new power of $\hat{\tau}$ are the slowest terms going to zero, and hence, for large enough values of the proper time $\hat{\tau}$ they will dominate. With regard to the kinematical quantities, in the case of the rotation the relation (37) remains valid, whereas in the case of the acceleration, the quantity $|\hat{a}^a|^2/\hat{\theta}^2$ has the same dependence on $\hat{\tau}$ that in the previous example (36), but with different coefficients.

Considering now the case in which the underlying FLRW model is the Einstein–de Sitter universe ($\gamma=0$), the analogous equation to Eq. (35) is

$$\hat{\Omega}_{\hat{\tau}}-1=(\hat{\Omega}_{\hat{\tau}_0}-1)\left\{\frac{F_1(\rho,\bar{\rho})}{F_1(\rho,\bar{\rho})+F_2(\rho,\bar{\rho})}\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{1/3}\right.$$
$$\left.+\frac{F_2(\rho,\bar{\rho})}{F_1(\rho,\bar{\rho})+F_2(\rho,\bar{\rho})}\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{2/3}\right\}, \quad (39)$$

where the explicit expression of the functions F_1 and F_2 can be obtained from Eq. (38).

We finish this section by comparing the results obtained in these two examples with the solutions in the linearized theory about a dust flat-FLRW background (see, e.g., [6]). In particular we focus on the solutions for the energy-density perturbations. In order to study the perturbations of this quantity, several approaches (see [6]) use some gauge-invariant variables related directly with the dimensionless density contrast.[4] In the covariant and gauge-invariant approach to the linearized perturbations of a FLRW cosmological model [5], one of the most used quantities is the *comoving fractional density gradient* (see [5,14] and references therein), which in our notation is $R\hat{\varrho}^{-1}\hat{\mathcal{D}}_a\hat{\varrho}\equiv\hat{\psi}_a$. In all these approaches, the solution of the evolution equation has two different modes, a growing mode which evolves like $\hat{\tau}^{3/2}$, and a decaying mode which evolves like $\hat{\tau}^{-1}$. In our case, if we consider the quantity $\hat{\Omega}-1$, which gives the information about the deviation of the density parameter from the unity (its value in a dust FLRW model), from expressions (35),(39), we can see that two differences appear: first, there are not decaying terms and second, there are two growing terms [in the first case (35) there is only one], the usual term proportional to $\hat{\tau}^{3/2}$ (that is, proportional to the scale factor) and another term proportional to $\hat{\tau}^{1/3}$. On the other hand, we can compute in our models the comoving fractional density gradient $\hat{\psi}_a$. As we can see below, this quantity is not a linear combination of powers of $\hat{\tau}$, which is a consequence of the fact that our study is nonperturbative, in contrast with the form of the solutions in the linearized theory, which come from the linearized dynamical equations. After some calculations, we get the following expression for $\hat{\psi}_a$:

$$\hat{\psi}_a=\frac{1}{1+Q_1(\hat{\tau}/\hat{\tau}_0)^{1/3}+Q_2(\hat{\tau}/\hat{\tau}_0)^{2/3}}\{P_{1a}(\hat{\tau}/\hat{\tau}_0)^{-1/3}+P_{2a}$$
$$+P_{3a}(\hat{\tau}/\hat{\tau}_0)^{1/3}+P_{4a}(\hat{\tau}/\hat{\tau}_0)^{2/3}\}, \quad (40)$$

where the form of the objects Q_1, Q_2, P_{1a}, P_{2a}, P_{3a}, and P_{4a} can be obtained from the expressions for the energy density (28),(38). In the case of the first example we find that $Q_2=P_{4a}=0$. In general, the scalars Q_1 and Q_2 are functions of ρ and $\bar{\rho}$ only, and therefore $\dot{Q}_1=\dot{Q}_2=0$. On the other hand, the one-forms P_{Ia} $(I=1,\ldots,4)$ satisfy $|P_{Ia}|^{\cdot}=\hat{u}^b\partial_b|P_{Ia}|=0$. Moreover, when $H\to 0$, all these objects tend to zero like H. As we can see, the scalar associated with this quantity (40), that is $|\hat{\psi}_a|$, has the following asymptotic behavior:

$$|\hat{\psi}_a|\overset{\hat{\tau}\to 0}{\longrightarrow}|P_{1a}|\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{-1/3},$$

$$|\hat{\psi}_a|\overset{\hat{\tau}\to\infty}{\longrightarrow}K, \quad \dot{K}=0.$$

The first equation shows how this quantity becomes singular at $\hat{\tau}=0$, like in the linearized theory. However, the typical behavior of a decaying mode in the linearized theory is of the

[4]The dimensionless density contrast is usually defined by $(\varrho_p-\varrho_b)/\varrho_b$, where ϱ_p and ϱ_b are the energy density of the perturbed and background spacetimes, respectively. However, this quantity is, in general, gauge dependent.

type $\hat{\tau}^{-1}$. The second equation, where K is $|P_{3a}|/|Q_1|$ in the first example and $|P_{4a}|/|Q_2|$ in the second example, shows how the growth of $|\hat{\psi}_a|$ is bounded, contrary to what happens in the linearized theory, where the growing mode, which evolves like $\hat{\tau}^{2/3}$, goes to infinity. Note also that K is for both examples the quotient of two quantities of first order in H and therefore, K will be a zero-order quantity in H, or in other words, in general it does not go to zero when H does.

IV. CONCLUSIONS AND DISCUSSION

In this paper we have constructed cosmological models which according to the covariant criterion exposed in the introduction can be seen as exact (nonlinear) perturbations of the standard FLRW cosmological models. The advantages of this approach are, on one hand, that we can study any physical quantity without having the problems of interpretation that some quantities have in the linearized theory of perturbations (specially those which are not gauge invariant), and on the other hand, the possibility of taking into account nonlinear effects. We have also seen how the special characteristics of the method used here provided the necessary tools to make an exhaustive study of the geometrical and physical properties of these models. In this study we have considered two particular examples, in which we have looked at some interesting quantities, and also we have been able to compare our models with the results of the linearized theory, showing the differences and similarities.

The models found in this work have some interesting properties. Among them we can mention the fact that in general they have no Killing vector field, but they have a conformal timelike vector field, which means that the models are compatible with the existence of isotropic radiation. Furthermore, the motion of the observers that would ''see'' the radiation as isotropic is in general rotating. Moreover, we have two free functions in order to fix a particular model. On one hand, the complex function G Eq. (11), in which can control, for instance, the rotation of the preferred observers, and on the other hand we have the conformal factor Φ, which plays the role of a scale factor [see Eq. (20)] and controls the underlying FLRW model. Apart from this freedom, we have also the arbitrary constants m and w, which control the deviation from the FLRW models. It is important to remark that we have seen that the quantities associated with the preferred observers (with unit velocity $\hat{\boldsymbol{u}}$) satisfy the inequalities of the criterion given in the introduction, provided the arbitrary constants m and w are chosen properly for a given ϵ.

With regard to the lacks of these cosmological models, we can mention that although they are very general in the sense that they do not possess any isometry, they are not so general from the point of view of the Petrov classification, since they are algebraically special (although they have a minimum degree of degeneration given that their Petrov type is II in general). With regard to the matter content, in the procedure here described we have not introduced any description of the matter, we have just decomposed the energy-momentum tensor like a general fluid with velocity $\hat{\boldsymbol{u}}$, and then we have seen how in the limit $H \rightarrow 0$ it reduces to a perfect fluid, and $\hat{\boldsymbol{u}}$ becomes the fluid velocity of the resulting FLRW spacetimes. However, we could try to give such a description, for instance, by means of the kinetic theory, or using a consistent thermodynamical scheme, or we can divide the total energy-momentum tensor in several components and then interpret each of them separately.

On the other hand, the procedure shown in this paper is an example of how we can get exact cosmological models which can describe deviations from the standard FLRW models. This procedure can be generalized in different ways. Within the scheme presented here, we could relax the condition of obtaining perfect-fluid solutions in the first step of the procedure (the GKS transformation, see Fig. 1), although it is possible that in such a case the remaining system of equations could not be integrated completely. On the other hand, there are other procedures similar to the GKS transformation, which can be exploited in order to get exact families of cosmological models with general properties (see, e.g., [16,32]). An example of these procedures can be seen in the paper by Bonanos [32]. In this work, the author considers the following transformation: $g_{ab} \rightarrow g_{ab} - 2\zeta_{(ab)} + \zeta_{ca}\zeta^c{}_b$ and $g^{ab} \rightarrow g^{ab} + 2\zeta^{(ab)} + \zeta^{ac}\zeta^b{}_c$, where ζ_{ab} is a tensor (in general nonsymmetric) associated with the initial spacetime and subject to the condition $\zeta^a{}_c\zeta^c{}_b = 0$. As we can see, it includes the GKS transformation ($\zeta_{ab} = -Hl_a l_b$). In [32] a particular case of this transformation is studied in the case of a Minkowski background. It is shown that this transformation has some special properties similar to those of the GKS transformation. In this sense, it could be interesting to study it in the case of a FLRW background.

There are other models in cosmology that can be seen as exact perturbations of the FLRW models. For instance, the perfect-fluid models given in [33], which have also been constructed using the GKS transformation (in that case $\boldsymbol{l}$ is geodesic but not shear free). In general, they only have one Killing vector field, but on the other hand, they include only some particular FLRW models (within the three classes $k = 0,1,-1$). Other examples are provided by the the well-known Szekeres dust models [34]: in [35] an analogy of the equations of these models with those of the linearized theory is given (see also [15,36,37]), and in [38] they are reobtained by using approximation methods based on the Hamilton-Jacobi approach to irrotational dust models.

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APPENDIX: NEWMAN-PENROSE AND ORTHONORMAL BASIS

In this appendix we give a Newman-Penrose (NP) and an orthonormal basis (see [39,20]) for each of the different spacetimes used in our construction. The NP basis will be adapted to the null vector field $\boldsymbol{l}$ Eq. (4) of the GKS transformation. First of all, we are going to consider the initial metrics (1). In that case the Newman-Penrose basis

$\{\boldsymbol{l},\boldsymbol{k},\boldsymbol{m},\overline{\boldsymbol{m}}\}$ is given by $\boldsymbol{l}$ Eq. (4) and

$$\boldsymbol{k}=\frac{-a}{\sqrt{2}(1+\Omega\overline{\Omega})}\Bigg[(1+\Omega\overline{\Omega})\frac{\boldsymbol{d}t}{a}+(1-\Omega\overline{\Omega})\boldsymbol{d}\chi - \frac{2\Sigma_k}{1+\xi\overline{\xi}}(\overline{\Omega}\boldsymbol{d}\xi+\Omega\boldsymbol{d}\overline{\boldsymbol{\xi}})\Bigg],$$

$$\boldsymbol{m}=\frac{\sqrt{2}a}{1+\Omega\overline{\Omega}}\Bigg[\sqrt{\Omega\overline{\Omega}}\boldsymbol{d}\chi+\Sigma_k\sqrt{\frac{\overline{\Omega}}{\Omega}}(\boldsymbol{d}\xi-\Omega^2\boldsymbol{d}\overline{\boldsymbol{\xi}})\Bigg].$$

As is obvious, $\{R(t)\boldsymbol{l},R(t)\boldsymbol{k},R(t)\boldsymbol{m},R(t)\overline{\boldsymbol{m}}\}$ is a Newman-Penrose basis for the FLRW metrics. An orthonormal basis $\{\boldsymbol{u},\boldsymbol{v},\boldsymbol{e}_1,\boldsymbol{e}_2\}$ [$\boldsymbol{u}$ is the timelike vector field given in Eq. (2)] for Eq. (1) can be constructed in the usual way

$$\boldsymbol{u}=\frac{1}{\sqrt{2}}(\boldsymbol{l}+\boldsymbol{k}),\quad \boldsymbol{v}=\frac{1}{\sqrt{2}}(\boldsymbol{l}-\boldsymbol{k}),\quad \boldsymbol{e}_1=\frac{1}{\sqrt{2}}(\boldsymbol{m}+\overline{\boldsymbol{m}}),$$

$$\boldsymbol{e}_2=\frac{1}{\sqrt{2}i}(\boldsymbol{m}-\overline{\boldsymbol{m}}),$$

and obviously, $\{R(t)\boldsymbol{u},R(t)\boldsymbol{v},R(t)\boldsymbol{e}_1,R(t)\boldsymbol{e}_2\}$ is an orthonormal basis for the FLRW metrics.

From the NP basis given above we can construct a NP basis for the metrics (7), namely $\{\tilde{\boldsymbol{l}},\tilde{\boldsymbol{k}},\tilde{\boldsymbol{m}},\bar{\tilde{\boldsymbol{m}}}\}$, which is given by

$$\tilde{l}^a=l^a,\quad \tilde{k}^a=k^a+Hl^a,\quad \tilde{m}^a=m^a,$$

$$\tilde{l}_a=l_a,\quad \tilde{k}_a=k_a-Hl_a,\quad \tilde{m}_a=m_a.$$

An orthonormal basis $\{\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}},\tilde{\boldsymbol{e}}_1,\tilde{\boldsymbol{e}}_2\}$ is given by

$$\tilde{\boldsymbol{u}}=\frac{(1-H)\tilde{\boldsymbol{l}}+\tilde{\boldsymbol{k}}}{\sqrt{2(1-H)}},\quad \tilde{\boldsymbol{v}}=\frac{(1-H)\tilde{\boldsymbol{l}}-\tilde{\boldsymbol{k}}}{\sqrt{2(1-H)}},$$

$$\tilde{\boldsymbol{e}}_1=\frac{1}{\sqrt{2}}(\tilde{\boldsymbol{m}}+\bar{\tilde{\boldsymbol{m}}}),\quad \tilde{\boldsymbol{e}}_2=\frac{1}{\sqrt{2}i}(\tilde{\boldsymbol{m}}-\bar{\tilde{\boldsymbol{m}}}).$$

It is clear that a NP basis for the final metrics (18), which we call $\{\hat{\boldsymbol{l}},\hat{\boldsymbol{k}},\hat{\boldsymbol{m}},\bar{\hat{\boldsymbol{m}}}\}$, is given by $\{R\tilde{\boldsymbol{l}},R\tilde{\boldsymbol{k}},R\tilde{\boldsymbol{m}},R\bar{\tilde{\boldsymbol{m}}}\}$. And in the same way, we have an orthonormal basis $\{\hat{\boldsymbol{u}},\hat{\boldsymbol{v}},\hat{\boldsymbol{e}}_1,\hat{\boldsymbol{e}}_2\}$ by taking $\{R\tilde{\boldsymbol{u}},R\tilde{\boldsymbol{v}},R\tilde{\boldsymbol{e}}_1,R\tilde{\boldsymbol{e}}_2\}$.

Finally, we introduce the differential operators associated with the NP basis $\{\boldsymbol{l},\boldsymbol{k},\boldsymbol{m},\overline{\boldsymbol{m}}\}$ (see, e.g., [20]):

$$D\equiv l^a\nabla_a,\quad \Delta\equiv k^a\nabla_a,\quad \delta\equiv m^a\nabla_a,\quad \overline{\delta}\equiv\overline{m}^a\nabla_a.$$

For a function f that does not depend on t, like H, we have $\Delta f=-Df$.

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