Thermodynamically admissible equations for causal dissipative cosmology, galaxy formation, and transport processes in a gravitational collapse

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(Received 12 February 1999; published 21 October 1999)

We formulate a set of equations for a self-gravitating imperfect fluid that satisfies all the principles of both general relativity and nonequilibrium thermodynamics, where the latter are condensed in the covariant version of the recently proposed general equation for the nonequilibrium reversible-irreversible coupling (GENERIC). In doing so, Einstein's field equation is supplemented by fundamental and clearly structured transport equations for the sources of gravitational fields. The GENERIC framework determines the selection of the appropriate variables and the structure of the field equations compatible with the fundamental laws of thermodynamics. A nonzero cosmological constant cannot be ruled out by thermodynamic consistency criteria. In order to discuss the relationship to previous approaches, the simplified equations for bulk viscous cosmology are presented in some detail. [S0556-2821(99)06322-5]

PACS number(s): 98.80.Hw, 04.20.Cv, 04.40.Nr, 05.70.Ln

I. INTRODUCTION

''The correct treatment of dissipative effects for relativistic fluids raises certain delicate questions of principle, which do not arise in the nonrelativistic case. For this reason, and also because dissipation plays an increasingly important role in theories of the early universe, it will be worth our while here to develop the outlines of the general theory of relativistic imperfect fluids.'' (Weinberg [1], p. 53.)

How can we guarantee that such a general theory of dissipative or imperfect relativistic fluids is consistent with all the principles of nonequilibrium thermodynamics? This is the question we want to address in the present paper in the light of the recently developed general equation for the nonequilibrium reversible-irreversible coupling (GENERIC) of nonequilibrium thermodynamics.

In the usual procedure, the thermodynamic admissibility of relativistic hydrodynamics is implemented by formulating an entropy balance equation and simply verifying that the source term therein is always non-negative. As the classical theories obtained in this way by Eckart [2] and by Landau and Lifshitz [3] have problems concerning causality and stability [4,5], most current analyses of dissipative phenomena are based on the causal second-order theory of Israel and Stewart [6–8] or on extended irreversible thermodynamics [9]. More far-reaching principles of nonequilibrium thermodynamics have recently been condensed in the GENERIC structure [10,11]. The purpose of this paper is to develop a set of equations for imperfect fluids which are in accordance with both the principles of general relativity and that very restrictive new framework of nonequilibrium thermodynamics.

The general relativistic hydrodynamic equations proposed in this paper are expected to be relevant to describing the expansion of the universe, the formation of galaxies, and the gravitational collapse of stars into neutron stars whenever one needs to go beyond the frequently used model of a perfect fluid (see, e.g., the standard textbooks [1,12], the recent references [13–20], and references therein). For example, the beginnings of galaxy formation can be studied by looking at perturbations of a spherically symmetric and homogeneous universe filled with an imperfect fluid (see Sec. 15.10 of [1]); clearly, the structure of the hydrodynamic equations strongly affects the stability of small perturbations. In any attempt to understand the high entropy of the present universe one needs to incorporate a dissipative mechanism into the equations for the expansion of the universe; within the spherically symmetric and homogeneous standard model of the universe one can only introduce bulk or dilatational viscosity effects (see Sec. 15.11 of [1]), while other dissipative mechanisms can only be introduced in more detailed, less symmetric descriptions of the universe. For such cosmological and astrophysical applications it obviously is of crucial importance to have a set of hydrodynamic equations for a self-gravitating fluid that is consistent with all the established principles of nonequilibrium thermodynamics.

In cosmological and astrophysical applications, Einstein's fundamental equation for the gravitational field needs to be supplemented by rather ambiguous transport equations for mass, momentum, and energy. (See, e.g., the Introduction of [13]: ''In the absence of any well founded theory of nonequilibrium thermodynamics at very high energies or far from equilibrium, the best current option appears to be to apply standard relativistic non-equilibrium thermodynamics in and beyond its own range. This option is not as straightforward as it may sound, since there are difficulties and subtleties involved in standard relativistic thermodynamics.'' Or see the Introduction of [19]: ''Moreover it is indispensable to calculate the bulk and shear viscous pressure. Unluckily the solution of the Einstein equations does not provide any information about it. This is why a set of transport equations must be adopted.'') In this paper, a covariant version of the GENERIC structure is first proposed and then used for constructing unambiguous, clearly structured transport equations for self-gravitating fluids, deeply rooted in nonequilibrium thermodynamics. Finally, the equations for bulk viscous cosmology are considered in some detail. In the companion paper [21], the predictions of the new equations for bulk viscous cosmology are compared to those of various previous theories.

II. COVARIANT GENERIC

In a recent series of papers, a general formalism for nonrelativistic nonequilibrium systems, GENERIC, was developed by considering the compatibility of two levels of description [10], by studying a large number of specific examples [11], and by a projection-operator derivation from Hamilton's equations of motion [22]. Although the core part of the GENERIC formalism is an explicit time-evolution equation, it was shown to contain a set of covariant equations for relativistic hydrodynamics as a special realization [23,24]. In formulating a relativistic hydrodynamic theory, one is forced to introduce additional generalized force variables (related to velocity and temperature gradients and hence eventually also to the fluxes of momentum and energy), which are known to be essential for a causal theory. Moreover, the structure of the transport equations is severely restricted by the combination of the rather complementary principles of relativity and GENERIC. The previous set of hydrodynamic equations could be generalized to self-gravitating imperfect fluids by replacing all partial derivatives by covariant derivatives (minimal coupling). One can even verify that the system of equations obtained by such a replacement and by including Einstein's equation for the additional gravitational field variables possesses the full GENERIC structure, as we have checked explicitly for the original noncovariant formalism. However, in order to establish confidence in the GENERIC formalism and the resulting equations for a self-gravitating imperfect fluid, we here first propose a covariant version of the formalism and then derive a much wider class of hydrodynamic equations obtained within the covariant formalism.

In the original formulation of GENERIC, equivalent Poisson operator and bracket representations have been proposed. Inspired by the work of Marsden *et al.* [25] for reversible relativistic field theories, we here propose a covariant formulation of the GENERIC idea based on the bracket formulation. In the proposed formalism, the field equations are expressed in the form

$$\{A,I\}=[A,J], \tag{1}$$

where I is an action integral, J is the integral of the entropy density over space and time, and A is an arbitrary functional of the fields. $\{A,B\}$ is a Poisson bracket, and $[A,B]$ is a Ginzburg-Landau or dissipative bracket with $[A,B]=[B,A]$ for arbitrary functionals A, B, and $[A,A]\geqslant 0$ for all A. The two contributions to the field equations generated by I and J in Eq. (1) are called the reversible and irreversible contributions, respectively. In the reversible situation considered by Marsden *et al.* [25], the right-hand side of Eq. (1) is zero.

While Eq. (1) replaces the fundamental time-evolution equation of the noncovariant GENERIC formalism, the mutual degeneracy requirements of that formalism are replaced by

$$\{A,J\}=0 \tag{2}$$

and

$$[A,I]=0, \tag{3}$$

where, again, A is an arbitrary functional of the fields. These requirements, which are strong formulations of the conservation of entropy by the reversible dynamics and of the conservation of energy by the irreversible dynamics, are known to be important parts of the GENERIC framework.

The covariant formulation (1)–(3) retains the key innovation in the GENERIC structure, that is, the use of two separate generators for the reversible and irreversible dynamics, together with the symmetric degeneracy requirements. This innovation is of crucial importance when treating systems without local equilibrium states, for example, systems described by Boltzmann's kinetic equation [26]. The example of Boltzmann's equation furthermore shows that, in spite of the linear appearance of Eqs. (1)–(3), the formalism is not limited to the linear response regime. While nonequilibrium dynamics is usually expressed in terms of a single generator (the effective Hamiltonian [27]), the two-generator idea, which leaves more flexibility in the choice of variables, allows us to formulate the mutual degeneracy conditions [10,11].

As a next step, we use the proposed formalism for constructing generally covariant equations for self-gravitating imperfect fluids. We first select the state variables and, after determining the building blocks of GENERIC, we write out all the field equations.

III. GENERALLY RELATIVISTIC HYDRODYNAMICS

Before presenting the thermodynamically admissible, generally covariant equations for relativistic imperfect fluids, some introductory remarks comparing the general procedure within the new thermodynamic framework and in the established theory of Israel and Stewart are in order (as outlined in [24] and [8], respectively). In the Israel-Stewart theory, the basic (thermodynamic flux) variables are introduced through a straightforward decomposition of the energy-momentum tensor; a quadratic expansion in the flux variables is then postulated for the entropy current vector, and evolution equations for the flux variables are chosen such that the divergence of the entropy current is non-negative (there is a number of possibilities of formulating such equations; see the note added in proof in [8]). Within the GENERIC framework, the choice of the basic (thermodynamic force) variables is strongly restricted by the Poissonian structure of the reversible dynamics (in particular, by the Jacobi identity expressing the time-structure invariance of the reversible dynamics). Furthermore, the dependence of the energy-momentum tensor and of the entropy current vector on these variables is determined by the GENERIC structure. While the second law of thermodynamics is crucial for formulating the Israel-Stewart equations, in the general approach to nonequilibrium thermodynamics it basically leads to stability conditions for the thermodynamic potential expressing the internal energy as a nontrivial function of the thermodynamic variables. The quadratic expression for the entropy current in the Israel-Stewart theory, for which there is no counterpart in the new thermodynamic approach, can be regarded as the result of an expansion around equilibrium; this is a severe restriction which is not shared by the approach of this paper. As a consequence of all these differences, the structure of the resulting equations is considerably different in general nonequilibrium situations, but coincides for the linearized equations [23,24]. (Note that only the linearized equations are relevant to the discussion of stability and causality [8].) In the companion paper, the solutions of the new equations are compared to those of previously proposed equations in the context of bulk viscous cosmology [21].

A. State variables

Before applying GENERIC, we first need to specify the list of state variables for our thermodynamic system of interest, that is, for a self-gravitating imperfect fluid. Throughout this paper, following Eckart's approach, the velocity four-vector u^μ is taken to be the velocity of particle transport. As the basic hydrodynamic variables we use the particle number density per unit rest volume, n, the entropy density per unit rest volume, s, and the momentum-density four-vector, M_μ ($\mu=0,1,2,3$).

We next need to introduce a four-vector w_μ closely related to the temperature gradient and a symmetric tensor variable $C_{\mu\nu}$ closely related to the velocity gradient tensor in order to render a thermodynamically admissible theory possible [23,24]. These additional variables are required by GENERIC, and they are crucial for the causality of the theory. For completeness, we here repeat some arguments used for introducing these additional variables for the example of the four-vector w_μ, which will turn out to describe an entropy-flux contribution that is not proportional to the velocity four-vector (further details can be found in [23,24]). The nonrelativistic description of dissipative phenomena, such as heat flow, typically involves second-order space derivatives, but only first-order time derivatives. For example, the energy equation involves the divergence of the heat flux, which is itself proportional to the gradient of temperature. The standard procedure for avoiding such unmatched higher-order space derivatives is to introduce additional variables which are closely related to these space derivatives. For example, we introduce a simple dissipative relaxation mechanism by which an additional vector variable rapidly converges to the temperature gradient, while the divergence of this variable occurs in the reversible contribution to the energy or entropy balance equation. If the new variable is supposed to play the role of temperature gradients, it needs to be a covariant vector variable of intensive nature.

In addition to the hydrodynamic fields n, M_μ, s, w_μ, and $C_{\mu\nu}$, we use the components of the contravariant symmetric tensor representing the dual metric, $g^{\mu\nu}$, and the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ as state variables. As usual, $g^{\mu\nu}$ is used for raising indices, while its inverse, the metric tensor $g_{\mu\nu}$, is used for lowering indices. The signature of the space-time metric $g_{\mu\nu}$ is $(-+++)$, and g is the absolute value of the determinant of $g_{\mu\nu}$. Semicolons (;) denote covariant derivatives compatible with the metric $g_{\mu\nu}$, and Einstein's summation convention is used.

B. Reversible contribution

As a first step, we generalize the Poisson bracket and action integral proposed by Marsden *et al.* [25] by including the variables w_μ and $C_{\mu\nu}$ [28]. We use the following bracket:

$$
\begin{aligned}
\{A,B\} = {} & \int \sqrt{g}\, n \left(\frac{\delta B}{\delta M_\mu} \partial_\mu \frac{\delta A}{\delta n} - \frac{\delta A}{\delta M_\mu} \partial_\mu \frac{\delta B}{\delta n} \right) d^4x + \int \sqrt{g}\, M_\nu \left(\frac{\delta B}{\delta M_\mu} \partial_\mu \frac{\delta A}{\delta M_\nu} - \frac{\delta A}{\delta M_\mu} \partial_\mu \frac{\delta B}{\delta M_\nu} \right) d^4x \\
& + \int \sqrt{g}\, s \left(\frac{\delta B}{\delta M_\mu} \partial_\mu \frac{\delta A}{\delta s} - \frac{\delta A}{\delta M_\mu} \partial_\mu \frac{\delta B}{\delta s} \right) d^4x + \int V^\lambda \left(\frac{\delta A}{\delta g^{\mu\nu}} \frac{\delta B}{\delta \Gamma^\lambda_{\mu\nu}} - \frac{\delta B}{\delta g^{\mu\nu}} \frac{\delta A}{\delta \Gamma^\lambda_{\mu\nu}} \right) d^4x \\
& + \int \sqrt{g}\, (\partial_\lambda w_\mu) \left(\frac{\delta A}{\delta M_\lambda} \frac{\delta B}{\delta w_\mu} - \frac{\delta B}{\delta M_\lambda} \frac{\delta A}{\delta w_\mu} \right) d^4x + \int \sqrt{g}\, w_\mu \left(\frac{\delta B}{\delta w_\lambda} \partial_\lambda \frac{\delta A}{\delta M_\mu} - \frac{\delta A}{\delta w_\lambda} \partial_\lambda \frac{\delta B}{\delta M_\mu} \right) d^4x \\
& + \int \sqrt{g} \left(\frac{\delta B}{\delta w_\mu} \partial_\mu \frac{\delta A}{\delta s} - \frac{\delta A}{\delta w_\mu} \partial_\mu \frac{\delta B}{\delta s} \right) d^4x + \int \sqrt{g}\, (\partial_\lambda C_{\mu\nu}) \left(\frac{\delta A}{\delta M_\lambda} \frac{\delta B}{\delta C_{\mu\nu}} - \frac{\delta B}{\delta M_\lambda} \frac{\delta A}{\delta C_{\mu\nu}} \right) d^4x \\
& + \int \sqrt{g}\, C_{\mu\nu} \left(\frac{\delta B}{\delta C_{\mu\lambda}} \partial_\lambda \frac{\delta A}{\delta M_\nu} - \frac{\delta A}{\delta C_{\mu\lambda}} \partial_\lambda \frac{\delta B}{\delta M_\nu} \right) d^4x + \int \sqrt{g}\, C_{\mu\nu} \left(\frac{\delta B}{\delta C_{\lambda\nu}} \partial_\lambda \frac{\delta A}{\delta M_\mu} - \frac{\delta A}{\delta C_{\lambda\nu}} \partial_\lambda \frac{\delta B}{\delta M_\mu} \right) d^4x. \qquad (4)
\end{aligned}
$$

The symbols ∂_μ denote partial derivatives with respect to the coordinates, which could equivalently be replaced by covariant derivatives in the proposed bracket expression. The functional derivatives in Eq. (4) are defined as scalars, vectors or tensors, not as densities; for example,

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} A(n+\lambda\,\delta n)=\int \delta n\,\frac{\delta A}{\delta n}\sqrt{g}\,d^4x. \tag{5}$$

The only exception is made in order to keep the Christoffel symbols as convenient state variables,

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} A(n+\lambda\,\delta\Gamma)=\int \delta\Gamma^{\lambda}_{\mu\nu}\frac{\delta A}{\delta\Gamma^{\lambda}_{\mu\nu}}\,d^4x. \tag{6}$$

The first three integrals in the Poisson bracket (4) represent the convection mechanism for the standard hydrodynamic fields. The fourth contribution involves only the variables of the gravitational field, $g^{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$, and an arbitrary vector field V^λ; the occurrence of V^λ is related to the freedom of arbitrary coordinate transformations in four-dimensional space. Two further integrals express the convection mechanism for the covariant four-vector field w_μ, and a further contribution, involving w_μ and s, establishes the relation between w_μ and the temperature gradients [23,24]. This coupling of w_μ and s is the only non-standard contribution in the bracket (4); it implies that w_μ has the dimensions of temperature (this differs by a factor of the speed of light, c, compared to previous work [23,24]). The last three integrals express the convection mechanism for the covariant four-tensor field $C_{\mu\nu}$.

The action of Marsden *et al.* [25],

$$I=\int \sqrt{g}\left(\frac{c^4}{16\pi G}g^{\mu\nu}R_{\mu\nu}-\frac{1}{2}g^{\mu\nu}\frac{M_\mu M_\nu}{U}+V\right)d^4x, \tag{7}$$

is modified only by letting the functions U and V depend not only on the scalar quantities n and s, but also on the further scalar variables $y=y(w_\mu,C_{\mu\nu},g^{\mu\nu})$ and $z=z(C_{\mu\nu},g^{\mu\nu})$ ($g^{\mu\nu}$ needs to be considered in addition to the thermodynamic force variables w_μ and $C_{\mu\nu}$ in order to make the formation of scalars possible). For example, one could use expressions of the form

$$y=a_1^y\,w_\mu g^{\mu\nu}w_\nu+a_2^y\,w_\mu g^{\mu\nu}C_{\nu\lambda}g^{\lambda\kappa}w_\kappa+\ \dots, \tag{8}$$

$$z=a_0^z+a_1^z\,g^{\mu\nu}C_{\nu\mu}+a_2^z\,g^{\mu\nu}C_{\nu\lambda}g^{\lambda\kappa}C_{\kappa\mu}+\ \dots, \tag{9}$$

where the quantities a_i^y and a_i^z are constant coefficients. If a_1^y is nonzero, then we assume that the normalization is such that $a_1^y=1$. In Eq. (7), G is Newton's gravitational constant, and the Ricci tensor

$$R_{\mu\nu}=\partial_\lambda\Gamma^{\lambda}_{\mu\nu}-\partial_\nu\Gamma^{\lambda}_{\mu\lambda}+\Gamma^{\lambda}_{\kappa\lambda}\Gamma^{\kappa}_{\mu\nu}-\Gamma^{\lambda}_{\kappa\nu}\Gamma^{\kappa}_{\mu\lambda} \tag{10}$$

depends on the Christoffel symbols alone.

For later convenience, we introduce the velocity four-vector as

$$u^\mu=-\frac{\delta I}{\delta M_\mu}=\frac{M^\mu}{U}, \tag{11}$$

which implies the constraint

$$g^{\mu\nu}M_\mu M_\nu=-U^2. \tag{12}$$

This constraint is to be imposed *after* all the functional derivatives are taken. The final continuity equation for the particle current [see Eq. (38) below] will show that this definition of the velocity four-vector u^μ is indeed consistent with the approach of Eckart adopted in this paper.

In calculating further functional derivatives of I, it is convenient to introduce the quantities

$$p=\frac{1}{2}U+V,\quad q=\frac{1}{2}U-V, \tag{13}$$

to be interpreted below, and the derivatives

$$T=\frac{\partial q}{\partial s}, \tag{14}$$

$$\sigma=2\,\frac{\partial q}{\partial y}, \tag{15}$$

$$\phi=4\,\frac{\partial q}{\partial z}. \tag{16}$$

As the functions U and V, the transformed functions p and q depend on n, s, y, and z. In the following, the derivatives T, σ, and ϕ are required to be non-negative.

C. Irreversible contribution

In constructing the dissipative bracket we make a quite general ansatz which includes our previous work on special relativistic hydrodynamics [23,24]. Only the entropy density, s, and the new variables, w_μ and $C_{\mu\nu}$, are assumed to be involved in the dissipative brackets. The following expression is obviously symmetric and covariant:

$$[A,B]=\int \sqrt{g}\left(\tilde{w}^\mu\frac{\delta A}{\delta s}-\frac{\delta A}{\delta w_\mu}\right)f_{\mu\nu}\left(\tilde{w}^\nu\frac{\delta B}{\delta s}-\frac{\delta B}{\delta w_\nu}\right)d^4x$$
$$+\int \sqrt{g}\left(\tilde{C}^{\mu\nu}\frac{\delta A}{\delta s}-\frac{\delta A}{\delta C_{\mu\nu}}\right)(\overline{f}_{\mu\nu}\overline{f}_{\kappa\lambda}+\mathring{f}_{\mu\kappa}\mathring{f}_{\lambda\nu})$$
$$\times\left(\tilde{C}^{\kappa\lambda}\frac{\delta B}{\delta s}-\frac{\delta B}{\delta C_{\kappa\lambda}}\right)d^4x, \tag{17}$$

where $\overline{f}_{\mu\nu}$ is arbitrary and $f_{\mu\nu}$, $\mathring{f}_{\mu\nu}$ are assumed to be positive semidefinite and symmetric. The remaining unknowns in Eq. (17) are determined by the degeneracy requirement (3),

$$\tilde{w}^\mu=\frac{\sigma}{2T}\frac{\partial y}{\partial w_\mu}, \tag{18}$$

$$\tilde{C}^{\mu\nu}=\frac{\sigma}{2T}\frac{\partial y}{\partial C_{\mu\nu}}+\frac{\phi}{4T}\frac{\partial z}{\partial C_{\mu\nu}}. \tag{19}$$

As the entropy density is among the state variables, we can immediately write the entropy integral J as

$$J=\int s\sqrt{g}\,d^4x. \tag{20}$$

While s is the entropy density per unit three space, the integration is not only over three space but also over time. The degeneracy requirement (2) is trivially fulfilled.

D. Field equations

After defining the brackets and the integrals I and J in the preceding subsections, we can now write out all the explicit field equations. If we apply the fundamental equation (1) to an arbitrary functional $A=A(\Gamma)$, we obtain the condition $\delta I/\delta g^{\mu\nu}=0$, which implies Einstein's field equation

$$G_{\mu\nu}=\frac{8\pi G}{c^4}T_{\mu\nu}, \tag{21}$$

where the Einstein tensor

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R,\quad R=g^{\mu\nu}R_{\mu\nu}, \tag{22}$$

is expressed in terms of the Ricci tensor and the curvature scalar. For the energy-momentum tensor $T_{\mu\nu}$ we find

$$T_{\mu\nu}=qu_\mu u_\nu+ph_{\mu\nu}+P_{\mu\nu}, \tag{23}$$

with

$$h_{\mu\nu}=g_{\mu\nu}+u_\mu u_\nu \tag{24}$$

and

$$P_{\mu\nu}=\sigma\frac{\partial y}{\partial g^{\mu\nu}}+\frac{1}{2}\phi\frac{\partial z}{\partial g^{\mu\nu}}. \tag{25}$$

Equation (23) allows us to identify p as the pressure and

$$\rho=q+u^\mu P_{\mu\nu}u^\nu \tag{26}$$

as the fluid mass-energy per unit rest three volume.

After obtaining an explicit expression for the energy-momentum tensor in Einstein's field equation, it is interesting to consider the equation for the momentum-density four-vector, as obtained by choosing $A=A(M)$. Equation (1) implies

$$T^{\mu\nu}{}_{;\mu}=0, \tag{27}$$

provided that the following condition holds:

$$p=n\frac{\partial q}{\partial n}+s\frac{\partial q}{\partial s}-q. \tag{28}$$

The proper choice of the pressure according to Eq. (28) hence guarantees the consistency of the field equation (21) with the Bianchi identities, which can be derived once the expression for the Einstein tensor in terms of the Christoffel symbols [see Eqs. (10) and (22)] is related to the metric tensor [see Eq. (37)]. In order to rewrite Eq. (28) in a more familiar form, we introduce the quantities

$$s_{\mathrm{f}}=s+\frac{1}{T}u^\mu P_{\mu\nu}u^\nu \tag{29}$$

and

$$y_{\mathrm{f}}=y+\frac{1}{\sigma}u^\mu P_{\mu\nu}u^\nu; \tag{30}$$

s_{f} is the conventional local-equilibrium entropy density which, in the presence of the thermodynamic force variable w_μ, can be different from the total entropy density s.

If we consider the total differential of $\rho=\rho(n,s_{\mathrm{f}},y_{\mathrm{f}},z)$ of Eq. (26) and impose the constraints

$$w_\nu u^\nu=-T \tag{31}$$

and

$$C_{\mu\nu}u^\nu=0, \tag{32}$$

then we obtain

$$T=\frac{\partial\rho}{\partial s_{\mathrm{f}}}, \tag{33}$$

$$\sigma=2\frac{\partial\rho}{\partial y_{\mathrm{f}}}, \tag{34}$$

$$\phi=4\frac{\partial\rho}{\partial z}. \tag{35}$$

These identities allow us to identify the variable T as the absolute temperature (defined for constant intensive variables y_{f} and z), and to rewrite Eq. (28) in the form

$$p=n\frac{\partial\rho}{\partial n}+s_{\mathrm{f}}\frac{\partial\rho}{\partial s_{\mathrm{f}}}-\rho. \tag{36}$$

Equation (36) has the form of the familiar Gibbs-Duhem relation between p and ρ. Note, however, that the functions p and ρ in Eq. (36) depend not only on n and s_{f} but also on y_{f} and z.

We can now list the remaining equations implied by the covariant GENERIC (1). Choosing $A=A(g)$ in Eq. (1) implies $\delta I/\delta\Gamma^\lambda_{\mu\nu}=0$, which can be rewritten as the anticipated expression for the Christoffel symbols,

$$\Gamma^\lambda_{\mu\nu}=\frac{1}{2}g^{\lambda\kappa}[\partial_\mu g_{\kappa\nu}+\partial_\nu g_{\mu\kappa}-\partial_\kappa g_{\mu\nu}]. \tag{37}$$

By choosing $A=A(n)$ and $A=A(s)$, we obtain the continuity equation

$$(nu^{\mu})_{;\mu}=0 \tag{38}$$

and the entropy balance equation

$$S^{\mu}{}_{;\mu}=\tilde{w}^{\mu}f_{\mu\nu}\tilde{w}^{\nu}+\tilde{C}^{\mu\nu}(\overline{f}_{\mu\nu}\overline{f}_{\kappa\lambda}+\mathring{f}_{\mu\kappa}\mathring{f}_{\lambda\nu})\tilde{C}^{\kappa\lambda}, \tag{39}$$

where

$$S^{\mu}=su^{\mu}+T\tilde{w}^{\mu} \tag{40}$$

is the total entropy four-vector. Equation (38) justifies the definition (11) of u^{μ} according to Eckart's approach. Equation (40) clarifies the physical role of w^{μ} in accounting for an entropy flux not proportional to the velocity four-vector.

Finally, choosing $A=A(w)$ and $A=A(C)$ in Eq. (1) gives the relaxation equations

$$u^{\lambda}(w_{\mu;\lambda}-w_{\lambda;\mu})=-f_{\mu\lambda}\,\tilde{w}^{\lambda} \tag{41}$$

and

$$u^{\lambda}(C_{\mu\nu;\lambda}-C_{\lambda\nu;\mu}-C_{\mu\lambda;\nu})=-\overline{f}_{\mu\nu}\overline{f}_{\kappa\lambda}\tilde{C}^{\kappa\lambda}-\mathring{f}_{\mu\kappa}\tilde{C}^{\kappa\lambda}\mathring{f}_{\lambda\nu}. \tag{42}$$

At this point, we have derived the complete set of equations [Einstein's field equation (21), (37) and the new hydrodynamic equations (27), (38), (39), (41), (42)] from the covariant GENERIC formalism. If we assume $u^{\mu}f_{\mu\nu}=u^{\mu}\overline{f}_{\mu\nu}=u^{\mu}\mathring{f}_{\mu\nu}=0$, then the constraints (12), (31) and (32) are compatible with the field equations [24].

E. Special case

The previous work on hydrodynamic equations satisfying the principles of special relativity and GENERIC [23,24] corresponds to the particular choices

$$y=w_{\mu}g^{\mu\nu}w_{\nu}, \tag{43}$$

$$z=g^{\mu\nu}g^{\kappa\lambda}C_{\mu\kappa}C_{\nu\lambda}+2g^{\mu\nu}C_{\mu\nu}+3. \tag{44}$$

We then find the energy-momentum tensor contribution

$$P_{\mu\nu}=\sigma w_{\mu}w_{\nu}+\phi(C_{\mu\lambda}C^{\lambda}{}_{\nu}+C_{\mu\nu}) \tag{45}$$

and the entropy four-vector

$$S^{\mu}=su^{\mu}+\sigma w^{\mu}=s_{\rm f}u^{\mu}+\sigma h^{\mu\nu}w_{\nu}. \tag{46}$$

The previously introduced relaxation processes for w_{μ} and $C_{\mu\nu}$ are recovered for

$$f_{\mu\nu}=\frac{T}{c\tau_1\sigma}h_{\mu\nu}, \tag{47}$$

$$\mathring{f}_{\mu\nu}=\sqrt{\frac{2T}{c\tau_2\phi}}h_{\mu\nu}, \tag{48}$$

and

$$\overline{f}_{\mu\nu}=\sqrt{\frac{2T}{3c\phi}\left(\frac{1}{\tau_0}-\frac{1}{\tau_2}\right)}\,h_{\mu\nu}, \tag{49}$$

where the relaxation times τ_0, τ_1, and τ_2 representing the dynamical material properties are assumed to be positive. The resulting equations

$$u^{\lambda}(w_{\mu;\lambda}-w_{\lambda;\mu})=-\frac{1}{c\tau_1}h_{\mu\lambda}\,w^{\lambda} \tag{50}$$

and

$$u^{\lambda}(C_{\mu\nu;\lambda}-C_{\lambda\nu;\mu}-C_{\mu\lambda;\nu})=-\frac{1}{c\tau_0}\overline{C}_{\mu\nu}-\frac{1}{c\tau_2}\mathring{C}_{\mu\nu}, \tag{51}$$

with the auxiliary tensors

$$\overline{C}_{\mu\nu}=\left(1+\frac{1}{3}C_{\kappa\lambda}h^{\kappa\lambda}\right)h_{\mu\nu}, \tag{52}$$

$$\mathring{C}_{\mu\nu}=h_{\mu\kappa}C^{\kappa\lambda}h_{\lambda\nu}-\frac{1}{3}C_{\kappa\lambda}h^{\kappa\lambda}\,h_{\mu\nu}, \tag{53}$$

can also be reproduced with the alternative dissipative bracket

$$\begin{aligned}[A,B]=&\int\sqrt{g}\frac{1}{c\tau_1}\frac{\sigma}{T}\left(w^{\mu}\frac{\delta A}{\delta s}-\frac{T}{\sigma}\frac{\delta A}{\delta w_{\mu}}\right)h_{\mu\nu}\left(w^{\nu}\frac{\delta B}{\delta s}\right.\\&\left.-\frac{T}{\sigma}\frac{\delta B}{\delta w_{\nu}}\right)d^4x+\int\sqrt{g}\frac{1}{c\tau_0}\frac{r_0}{T}\left(\frac{\delta A}{\delta s}-\frac{T}{r_0}\overline{C}_{\mu\nu}\frac{\delta A}{\delta C_{\mu\nu}}\right)\\&\times\left(\frac{\delta B}{\delta s}-\frac{T}{r_0}\overline{C}_{\kappa\lambda}\frac{\delta B}{\delta C_{\kappa\lambda}}\right)d^4x+\int\sqrt{g}\frac{1}{c\tau_2}\frac{r_2}{T}\\&\times\left(\frac{\delta A}{\delta s}-\frac{T}{r_2}\mathring{C}_{\mu\nu}\frac{\delta A}{\delta C_{\mu\nu}}\right)\left(\frac{\delta B}{\delta s}-\frac{T}{r_2}\mathring{C}_{\kappa\lambda}\frac{\delta B}{\delta C_{\kappa\lambda}}\right)d^4x,\end{aligned} \tag{54}$$

where

$$r_0=\frac{1}{2}\phi\,\overline{C}_{\mu\nu}\overline{C}^{\mu\nu} \tag{55}$$

and

$$r_2=\frac{1}{2}\phi\,\mathring{C}_{\mu\nu}\mathring{C}^{\mu\nu}. \tag{56}$$

For small relaxation times τ_i, the solutions to Eqs. (50) and (51) can be expanded in terms of τ_i, thus eliminating the thermodynamic forces w_{μ} and $C_{\mu\nu}$ as dynamic variables. To first order we obtain the explicit expressions

$$w_{\mu}=Tu_{\mu}-c\tau_1h_{\mu}^{\nu}T_{;\nu}-c\tau_1Tu_{\mu;\nu}u^{\nu} \tag{57}$$

and

$$C_{\mu\nu}=-h_{\mu\nu}+\frac{2}{3}c(\tau_0-\tau_2)u^{\lambda}{}_{;\lambda}h_{\mu\nu}+c\tau_2 h_{\mu\kappa}[u^{\kappa;\lambda}+u^{\lambda;\kappa}]h_{\lambda\nu}. \tag{58}$$

These expressions clarify the role of the variables w_μ and $C_{\mu\nu}$ as the thermodynamic forces related to temperature and velocity gradients. For example, if the expression (57) is inserted into the entropy four-vector (46), we find the expected contribution to the entropy flux due to temperature gradients. More precisely, the expressions (57) and (58) can be rewritten in terms of the heat-flow vector Q_μ and the shear tensor $W_{\mu\nu}$ introduced by Weinberg (see Secs. 2.11 and 15.10 of [1]),

$$w_\mu=Tu_\mu-c\tau_1 h_{\mu\nu}Q^\nu \tag{59}$$

and

$$C_{\mu\nu}=-h_{\mu\nu}+\frac{2}{3}c\tau_0 u^{\lambda}{}_{;\lambda}h_{\mu\nu}+c\tau_2 h_{\mu\kappa}W^{\kappa\lambda}h_{\lambda\nu}. \tag{60}$$

To first order in τ_i, the resulting energy-momentum tensor and entropy four-vector coincide with the expressions given by Weinberg, and we can identify the heat conduction coefficient $\tau_1\sigma T/c^2$, the shear viscosity $\tau_2\phi$, and the bulk viscosity $2\tau_0\phi/3$.

F. Cosmological constant

Although there is no interest in the static Einstein universe as a realistic cosmological model, ‘‘the existence of a cosmological constant remains a logical possibility, and cosmologists have thoroughly explored the dynamics of expanding universes with a cosmological constant’’ (Weinberg [1], p. 614). In view of the recently revived interest in this possibility, we mention that Einstein's modified field equation

$$G_{\mu\nu}+\Lambda g_{\mu\nu}=\frac{8\pi G}{c^4}T_{\mu\nu} \tag{61}$$

with the cosmological constant Λ possesses the full GENERIC structure.

For a reversible realization of the cosmological term, we need to modify the action properly. In fact, we only need to add the contribution $-\int\sqrt{g}\Lambda c^4/(8\pi G)d^4x$ to the action (7) in order to reproduce the field equation (61). A reversible cosmological term hence fits very naturally into the general thermodynamic framework.

IV. BULK VISCOUS COSMOLOGY

Bulk viscosity is a particularly interesting dissipative effect because it can occur in an isotropic, spatially homogeneous universe. While the bulk viscosity vanishes for a fluid with a purely relativistic equation of state, it may be important for mixtures of radiation and matter. Also, particle production processes may be phenomenologically described in terms of an effective bulk viscosity [14]. We hence study this important phenomenon separately.

In an isotropically expanding universe, $C_{\mu\nu}$ must be of the form

$$C_{\mu\nu}=(F-1)h_{\mu\nu}, \tag{62}$$

and Eq. (51) implies the following time-evolution equation for F,

$$\dot{F}=cu^\lambda\partial_\lambda F=\frac{2}{3}(1-F)\,c\,u^\mu{}_{;\mu}-\frac{1}{\tau_0}F. \tag{63}$$

To the lowest order in the time scale τ_0, we hence obtain the following physical interpretation of F as the ratio of the time scales for the relaxation of the bulk viscous stresses and for the expansion of the universe:

$$F=\frac{2}{3}\tau_0\,c\,u^\mu{}_{;\mu}. \tag{64}$$

More generally, the dimensionless expansion rate F may be regarded as the thermodynamic force variable associated with bulk viscous stresses.

We can rewrite Eq. (63) as

$$\phi(F+\tau_0\dot{F})=\xi(1-F)\,c\,u^\mu{}_{;\mu}, \tag{65}$$

where the bulk or dilatational viscosity $\xi=2\tau_0\phi/3$ has been introduced (see [24]). The mass-energy density

$$\rho=\rho(n,s,z) \tag{66}$$

can alternatively be expressed in terms of n, s, and

$$F^2=\frac{1}{3}z. \tag{67}$$

Furthermore, we can rewrite the entropy balance equation (39) as

$$S^\mu{}_{;\mu}=\frac{(\phi F)^2}{c\,\xi T} \tag{68}$$

and the energy-momentum tensor (23) as

$$T_{\mu\nu}=\rho u_\mu u_\nu+[p-\phi F(1-F)]\,h_{\mu\nu}. \tag{69}$$

For the lowest-order approximation in the small quantity F, Eq. (65) for the bulk viscous pressure contribution, $-\phi F$, agrees with the often assumed Maxwell-Cattaneo type equation (cf. the ‘‘truncated’’ equations of [13–16,18,19]). Beyond that approximation, however, the usual equations based on the second-order theory of Israel and Stewart or on extended irreversible thermodynamics differ from the present equation (65), which is not directly for the bulk viscous pressure but for the related generalized thermodynamic force F. While the usual equations are often truncated for reasons of tractability, the full time-evolution equation (65) is of a relatively simple form. A detailed comparison of various approaches to bulk viscous cosmology can be found in the companion paper [21].

In a highly nonlinear situation, in which ρ might be a complicated function of n, s, and $z=3F^2$, the GENERIC structure implies certain Maxwell-type relationships between the appropriate derivatives of p, T, and ϕ with respect to the independent variables [cf. Eqs. (33), (35), and (36)]. All the material information required to obtain a closed set of equations can consistently be condensed into the assumed functional form of ρ (instead of assuming several ''independent'' equations of state).

V. SUMMARY AND DISCUSSION

We have proposed and applied a covariant version of the GENERIC structure, which represents the most restrictive set of principles of nonequilibrium thermodynamics ever applied to a general relativistic imperfect fluid. As a benefit of the thermodynamic framework, the necessity of additional generalized force variables related to velocity and temperature gradients, which are known to be essential for a stable and causal theory, is here motivated by the structure of thermodynamically admissible equations rather than by the behavior of their solutions.

As an application of the covariant GENERIC formalism, we have developed a set of generally covariant equations for a self-gravitating imperfect fluid—the generalized hydrodynamic equations (27), (38), (39), (41), (42) together with Einstein's field equation (21) and the energy-momentum tensor expression (23). The various material functions [Eqs. (14)–(16), (28)] occurring in the energy-momentum tensor (23) can be expressed as derivatives of the single generalized thermodynamic relationship for the mass-energy density. This observation implies (and actually expresses) a pronounced underlying structure and hence predictive power of the proposed equations.

The GENERIC framework provides a more structured set of dissipative transport equations for the sources of gravitational fields. The corresponding structure can be established without changing Einstein's famous field equation, however, the previous generalized hydrodynamic equations need to be modified. The consequences of this modification of the full theory of gravitation remain to be explored in more detail. As a starting point, we have derived the explicit equations for an isotropic, spatially homogeneous model of the universe with bulk viscosity (see also [21]).

Within the thermodynamic framework, all the time-evolution equations for a self-gravitating imperfect fluid are completely determined once the following properties for a specific material are detailed: the generalized thermodynamic relationship for the mass-energy density and the relaxation times τ_0, τ_1, τ_2 in Eqs. (41) and (42). All other equilibrium and nonequilibrium properties of the material can be determined from these basic inputs. The formulation of nonequilibrium thermodynamics in terms of the building blocks has the same great advantages as the use of thermodynamic potentials in equilibrium thermodynamics: in an experimental or kinetic theory investigation of the system, one knows all the redundancies in the various material properties, and one can focus on the determination of the minimum required independent properties (building blocks). Actually, the experimental procedure or kinetic theory calculations [29] could be tailored to the structure of the equations proposed here.

ACKNOWLEDGMENTS

I am very grateful to Roy Maartens for helpful comments on a first version of this manuscript.

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