Stress effects in structure formation

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Residual velocity dispersion in cold dark matter induces stresses which lead to effects that are absent in the idealized dust model. A previous Newtonian analysis showed how this approach can provide a theoretical foundation for the phenomenological adhesion model. We develop a relativistic kinetic theory generalization which also incorporates the anisotropic velocity dispersion that will typically be present. In addition to density perturbations, we consider the rotational and shape distortion properties of clustering. These quantities together characterize the linear development of density inhomogeneity and we find exact solutions for their evolution. As expected, the corrections are small and arise only in the decaying modes, but their effect is interesting. One of the modes for density perturbations decays less rapidly than the standard decaying mode. The new rotational mode generates precession of the axis of rotation. The new shape modes produce additional distortion that persists during the subsequent (linear) evolution, despite the rapid decay of the terms that caused it.
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I. INTRODUCTION

The cold dark matter (CDM) model has had considerable success, based on using adiabatic perturbations of pressure-free dust on a Friedmann-Robertson-Walker (FRW) background to study the growth of structure in the matter distribution. The model is simple and the solutions are easy to interpret (see, e.g., [1]). The idealized dust assumption, i.e., exactly zero velocity dispersion, breaks down when density fluctuations begin to go nonlinear; caustics and infinite density layers form through shell crossing, precisely because velocity dispersion is forced to vanish. Theoretical modifications to complement the extensive numerical simulations and deal with the multi-stream flow problem are few in number. One of the most successful is the adhesion model [2]. This model has relied on a phenomenological justification rather than a theoretical derivation. Recently Buchert and Domínguez [3] developed theoretical models which contain the adhesion one as a special case. Furthermore, their models do not have the problem of the possible non-conservation of momentum which occurs in the adhesion model.

In a Newtonian framework, with comoving coordinates on an expanding background, they use the Poisson-Vlasov equations to obtain consistent models of a self-gravitating collisionless gas. The models are designed to allow for a small amount of velocity dispersion in the gas. The outcome of their approximation scheme is a system of equations that includes an effective viscosity term which is more general than the adhesion term, but which can be specialized to it. They point out that the inclusion of the velocity dispersion allows access to smaller spatial scales than previous models permit and could be used to connect studies of large scale structures with those of smaller ones. This aspect remains to be investigated.

A well known problem with the Vlasov hierarchy of moment equations is that it is infinite [4], and some additional information has to be provided. Without collisions, there is in general no mechanism for eliminating the quadrupole and higher moments. The simplest approach is to truncate above the dipole and use a dust model, but this has no velocity dispersion. A physically reasonable model is obtained in [3] by assuming small velocity dispersion, leading to truncation above the quadrupole. This closes the hierarchy and allows limited velocity dispersion.

In this paper, we develop a relativistic generalization of their approach, based on the Einstein-Liouville equations. One limitation of their model is that they assume the velocity dispersion is isotropic, so that the stress (pressure) is purely isotropic. Our generalization eliminates the isotropy assumption, allowing for anisotropic stress.[1] We are able to find self-consistent (i.e., based on kinetic theory, rather than *ad hoc* phenomenology) evolution equations for the isotropic and anisotropic stresses. Furthermore, we use a covariant gauge-invariant approach [6,7] to describe not only the magnitude of density inhomogeneities, i.e., the density perturba-

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[1]Recently Hu and Eisenstein [5] have investigated anisotropic stress effects in general, by postulating phenomenological parametrizations of stress evolution. Our model is of more limited applicability, but is self-consistent, since the stress evolution is governed by the kinetic-theory Liouville equation.

tions, but also their rotational and shape distortion properties. This leads to a unified system of equations governing the linear evolution of density inhomogeneity in a physically realistic model of dark matter. We find the exact solutions of these equations for a flat background (i.e., $\Omega_{\text{cdm}} = 1$). These solutions are relevant for the study of dark matter halo formation, neglecting baryons and assuming zero cosmological constant.

Free-streaming effects tend to smooth density fluctuations, while the energy density supported by stresses can enhance them. The exact solution shows that the growing mode of density perturbations is unchanged, while there are two extra decaying modes, one decaying less rapidly than the standard dust mode, and one more rapidly. Velocity dispersion will have a purely dissipative effect on angular momentum, and this is confirmed by the exact solution of the rotational equation, which shows an extra decaying mode that decays more rapidly than the standard mode. However, this new mode has the interesting effect of changing the direction of the axis of rotation.

The main impact of velocity dispersion is on the evolution of shape-distortion in the density distribution. The stresses, though small and decaying, have a significant effect, producing "active" distortion in addition to the inertial distortion that arises in dust models purely from the shear anisotropy. Despite being sourced by decaying terms, these distortions remain during the subsequent evolution (until the nonlinear regime).

In Sec. II we develop the self-consistent kinetic theory analysis of stress in CDM. Section III presents the covariant evolution equations for density perturbations, rotation and shape distortion, and gives the exact solutions of these equations. Finally, concluding remarks are made in Sec. IV. We follow the notation of [7,8]. The signature is $(-+++)$, units are such that $8\pi G = 1 = c$ and $k_B = 1$, spacetime indices are $a, b, \cdots$, and (square) round brackets enclosing indices denote (anti-) symmetrization. The spacetime metric is g_{ab}, and the spacetime alternating tensor is $\eta_{abcd} = -\sqrt{-g}\,\delta_{[a}{}^0\delta_b{}^1\delta_c{}^2\delta_{d]}{}^3$.

II. KINETIC MODEL OF CDM STRESSES

We use the covariant Lagrangian approach to relativistic kinetic theory [9,10,8,11], in which all the variables are physically measurable and which allows for a clear Newtonian interpretation. Given a 4-velocity field u^a, we decompose the 4-momentum p^a of a particle of mass m as

$$p^a = Eu^a + \lambda^a, \qquad (1)$$

where E is the particle energy relative to comoving observers, and

$$\lambda^a = \lambda e^a = m\gamma(v)v^a$$

is the particle 3-momentum, with $e_a e^a = 1$, $e^a u_a = 0$, and $\lambda = mv(1 - v_a v^a)^{-1/2} = (E^2 - m^2)^{1/2}$. The covariant volume element in momentum space is

$$\frac{d^3\lambda}{E} = \frac{\lambda^2 d\lambda\, d\Omega}{E} = \lambda\, dE\, d\Omega,$$

where $d\Omega$ is the solid angle spanned by two independent de^a. The distribution function $f(x, E, e^a)$ can be expanded in tensor multipoles $F_{a_1 \cdots a_l}(x, E)$, i.e.,

$$f = F + F_a e^a + F_{ab} e^a e^b + F_{abc} e^a e^b e^c + \cdots. \qquad (2)$$

This is the covariant generalization of the spherical harmonic expansion $f = \Sigma f_{lm} Y_{lm}$. The covariant multipoles are irreducible, i.e., $F_{a \cdots b} = F_{\langle a \cdots b \rangle}$, where the angled brackets denote the spatially projected symmetric tracefree (PSTF) part. They are given by

$$F_{a_1 \cdots a_l} = \frac{(2l+1)!}{4\pi(l!)^2 2^l} \int f e_{\langle a_1} e_{a_2} \cdots e_{a_l \rangle} d\Omega. \qquad (3)$$

The energy-momentum tensor is

$$T_{ab} = \int f p_a p_b \frac{d^3\lambda}{E} = \rho u_a u_b + p h_{ab} + 2q_{(a} u_{b)} + \pi_{ab}, \qquad (4)$$

where $h_{ab} = g_{ab} + u_a u_b$ is the spatial projector. The energy density, pressure, energy flux (momentum density), and anisotropic stress are given by

$$\rho = 4\pi \int_m^\infty E^2 \lambda F\, dE, \qquad (5)$$

$$p = \frac{4\pi}{3} \int_m^\infty \lambda^3 F\, dE, \qquad (6)$$

$$q_a = \frac{4\pi}{3} \int_m^\infty E\lambda^2 F_a\, dE, \qquad (7)$$

$$\pi_{ab} = \frac{8\pi}{15} \int_m^\infty \lambda^3 F_{ab}\, dE. \qquad (8)$$

Higher-level dynamical anisotropy than π_{ab} is defined via the $l \geq 3$ multipoles. For example, the octopole anisotropy is

$$\zeta_{abc} = \frac{8\pi}{35} \int_m^\infty E\lambda^2 F_{abc}\, dE. \qquad (9)$$

The number density is given by [10]

$$n = 4\pi \int_m^\infty E\lambda F\, dE,$$

and we can derive a useful relation between the monopole dynamical terms (compare the similar relation found in [11]):

$$mn + \tfrac{3}{2}p = \rho - \tfrac{1}{2}\mathcal{M} \qquad (10)$$

where

$$\mathcal{M} = 4\pi \int_m^\infty \left(1 - \frac{m}{E}\right)^2 E^2 \lambda F dE.$$

In the massless limit $m \to 0$, we have $\mathcal{M} \to \rho$, and Eq. (10) reduces to $p = \frac{1}{3}\rho$. Equation (10) is a generalized "equation

of state," which adopts a simple form in the massless limit and the limit of low velocity dispersion (see below), but which is more complicated in intermediate regimes.

The energy-momentum conservation equations $\nabla^b T_{ab} = 0$ follow from the Liouville (collisionless Boltzmann) equation, and are [10]

$$\dot{\rho} + (\rho + p)\Theta + D^a q_a = -2A^a q_a - \sigma^{ab}\pi_{ab}, \tag{11}$$

$$\dot{q}_{\langle a \rangle} + \frac{4}{3}\Theta q_a + (\rho + p)A_a + D_a p + D^b \pi_{ab} = -\sigma_{ab}q^b + \varepsilon_{abc}\omega^b q^c - A^b \pi_{ab}. \tag{12}$$

Here D_a is the spatially projected covariant derivative, i.e.,

$$D_a S_{b \cdots c} = h_a{}^d h_b{}^e \cdots h_c{}^f \nabla_d S_{e \cdots f},$$

an overdot is the covariant time derivative, i.e., $\dot{S}_{a \cdots b} = u^c \nabla_c S_{a \cdots b}$, and $\varepsilon_{abc} = \eta_{abcd}u^d$ is the spatial alternating tensor. The expansion, acceleration, vorticity and shear of the 4-velocity u^a are given by

$$\Theta = D^a u_a, \quad A_a = \dot{u}_a, \quad \omega_a = -\frac{1}{2}\text{curl } u_a, \quad \sigma_{ab} = D_{\langle a} u_{b \rangle}.$$

The covariant spatial curl of vectors and rank-2 tensors is defined by [12]

$$\text{curl } V_a = \varepsilon_{abc}D^b V^c, \quad \text{curl } S_{ab} = \varepsilon_{cd(a}D^c S^d{}_{b)}.$$

We are free to choose the 4-velocity u^a so that $q_a = 0$, i.e., so that in the comoving frame, no energy flux is observed [10,13]. In general, there will be a non-vanishing particle drift in this frame. To maintain vanishing energy flux, the momentum conservation equation (12) shows that

$$(\rho + p)A_a + D_a p + D^b \pi_{ab} = -A^b \pi_{ab}. \tag{13}$$

Thus the evolution equation (12) for q_a becomes a constraint equation (13) for the acceleration. Note also that in the energy frame, the dipole F_a will satisfy $\int E \lambda^2 F_a dE = 0$, from Eq. (7). From now on, we assume that the energy frame is chosen, i.e. $q_a = 0$.

In a universe that is close to an FRW model, i.e. with small inhomogeneity and anisotropy, we have that [6,14]

$$\frac{A_a}{\Theta}, \frac{\omega_a}{\Theta}, \frac{\sigma_{ab}}{\Theta}, \frac{\pi_{ab}}{\rho}, \frac{aD_a p}{\rho}, \frac{aD_a \rho}{\rho}, \frac{aD_a \Theta}{\Theta} = \mathcal{O}(\epsilon),$$

where ϵ is a dimensionless smallness parameter, and a is the cosmic scale factor in the FRW background. (In the background, $\Theta = 3H$, where H is the Hubble rate.) The higher-level dynamical anisotropy tensors are also $\mathcal{O}(\epsilon)$:

$$\frac{\zeta_{abc}}{\rho}, \cdots = \mathcal{O}(\epsilon).$$

To linear order in such a universe, the conservation equations (11) and (13) reduce to

$$\dot{\rho} + (\rho + p)\Theta = 0, \tag{14}$$

$$(\rho + p)A_a + D_a p + D^b \pi_{ab} = 0. \tag{15}$$

From now on, we will consider a universe that is close to FRW, i.e., we drop all $\mathcal{O}(\epsilon^2)$ terms. The Liouville equation may be decomposed into multipole evolution equations that are PSTF [8]. The monopole, dipole and quadrupole evolution equations are

$$E\dot{F} + \frac{1}{3}\lambda D^a F_a - \frac{1}{3}\lambda^2 \Theta \frac{\partial F}{\partial E} = 0, \tag{16}$$

$$E\dot{F}_a + \frac{2}{5}\lambda D^b F_{ab} - \frac{1}{3}\lambda^2 \Theta \frac{\partial F_a}{\partial E} + \lambda D_a F - \lambda E \frac{\partial F}{\partial E}A_a = 0, \tag{17}$$

$$E\dot{F}_{ab} + \frac{3}{7}\lambda D^c F_{abc} - \frac{1}{3}\lambda^2 \Theta \frac{\partial F_{ab}}{\partial E} + \lambda D_{\langle a}F_{b \rangle} - \lambda^2 \frac{\partial F}{\partial E}\sigma_{ab} = 0. \tag{18}$$

(Note that the vorticity does not enter the Liouville multipoles at the linear level.) Multiplying Eq. (16) by $E\lambda$ and integrating over all energies, and using the energy frame condition $\int E \lambda^2 F_a dE = 0$, we derive the energy conservation equation (14). Similarly, multiplying Eq. (17) by λ^2 and integrating, we arrive at the momentum conservation equation (15). When integrating by parts to obtain some of these terms, we use the assumption that as $E \to \infty$, $F_{a_1 \cdots a_l}$ ($l \geq 0$) tends to zero more rapidly than E^n for any $n < 0$.

We can derive a new evolution equation for the pressure after multiplying the monopole equation (16) by λ^3/E:

$$\dot{p} + \frac{5}{3}\Theta p = \frac{1}{3}\Theta \mathcal{P} - \frac{1}{3}D^a \mathcal{Q}_a, \tag{19}$$

where

$$\mathcal{P} = \frac{4\pi}{3}\int_m^\infty \left(1 - \frac{m^2}{E^2}\right)\lambda^3 F dE,$$

$$Q_a = \frac{4\pi}{3} \int_m^\infty \left(1 - \frac{m^2}{E^2}\right) E\lambda^2 F_a dE.$$

In the massless limit, Eq. (19) reduces to the energy conservation equation. But in general, Eq. (19) is a new and non-trivial evolution equation arising from the Liouville equation.

A new evolution equation for the anisotropic stress π_{ab} may also be found after multiplying the quadrupole equation (18) by λ^3/E:

$$\dot{\pi}_{ab} + \tfrac{5}{3}\Theta\pi_{ab} + 2p\sigma_{ab} = -\tfrac{2}{5}\mathcal{P}\sigma_{ab} - \tfrac{2}{5}D_{\langle a}Q_{b\rangle}$$
$$+ \tfrac{1}{3}\Theta\mathcal{R}_{ab} - D^c\mathcal{S}_{abc}, \quad (20)$$

where

$$\mathcal{R}_{ab} = \frac{8\pi}{15} \int_m^\infty \left(1 - \frac{m^2}{E^2}\right)\lambda^3 F_{ab} dE,$$

$$\mathcal{S}_{abc} = \frac{8\pi}{35} \int_m^\infty \left(1 - \frac{m^2}{E^2}\right) E\lambda^2 F_{abc} dE.$$

In the massless limit, we have $Q_a \to q_a (= 0)$, $\mathcal{R}_{ab} \to \pi_{ab}$, $\mathcal{S}_{abc} \to \zeta_{abc}$, and Eq. (20) reduces to the evolution equation for free-streaming radiation that was found in [14].

The Liouville multipole equations (16)–(18) are the beginning of an infinite hierarchy. (See [14,15] for the corresponding equations in the massless case, which is much simpler.) The evolution equation (20) for anisotropic stress contains the spatial divergence of the octopole, and the octopole evolution equation will contain the divergence of the hexadecapole, and so on. In general, the evolution equation for the l-pole has the spatial divergence of the $(l+1)$-pole as an effective source term, so that power is transmitted across levels of the hierarchy. Thus the multipoles above the quadrupole affect dynamical evolution, even though they do not directly enter the Einstein field equations. The Liouville hierarchy cannot in general be truncated, without some approximation scheme to close the truncated system. For a collisional gas, one expects on physical grounds that interactions tend to thermalize, and the higher multipoles will tend to be suppressed. For a collisionless and massless gas, anisotropy in the higher multipoles does not in general disappear through free-streaming in an expanding universe, since the velocity of particles is not affected by redshifting. On the other hand, redshifting the momentum of massive particles reduces the peculiar velocity v.

Up to this point, our results apply to any collisionless gas in a nearly FRW universe. Now we need to specialize to the case of CDM, for which the velocity dispersion is small. This allows us to develop a consistent approximation scheme for truncating the Liouville hierarchy, following an approach similar to that of [3]. Small velocity dispersion means that there is a small effective maximum velocity v_*, above which the distribution is effectively vanishing. More precisely,

$$v_*^2 = \mathcal{O}(\epsilon) \quad \text{and} \quad \frac{1}{\rho}\int_{E_*}^\infty E^{2-n}\lambda^{n+1} F_{a_1 \cdots a_l} dE = \mathcal{O}(\epsilon^2) \quad \text{for} \quad l \geq 0, \quad n = 0,1,2.$$

We assume that the derivatives of the distribution multipoles are similarly restricted. With the small velocity dispersion approximation, we can show that many of the terms in the equations above are second-order. For example,

$$p = \frac{4\pi}{3}\int_m^\infty \frac{\lambda^2}{E^2} E^2\lambda F dE$$

$$= \frac{4\pi}{3}\int_m^\infty [v^2 + \mathcal{O}(v^4)]E^2\lambda F dE$$

$$\leq \tfrac{1}{3}v_*^2\left(4\pi\int_m^\infty E^2\lambda F dE\right) + \rho\mathcal{O}(\epsilon^2),$$

so that

$$\frac{p}{\rho} \leq \tfrac{1}{3}v_*^2 + \mathcal{O}(\epsilon^2).$$

Similarly, we find that

$$\frac{\mathcal{M}}{\rho} \leq v_*^4 + \mathcal{O}(\epsilon^2),$$

$$\frac{\mathcal{P}}{\rho} \leq \tfrac{1}{3}v_*^4 + \mathcal{O}(\epsilon^2),$$

$$\frac{|Q_a|}{\rho} \leq v_*^2\frac{|q_a|}{\rho} + \mathcal{O}(\epsilon^2),$$

$$\frac{|\mathcal{R}_{ab}|}{\rho} \leq v_*^2\frac{|\pi_{ab}|}{\rho} + \mathcal{O}(\epsilon^2),$$

$$\frac{|\mathcal{S}_{abc}|}{\rho} \leq v_*^2\frac{|\zeta_{abc}|}{\rho} + \mathcal{O}(\epsilon^2).$$

To linear order, it follows that Eq. (10) produces the equation of state

$$\rho = mn + \tfrac{3}{2}p, \quad (21)$$

which simply expresses that each particle has rest mass m and kinetic energy $\frac{1}{2}mv^2$ to lowest order. The stress evolution equations (19) and (20) reduce to

$$\dot{p} + \tfrac{5}{3}\Theta p = 0, \tag{22}$$

$$\dot{\pi}_{ab} + \tfrac{5}{3}\Theta\,\pi_{ab} = 0. \tag{23}$$

Since $p/\rho = \mathcal{O}(\epsilon)$, the term $p\sigma_{ab}$ is second order and falls away from the stress evolution equation (23). The octopole anisotropy does not contribute to the stress evolution at linear order, so that the multipole hierarchy can be truncated after the quadrupole. The higher-multipole evolution equations are decoupled from the Einstein-Liouville system at linear order. Equations (14), (22) and (23) form a closed system of evolution equations for the dynamical quantities ρ, p and π_{ab}.

Our approximation scheme extends that of [3] from a Newtonian to a relativistic treatment, but it also generalizes the description of the matter. In [3], it is assumed that $\pi_{ab} = 0$, implying the very restrictive condition of *isotropic velocity dispersion*. We do not make this assumption; on the contrary, the anisotropic stress π_{ab} plays a crucial role in our analysis.

There are some formal similarities here to the Grad 14-moment method as applied in the hydrodynamic near-equilibrium regime. In that context, the Boltzmann hierarchy is also truncated beyond the quadrupole, and anisotropic stress obeys the Israel-Stewart transport equation [13]

$$\tau\dot{\pi}_{ab} + \pi_{ab} = -2\eta\sigma_{ab},$$

where τ is a relaxation timescale, and η is the shear viscosity. This transport equation has a similar form to our equation (23). However, the Israel-Stewart transport equation, and the relativistic Grad method on which it is based, apply to a *collision-dominated* gas, whereas we are dealing with a collision-free gas.

Note that since $p/\rho = \mathcal{O}(\epsilon)$, the momentum constraint equation (15) reduces to

$$\rho A_a + D_a p + D^b\pi_{ab} = 0. \tag{24}$$

In the background ($\epsilon \to 0$), we have $p \to 0$. This means that the background distribution function reduces to a delta-function, since there is no velocity dispersion, and we have the kinetic theory form of the dust model [10,16]. In the inhomogeneous perturbed universe, the monopole F of the distribution function is not a delta-function, since there is velocity dispersion. Thus perturbation of the background not only produces nonzero dipole and higher multipoles, but also changes the monopole.

III. COVARIANT ANALYSIS OF DENSITY INHOMOGENEITY

In this section we provide the basic equations governing the evolution of density inhomogeneity in cold dark matter when isotropic and anisotropic stresses are incorporated. The full set of covariant and gauge-invariant perturbation equations for a general energy-momentum tensor is derived and discussed in [6,7]. The formalism is based on constructing covariant quantities which vanish in the background, thus ensuring that they are gauge-invariant. Density inhomogeneity is described by the comoving fractional density gradient (which fulfills the above requirements):

$$\delta_a = \frac{aD_a\rho}{\rho}, \tag{25}$$

where a is the background scale factor.

This quantity carries information about the magnitude, rotational and shape-distortion properties of inhomogeneity, obtained by irreducibly splitting its comoving gradient [6]:

$$aD_b\delta_a = (\tfrac{1}{3}\delta)h_{ab} + \varepsilon_{abc}W^c + \xi_{ab}. \tag{26}$$

Here

$$\delta \equiv aD^a\delta_a = \frac{(aD)^2\rho}{\rho}$$

corresponds to the gauge-invariant density perturbation scalar ϵ_m in the metric-based formalism [17]. The quantity

$$W_a = -\tfrac{1}{2}a\,\mathrm{curl}\,\delta_a$$

describes the rotational properties of inhomogeneous clustering, and it is proportional to the vorticity ω_a. Finally,

$$\xi_{ab} = aD_{\langle a}\delta_{b\rangle}$$

describes the volume-true distortion of inhomogeneous clustering.

These quantities completely and covariantly describe infinitesimal inhomogeneities in the density. They obey evolution equations in which the stresses are source terms. Since the pressure p is $\mathcal{O}(\epsilon)\rho$, we may neglect it in the background. We assume a flat (i.e., Einstein–de Sitter) background, neglecting the baryonic component. Thus we are investigating density inhomogeneity in CDM in the linear regime, with potential applications to dark matter halo formation. The background field equations give

$$\rho = 3H^2, \quad H = \frac{2}{3t}, \quad a = a_0\left(\frac{t}{t_0}\right)^{2/3}. \tag{27}$$

The evolution equations (22) and (23) for CDM stresses in a nearly FRW universe can be integrated to give

$$\pi_{ab} = \pi_{ab}^{(0)}\left(\frac{a_0}{a}\right)^5, \tag{28}$$

$$p = p_0\left(\frac{a_0}{a}\right)^5, \tag{29}$$

where

$$\dot{\pi}_{ab}^{(0)} = 0 = \dot{p}_0.$$

Using the linearized identity

$$(a D_a S_{b \ldots c})^{\cdot} = a D_a \dot{S}_{b \ldots c},$$

which holds for any tensor $S_{a \ldots b}$ that vanishes in the background, it follows that

$$(a^n D_{a_1} \cdots D_{a_n} \pi_{ab}^{(0)})^{\cdot} = 0 = (a^n D_{a_1} \cdots D_{a_n} p_0)^{\cdot}, \quad (30)$$

for any positive integer n.

A. Density perturbations

We consider first the effect of stresses on density perturbations. The evolution equation for δ, as given by Eq. (28) of [7], reduces to

$$\ddot{\delta} + 2H\dot{\delta} - \tfrac{3}{2}H^2 \delta = \frac{a^2}{\rho} D^2 (D^2 p) + 3H\dot{S} - 3H^2 S + D^2 S, \quad (31)$$

where the anisotropic stress term is

$$S \equiv \frac{a^2 D^a D^b \pi_{ab}}{\rho}, \quad (32)$$

and H and ρ are given by Eq. (27). Note that the isotropic stress p occurs only via the gradient term $D_a p$. Using Eqs. (27)–(30), we find that

$$S = \tfrac{3}{4} t_0^2 S_0 \left(\frac{a_0}{a}\right)^2, \quad \frac{a^2}{\rho} D^4 p = \tfrac{3}{4} \left(\frac{t_0}{a_0}\right)^2 P_0 \left(\frac{a_0}{a}\right)^4$$

where $\dot{S}_0 = 0 = \dot{P}_0$ and

$$S_0 \equiv a^2 D^a D^b \pi_{ab}^{(0)}, \quad P_0 \equiv a^4 D^4 p_0. \quad (33)$$

Thus the evolution equation (31) becomes

$$\ddot{\delta} + \left(\frac{4}{3t}\right) \dot{\delta} - \left(\frac{2}{3t^2}\right) \delta = \frac{3}{4} \left(\frac{t_0}{a_0}\right)^2 (P_0 + R_0) \left(\frac{t_0}{t}\right)^{8/3}$$
$$- 3S_0 \left(\frac{t_0}{t}\right)^{10/3},$$

where

$$R_0 \equiv a^2 D^2 S_0, \quad (34)$$

so that $\dot{R}_0 = 0$ by Eq. (30). The solution is

$$\delta = C^{(+)} \left(\frac{t}{t_0}\right)^{2/3} + C^{(-)} \left(\frac{t}{t_0}\right)^{-1}$$
$$- \left[3t_0^2 \left(\frac{3t_0}{4a_0}\right)^2 (P_0 + R_0)\right] \left(\frac{t}{t_0}\right)^{-2/3} - \left[\tfrac{9}{2} t_0^2 S_0\right] \left(\frac{t}{t_0}\right)^{-4/3}, \quad (35)$$

where $\dot{C}^{(\pm)} = 0$.

The standard dust solution is given by the growing $C^{(+)}$ and decaying $C^{(-)}$ terms. The effects of stress (sourced in velocity dispersion) are encoded in the following two decay-

ing terms. Note that one of the two new decaying modes decays less rapidly than the standard decaying mode, and the other decays more rapidly. The second, more rapidly decaying, term, is a purely anisotropic stress term, whereas the first term has isotropic (P_0) and anisotropic (R_0) stress contributions and affects only small scales. When velocity dispersion is forced to vanish exactly in the dust model, it is possible to remove the decaying mode by choosing $C^{(-)} = 0$. When velocity dispersion is incorporated, it is no longer possible to remove decaying modes by choice of initial conditions. This is related to the fact that the perturbations are no longer adiabatic, given that the stresses are neglected in the background. The new decaying terms depend on the initial spatial distribution of stresses, as described by the quantities P_0, R_0 and S_0, defined in Eqs. (33) and (34). By the momentum conservation equation (24), we can replace $P_0 + R_0$ by a term proportional to the Laplacian of the divergence of the 4-acceleration:

$$P_0 + R_0 = - \rho_0 a_0 \left(\frac{t}{t_0}\right)^2 (aD)^2 (a D^a A_a).$$

B. Rotational instability

The evolution equation for the rotational part W_a of density inhomogeneity is given in [7]:

$$\dot{W}_a + \tfrac{3}{2} H W_a = - \left(\frac{3H}{2\rho}\right) a^2 \, \text{curl} \, D^b \pi_{ab}. \quad (36)$$

Using Eqs. (27)–(30), this becomes

$$\dot{W}_a + \left(\frac{1}{t}\right) W_a = - \tfrac{3}{4} t_0 N_a \left(\frac{t_0}{t}\right)^{7/3}, \quad (37)$$

where

$$N_a \equiv a^2 \, \text{curl} \, D^b \pi_{ab}^{(0)}, \quad (38)$$

so that $\dot{N}_a = 0$. Note that we can use the linearized form of the differential identities in [12] to rewrite this as

$$N_a = 2a^2 D^b \text{curl} \pi_{ab}^{(0)}.$$

The solution of Eq. (37) is

$$W_a = C_a^{(-)} \left(\frac{t}{t_0}\right)^{-1} + \left[\tfrac{9}{4} t_0^2 N_a\right] \left(\frac{t}{t_0}\right)^{-4/3}, \quad (39)$$

where $\dot{C}_a^{(-)} = 0$. The standard dust solution is the $C_a^{(-)}$ term, and the effect of velocity dispersion is to introduce another decaying mode, which decays more rapidly. The main effect of this new mode is to break the constancy of direction of the axis of rotation. It follows from Eq. (39) that

$$(W_a)|_{t_0} = C_a^{(-)} + \tfrac{9}{4} t_0^2 N_a, \quad (\dot{W}_a)|_{t_0} = - t_0^{-1} [C_a^{(-)} + 3 t_0^2 N_a].$$

In the absence of anisotropic stress (or for anisotropic stress with curl-free divergence), $\dot{W}_a$ remains parallel to W_a, and the direction of the axis of rotation is constant along u^a.

When anisotropic stresses are incorporated, $N_a \neq 0$ in general, so that $\dot{W}_a$ is no longer parallel to W_a, and the direction of the axis evolves in time.

C. Shape distortion

From [7], the shape distortion part ξ_{ab} obeys the evolution equation

$$\ddot{\xi}_{ab} + 2H\dot{\xi}_{ab} - \tfrac{3}{2}H^2\xi_{ab} = \frac{a^2}{\rho}D_{\langle a}D_{b\rangle}D^2 p + \frac{a^2}{\rho}[3HD_{\langle a}D^c\dot{\pi}_{b\rangle c}$$

$$+ 6H^2 D_{\langle a}D^c\pi_{b\rangle c}$$

$$+ D_{\langle a}D_{b\rangle}D^cD^d\pi_{cd}]. \tag{40}$$

Using again Eqs. (27)–(30), we find that

$$\ddot{\xi}_{ab} + \left(\frac{4}{3t}\right)\dot{\xi}_{ab} - \left(\frac{2}{3t^2}\right)\xi_{ab} = -\frac{3}{4}\left(\frac{t_0}{a_0}\right)^2(P_{ab} + R_{ab})\left(\frac{t_0}{t}\right)^{8/3}$$

$$- 3S_{ab}\left(\frac{t_0}{t}\right)^{10/3}, \tag{41}$$

where we have defined

$$P_{ab} \equiv a^4 D_{\langle a}D_{b\rangle}D^2 p_0,$$

$$R_{ab} \equiv a^4 D_{\langle a}D_{b\rangle}D^cD^d\pi_{cd}^{(0)}, \tag{42}$$

$$S_{ab} \equiv a^2 D_{\langle a}D^c\pi^{(0)}_{b\rangle c},$$

so that

$$\dot{P}_{ab} = \dot{R}_{ab} = \dot{S}_{ab} = 0.$$

Then, as in the scalar case, Eq. (41) can be solved to give

$$\xi_{ab} = C_{ab}^{(+)}\left(\frac{t}{t_0}\right)^{2/3} + C_{ab}^{(-)}\left(\frac{t}{t_0}\right)^{-1}$$

$$- \left[3t_0^2\left(\frac{3t_0}{4a_0}\right)^2(P_{ab} + R_{ab})\right]\left(\frac{t}{t_0}\right)^{-2/3}$$

$$- \left[\tfrac{9}{2}t_0^2 S_{ab}\right]\left(\frac{t}{t_0}\right)^{-4/3}, \tag{43}$$

where $\dot{C}_{ab}^{(\pm)} = 0$. Again we see the occurrence of new decaying modes arising from stress effects. One of the new terms decays more slowly than the standard decaying term which arises in the dust case.

These new terms have the following important implication. We consider an initially isotropic infinitesimal fluctuation at a point $\vec{x}_0$, and follow its evolution along u^a. The initial velocity is described via the PSTF and constant tensor V_{ab}, i.e.

$$\xi_{ab}(t_0, \vec{x}_0) = 0, \quad \dot{\xi}_{ab}(t_0, \vec{x}_0) = H_0 V_{ab}. \tag{44}$$

Let $\tau \equiv t/t_0$, and define the constant PSTF tensors

$$J_{ab} \equiv \tfrac{3}{80}t_0^2\left(\frac{t_0}{a_0}\right)^2[P_{ab}(\vec{x}_0) + R_{ab}(\vec{x}_0)],$$

$$K_{ab} \equiv \tfrac{1}{10}t_0^2 S_{ab}(\vec{x}_0).$$

Then Eq. (43) gives

$$\xi_{ab} = [\tfrac{2}{5}\tau^{2/3}(1 - \tau^{-5/3})]V_{ab}$$

$$+ [\tau^{2/3}(4 - 45\tau^{-4/3} + 41\tau^{-5/3})]J_{ab}$$

$$+ [\tau^{2/3}(-4 + 49\tau^{-5/3} - 45\tau^{-2})]K_{ab}. \tag{45}$$

Thus for a dust model, in which $J_{ab} = 0 = K_{ab}$, the evolution of shape distortion is purely inertial, i.e., it is fixed by the initial velocity ellipsoid V_{ab}, and no further distortion can develop as the fluctuation evolves (compare [18]). All the covariant time derivatives of ξ_{ab} are proportional to V_{ab}:

$$\xi_{ab} \propto \dot{\xi}_{ab} \propto \ddot{\xi}_{ab} \propto \cdots \propto V_{ab}. \tag{46}$$

By contrast, when velocity dispersion is incorporated via stress effects, the same initial conditions in Eq. (44) lead to a non-trivial evolution of distortion, away from that initially determined by the velocity ellipsoid V_{ab}. The simple relation in Eq. (46) is broken, and the evolution of distortion is no longer fixed by the initial velocity ellipsoid. Although the stress terms that cause the additional ''non-inertial'' distortion are small and decaying, once the extra distortion is introduced, there is no mechanism for removing it, at least during the linear regime. Thus the distortion persists during the subsequent linear evolution.

The impact of stress on shape distortion is reminiscent of the impact of stress on shear decay: for radiative anisotropic stress, Barrow and Maartens [19] have shown that the decay of shear due to expansion is slowed down. We expect that the same qualitative result holds in the case of non-radiative anisotropic stress, such as considered here.

IV. CONCLUSION

Density perturbation theory for the growth of structures in a CDM framework has been generalized in a covariant form which self-consistently incorporates small velocity dispersion. The analysis generalizes the Newtonian approach of Buchert and Domínguez [3] to general relativity; furthermore, it dispenses with their isotropic dispersion assumption, and considers the rotational and shape distortion properties of density inhomogeneity, in addition to the density perturbations. The evolution equations are integrated exactly for all these parts of density inhomogeneity in the linear regime.

As a special case ($\pi_{ab} = 0$), our results contain the generalization of the adhesion model, as shown in [3]. More generally, our solutions show explicitly how the decaying modes are modified by stress effects induced via velocity dispersion. These modifications are small, but they have some important implications.

(1) First, as argued in [3], the presence of velocity dispersion avoids some of the problems that arise in the dust

model, which is pathological in enforcing strictly zero dispersion.

(2) Second, the new decaying modes of density perturbations reflect non-adiabatic features introduced by the stresses. One of these modes decays less rapidly than the standard decaying mode.

(3) Third, the new decaying mode in the rotational part of density inhomogeneity has the effect of breaking the constancy of the direction of rotation axis.

(4) Fourth, the new decaying modes in the shape distortion mean that additional ''non-inertial'' distortion is generated, which is not present in the dust (purely inertial) model. The additional distortion remains during the linear regime, despite the decaying nature of the source terms, since there is no (linear) mechanism to reverse it. Although the dominant distortion effects will take place in the nonlinear regime, this linear effect has some interest, and it may be worth investigating the statistics of the phenomenon in order to be able to make more general assertions about the distortion conditions at the onset of nonlinear structure formation.

The stress effects on rotational and shape-distortion properties of the density distribution are qualitatively similar to the effects of a magnetic field [20].

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