# Resonance in strong *WW* rescattering in massive SU(2) gauge theory

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We investigate the effects of *WW* rescattering through strong anomalous four-vector boson couplings. In the $I=1$, $J=1$ channel, we find a resonance with a mass of approximately 200 GeV and a width of less than 12 GeV. In an application to pion physics we find a small correction to the Kawarabayashi-Suzuki-Riàzuddin-Fayyazuddin relation. [S0556-2821(99)09019-0]

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## I. INTRODUCTION

Within the standard model of electroweak interactions the gauge principle fully determines the self-couplings of the vector bosons. Therefore the measurement of these couplings is of prime importance, as deviations would be an indication of new physics. The gauge principle predicts well-defined three- and four-vector boson self-couplings. Deviations of these values can be described in a gauge-invariant way in the Stückelberg formalism [1]. Within the Stückelberg formalism the standard model is described as a gauged nonlinear sigma model. This implicitly assumes that the Higgs particle does not play a fundamental role. Alternatively, one can see it as the $m_H \to \infty$ limit of the standard model. If indeed anomalous vector boson couplings are present, this should be a reasonable assumption, since in that case strong interactions should be present. The triple vector boson couplings are severely constrained by the measurements at the CERN $e^+e^-$ collider LEP-200 and the Fermilab Tevatron [2]. Also, indirect limits from LEP-100 and $(g-2)_\mu$ exist. Altogether, experiments indicate they should be small. This is not too surprising, as it is very hard to construct a model that would give rise to large effects. The reason is that within the three-vector boson couplings there is always an interplay between longitudinal and transversal vector bosons.

For the four-vector boson couplings the situation is somewhat different. Here one can write down vertices that contain longitudinal vector bosons only. These vertices correspond to the Goldstone boson sector of the theory, and it is much easier to generate strong interactions in this sector. Such results come typically through intermediate heavy Higgs boson exchange. An example of such a model is given in [3], where the strong interactions are generated via singlet effects in the Higgs sector. Also, in the standard model the two-loop heavy Higgs correction in the four-vector boson [4] couplings is an order of magnitude larger than in the three-vector boson couplings [5]. About the four-vector boson couplings much less is known than about the three-vector boson couplings. Direct experiments probing these interactions do not exist at present. They can at the moment only be tested through radiative corrections in the $\rho$ parameter. These corrections can be calculated within perturbation theory with a cutoff $\Lambda$. Within the four-vector boson couplings one should distinguish between two types. In the standard model there is an extra global $SU_R(2)$ symmetry when the hypercharge is turned off. For the anomalous couplings this is not necessarily the case. The couplings that violate $SU_R(2)$ even in the absence of hypercharge give quartically divergent corrections to $\delta\rho$ [6,7] and should therefore be negligibly small. Physically, this means that the underlying strong interactions, generating the anomalous couplings, should respect the $SU_R(2)$ invariance. This leaves only two operators that preserve $SU_R(2)$ invariance in the absence of hypercharge. If one could use simple cutoff perturbation theory, these operators still give quadratically divergent corrections to $\delta\rho$ [6,7] and are sufficiently suppressed to give only small effects in future colliders. In [8] the quadratic divergences were ignored and therefore the limits are weak.

This leaves only the possibility that the anomalous couplings are so large that perturbation theory cannot be trusted, and therefore the low energy limits are invalidated. It is precisely this case that we study in this paper. We assume that anomalous couplings are present which preserve the $SU_R(2)$ symmetry of the theory and assume the coefficients for the corresponding operators to be large. To simplify the calculation we actually ignore hypercharge altogether and work in the $SU_L(2)\times SU_R(2)$ model. This leads to a simplification because we can study different isospin channels separately. Because the interactions are assumed to be strong, vector boson scattering cannot be described by the tree-level vertex. We therefore perform a resummation of loop graphs. In most channels we find no particularly interesting effect. The only exception is the $I=1$, $J=1$ channel, where a resonance is found. In the case of very strong anomalous couplings, which we assume, the resonance can be quite close to the two-vector boson threshold. The coupling of the resonance to the vector bosons is found to be suppressed by the cutoff and could be small. Dependent on the parameters, the resonance could be visible even at LEP-200.

The paper is organized as follows. In Sec. II we present the model. In Sec. III we perform the calculation of the bubble sum. In Sec. IV we discuss the results. The Appendix contains technical details.

## II. MODEL

### A. Lagrangian

We work in a pure massive SU(2) gauge theory and introduce the anomalous couplings in a gauge-invariant way using the Stückelberg formalism [1]. That is, we write the theory as a gauged nonlinear sigma model:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}\text{Tr}(\mathbf{W}_{u\nu}\mathbf{W}^{\mu\nu}) + m_{W0}^2\,\text{Tr}(\mathbf{V}_\mu\mathbf{V}^\mu), \tag{1}$$

with

$$\mathbf{W}_\mu \equiv \frac{1}{2}\tau_a W_\mu^a = \mathbf{W}_\mu^\dagger, \tag{2}$$

$$\mathbf{W}_{\mu\nu} \equiv \frac{1}{2}\tau_a W_{\mu\nu}^a = \partial_\mu \mathbf{W}_\nu - \partial_\nu \mathbf{W}_\mu + ig[\mathbf{W}_\mu, \mathbf{W}_\nu] = \mathbf{W}_{\mu\nu}^\dagger, \tag{3}$$

$$\mathbf{V}_\mu \equiv -\frac{i}{g}(D_\mu U)U^\dagger = V_\mu^\dagger, \tag{4}$$

$$D_\mu U \equiv \partial_\mu U + ig\mathbf{W}_\mu U, \tag{5}$$

$$U = \exp(i v_a \tau_a), \tag{6}$$

with real fields $v_a$; i.e., the SU(2)-valued field $U$ describes the Goldstone degrees of freedom. $m_{W0}$ is the mass of the three gauge bosons in absence of the higher covariant derivative terms to be added below. The anomalous couplings are then introduced as

$$\begin{aligned}\mathcal{L}_{\text{ano}} &= g_4(\text{Tr}[\mathbf{V}_\mu\mathbf{V}_\nu])^2 + g_5(\text{Tr}[\mathbf{V}^\mu\mathbf{V}_\mu])^2\\ &= g_4(\text{Tr}[\mathbf{V}_\mu\mathbf{V}_\nu^\dagger])^2 + g_5(\text{Tr}[\mathbf{V}^\mu\mathbf{V}_\mu^\dagger])^2\\ &= g_4 g^{-4}\{\text{Tr}[(D_\mu U)(D_\nu U)^\dagger]\}^2\\ &\quad + g_5 g^{-4}\{\text{Tr}[(D^\mu U)(D_\mu U)^\dagger]\}^2.\end{aligned} \tag{7}$$

Their form is determined by the requirement that they conserve *CP*, are not accompanied by three-vector boson couplings (on which the limits are much more stringent, as was mentioned in the Introduction), and that they are invariant under the custodial $\text{SU}_R(2)$ symmetry $U \to UU_R$ with $U \in \text{SU}(2)$.

To regulate higher-than-logarithmic divergences, we introduce higher covariant derivative terms through

$$\begin{aligned}\mathcal{L}_{\text{hcd}} &= \frac{1}{2\Lambda_W^2}\text{Tr}[(D_\alpha \mathbf{W}_{\mu\nu})(D^\alpha \mathbf{W}^{\mu\nu})]\\ &\quad - \frac{m_{W0}^2}{\Lambda_V^2}\text{Tr}[(D_\alpha \mathbf{V}_\mu)(D^\alpha \mathbf{V}^\mu)],\end{aligned} \tag{8}$$

with

$$D_\alpha \mathbf{W}_{\mu\nu} = \partial_\alpha \mathbf{W}_{\mu\nu} + ig[\mathbf{W}_\alpha, \mathbf{W}_{\mu\nu}], \tag{9}$$

$\Lambda_W$ and $\Lambda_V$ effectively being momentum cutoffs. These are the unique dimension-6 higher-derivative propagator terms and are further discussed in [7].

We work in unitary gauge, where we have $U = \mathbf{1}$, $v_a = 0$, and therefore

$$\mathbf{V}_\mu = \mathbf{W}_\mu. \tag{10}$$

The signature of our metric is

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1). \tag{11}$$

### B. Feynman rules

The $W$ propagator in unitary gauge with higher covariant derivatives is

$$\Delta_{\mu\nu}^W(k) = \Delta_{\text{tr}}^W(k^2)\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \Delta_{lg}^W(k^2)\frac{k_\mu k_\nu}{k^2}, \tag{12}$$

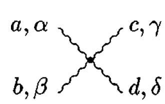
with

$$\Delta_{\text{tr}}^W(k^2) = \frac{i\Lambda_W^2}{(k^2 - m_-^2)(k^2 - m_+^2)}, \tag{13}$$

$$\Delta_{lg}^W(k^2) = -\frac{i\Lambda_V^2}{m_{W0}^2}\frac{1}{k^2 - \Lambda_V^2}, \tag{14}$$

$$\begin{aligned}m_\pm^2 &= \frac{\Lambda_W^2}{2}\left[\left(1 + \frac{m_{W0}^2}{\Lambda_V^2}\right) \pm \sqrt{\left(1 + \frac{m_{W0}^2}{\Lambda_V^2}\right)^2 - \frac{4m_{W0}^2}{\Lambda_W^2}}\right]\\ &= \begin{cases}\Lambda_W^2 + \mathcal{O}(\Lambda^0),\\ m_{W0}^2 + \mathcal{O}(\Lambda^{-2}).\end{cases}\end{aligned} \tag{15}$$

The Feynman rule for the anomalous four-vector boson couplings is

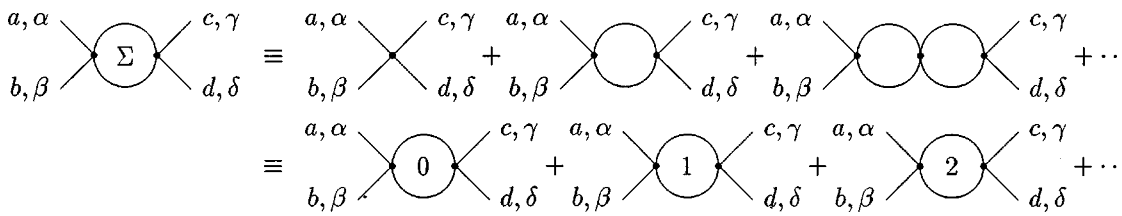



$$\begin{aligned}= i\{&\delta_{ab}\delta_{cd}[2g_5 g_{\alpha\beta}g_{\gamma\delta} + g_4(g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma})]\\ &+\delta_{ac}\delta_{bd}[2g_5 g_{\alpha\gamma}g_{\beta\delta} + g_4(g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\delta}g_{\beta\gamma})]\\ &+\delta_{ad}\delta_{bc}[2g_5 g_{\alpha\delta}g_{\beta\gamma} + g_4(g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\beta}g_{\gamma\delta})]\}\\ \equiv\ & V_{\alpha\beta\gamma\delta}^{abcd}.\end{aligned} \tag{16}$$

## III. BUBBLE SUM

### A. Definitions

Assuming the anomalous couplings to dominate the gauge coupling, we compute the ''bubble sum''




$$\equiv\; B^{(0)abcd}_{\alpha\beta\gamma\delta} + B^{(1)abcd}_{\alpha\beta\gamma\delta} + B^{(2)abcd}_{\alpha\beta\gamma\delta} + \cdots$$

$$\equiv\; \Sigma^{abcd}_{\alpha\beta\gamma\delta}\,, \tag{17}$$

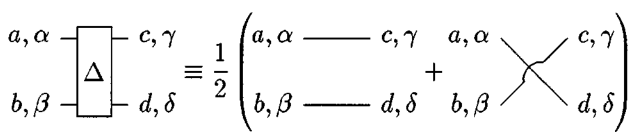
where all lines represent gauge bosons and where the vertices contain only the anomalous part. Note that Eq. (17) is invariant under each of

$$(a,\alpha)\leftrightarrow(b,\beta), \tag{18}$$

$$(c,\gamma)\leftrightarrow(d,\delta), \tag{19}$$

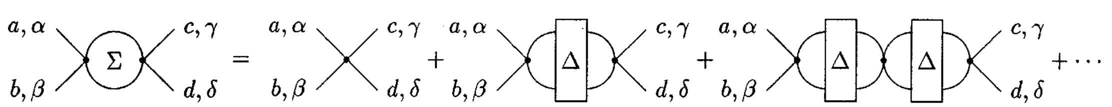
$$(a,\alpha,b,\beta)\leftrightarrow(c,\gamma,d,\delta). \tag{20}$$

To express Eq. (17) in a purely algebraic form, we split each additional bubble into a vertex part (16) and a propagator part




(21)

where we have imposed the relevant symmetries (18)–(20). Equation (17) can now be written as



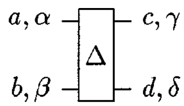

(22)

Since the vertex is independent of any momenta, the integration over the loop momenta can be done in the propagator part. Abbreviate

$$\int_p \equiv \mu^{4-d}\int \frac{d^d p}{(2\pi)^d}, \tag{23}$$

with the renormalization scale $\mu$, and use dimensional regularization together with the modified minimal subtraction ($\overline{MS}$) scheme throughout. Define

$$P^{abcd}_{\alpha\beta\gamma\delta} \equiv \tfrac{1}{2}\int_p$$



$$\begin{aligned}
&= \tfrac{1}{4}\int_p \Big\{ \delta_{ac}\delta_{bd}\left[\Delta^W_{tr}(p^2)\left(g_{\alpha\gamma}-\tfrac{p_\alpha p_\gamma}{p^2}\right)+\Delta^W_{lg}(p^2)\tfrac{p_\alpha p_\gamma}{p^2}\right] \\
&\qquad \times\left[\Delta^W_{tr}((p+k)^2)\left(g_{\beta\delta}-\tfrac{(p+k)_\beta(p+k)_\delta}{(p+k)^2}\right)+\Delta^W_{lg}((p+k)^2)\tfrac{(p+k)_\beta(p+k)_\delta}{(p+k)^2}\right] \\
&\qquad +\delta_{ad}\delta_{bc}\left[\Delta^W_{tr}(p^2)\left(g_{\alpha\delta}-\tfrac{p_\alpha p_\delta}{p^2}\right)+\Delta^W_{lg}(p^2)\tfrac{p_\alpha p_\delta}{p^2}\right] \\
&\qquad \times\left[\Delta^W_{tr}((p+k)^2)\left(g_{\beta\gamma}-\tfrac{(p+k)_\beta(p+k)_\gamma}{(p+k)^2}\right)+\Delta^W_{lg}((p+k)^2)\tfrac{(p+k)_\beta(p+k)_\gamma}{(p+k)^2}\right]\Big\} \\
&\equiv \Big\{ \delta_{ac}\delta_{bd}\Big[\Delta_1(k^2)g_{\alpha\gamma}g_{\beta\delta}+\Delta_2(k^2)(g_{\alpha\beta}g_{\gamma\delta}+g_{\alpha\delta}g_{\beta\gamma})+\Delta_3(k^2)(g_{\alpha\gamma}k_\beta k_\delta+g_{\beta\delta}k_\alpha k_\gamma)/k^2 \\
&\qquad +\Delta_4(k^2)(g_{\alpha\beta}k_\gamma k_\delta+g_{\gamma\delta}k_\alpha k_\beta+g_{\alpha\delta}k_\beta k_\gamma+g_{\beta\gamma}k_\alpha k_\delta)/k^2+\Delta_5(k^2)k_\alpha k_\beta k_\gamma k_\delta/k^4\Big] \\
&\qquad +\delta_{ad}\delta_{bc}\Big[\Delta_1(k^2)g_{\alpha\delta}g_{\beta\gamma}+\Delta_2(k^2)(g_{\alpha\beta}g_{\gamma\delta}+g_{\alpha\gamma}g_{\beta\delta})+\Delta_3(k^2)(g_{\alpha\delta}k_\beta k_\gamma+g_{\beta\gamma}k_\alpha k_\delta)/k^2 \\
&\qquad +\Delta_4(k^2)(g_{\alpha\beta}k_\gamma k_\delta+g_{\gamma\delta}k_\alpha k_\beta+g_{\alpha\gamma}k_\beta k_\delta+g_{\beta\delta}k_\alpha k_\gamma)/k^2+\Delta_5(k^2)k_\alpha k_\beta k_\gamma k_\delta/k^4\Big]\Big\}, \qquad (24)
\end{aligned}$$

where an additional factor of 1/2 has been introduced to account for the implicit symmetry factors in the bubble sum (17), which can now be written as

$$\begin{aligned}
\Sigma^{abcd}_{\alpha\beta\gamma\delta} &= V^{abcd}_{\alpha\beta\gamma\delta} + V^{abmn}_{\alpha\beta\mu\nu}P^{mnm'n'}_{\mu\nu\mu'\nu'}V^{m'n'cd}_{\mu'\nu'\gamma\delta} \\
&\quad + V^{abmn}_{\alpha\beta\mu\nu}P^{mnm'n'}_{\mu\nu\mu'\nu'}V^{m'n'm''n''}_{\mu'\nu'\mu''\nu''}P^{m''n''m'''n'''}_{\mu''\nu''\mu'''\nu'''}V^{m'''n'''cd}_{\mu'''\nu'''\gamma\delta} \\
&\quad + \cdots . \qquad (25)
\end{aligned}$$

The strategy for computing $\Delta_1$–$\Delta_5$ is given in the Appendix. Keeping only at least quadratically divergent, i.e., $\mathcal{O}(\Lambda^2)$, terms for the total $\Delta_1$–$\Delta_5$, and $\mathcal{O}(\Lambda^0)$ terms for their real parts, the results are

$$\begin{aligned}
\Delta_1(k^2) &= -\frac{i\Lambda_V^4}{96(4\pi)^2 m_{W0}^4}\left[\frac{1}{\epsilon}+\frac{5}{6}-\ln\frac{\Lambda_V^2}{\overline{\mu}^2}\right] \\
&\quad -\frac{5i\Lambda_W^2\Lambda_V^2}{48(4\pi)^2(\Lambda_W^2-\Lambda_V^2)m_{W0}^2}\ln\frac{\Lambda_W^2}{\Lambda_V^2} \\
&\quad -\frac{7i\Lambda_V^2}{1152(4\pi)^2 m_{W0}^2}\left(\frac{k^2}{m_{W0}^2}\right)+\mathcal{O}(\Lambda^0), \qquad (26)
\end{aligned}$$

$$\begin{aligned}
\operatorname{Re}\Delta_1(k^2) &= \frac{\pi\sqrt{1-4m_{W0}^2/k^2}}{60(4\pi)^2}\left[\left(\frac{k^2}{4m_{W0}^2}\right)^2+8\left(\frac{k^2}{4m_{W0}^2}\right)+6\right] \\
&\quad +\mathcal{O}(\Lambda^{-2}), \qquad (27)
\end{aligned}$$

$$\begin{aligned}
\Delta_2(k^2) &= -\frac{i\Lambda_V^4}{96(4\pi)^2 m_{W0}^4}\left[\frac{1}{\epsilon}+\frac{5}{6}-\ln\frac{\Lambda_V^2}{\overline{\mu}^2}\right] \\
&\quad +\frac{i\Lambda_W^2\Lambda_V^2}{48(4\pi)^2(\Lambda_W^2-\Lambda_V^2)m_{W0}^2}\ln\frac{\Lambda_W^2}{\Lambda_V^2} \\
&\quad -\frac{7i\Lambda_V^2}{1152(4\pi)^2 m_{W0}^2}\left(\frac{k^2}{m_{W0}^2}\right)+\mathcal{O}(\Lambda^0), \qquad (28)
\end{aligned}$$

$$\operatorname{Re}\Delta_2(k^2) = \frac{\pi\sqrt{1-4m_{W0}^2/k^2}}{60(4\pi)^2}\left[\left(\frac{k^2}{4m_{W0}^2}\right)-1\right]^2+\mathcal{O}(\Lambda^{-2}), \qquad (29)$$

$$\Delta_3(k^2) = \frac{11i\Lambda_V^2}{576(4\pi)^2 m_{W0}^2}\left(\frac{k^2}{m_{W0}^2}\right)+\mathcal{O}(\Lambda^0), \qquad (30)$$

$$\begin{aligned}
\operatorname{Re}\Delta_3(k^2) &= \frac{\pi\sqrt{1-4m_{W0}^2/k^2}}{60(4\pi)^2}\left[-6\left(\frac{k^2}{4m_{W0}^2}\right)^2\right. \\
&\quad \left.-13\left(\frac{k^2}{4m_{W0}^2}\right)+4\right]+\mathcal{O}(\Lambda^{-2}), \qquad (31)
\end{aligned}$$

$$\Delta_4(k^2) = -\frac{7i\Lambda_V^2}{576(4\pi)^2 m_{W0}^2}\left(\frac{k^2}{m_{W0}^2}\right)+\mathcal{O}(\Lambda^0), \qquad (32)$$

$$\begin{aligned}
\operatorname{Re}\Delta_4(k^2) &= \frac{\pi\sqrt{1-4m_{W0}^2/k^2}}{60(4\pi)^2}\left[\left(\frac{k^2}{4m_{W0}^2}\right)-1\right] \\
&\quad \times\left[4\left(\frac{k^2}{4m_{W0}^2}\right)+1\right]+\mathcal{O}(\Lambda^{-2}), \qquad (33)
\end{aligned}$$

$$\Delta_5(k^2)=\mathcal{O}(\Lambda^0), \tag{34}$$

$$\mathrm{Re}\,\Delta_5(k^2)=\frac{\pi\sqrt{1-4m_{W0}^2/k^2}}{60(4\pi)^2}\left[8\left(\frac{k^2}{4m_{W0}^2}\right)^2+4\left(\frac{k^2}{4m_{W0}^2}\right)+3\right]+\mathcal{O}(\Lambda^{-2}), \tag{35}$$

where $\epsilon$ is defined by $d=4-2\epsilon$ with spacetime dimension $d$. Note that $\Delta_1(k^2)$ and $\Delta_2(k^2)$ have the same quartically divergent part

$$\Delta^{(4)}\equiv-\frac{i\Lambda_V^4}{96(4\pi)^2m_{W0}^4}\left[\frac{1}{\epsilon}+\frac{5}{6}-\ln\frac{\Lambda_V^2}{\bar{\mu}^2}\right]. \tag{36}$$

Note further that $\Delta_3(k^2)$, $\Delta_4(k^2)$ are only quadratically divergent and that $\Delta_5(k^2)$ is at most logarithmically divergent. In Sec. III C we will need the following linear combinations which are only quadratically divergent:

$$-4i[\Delta_1(k^2)-\Delta_2(k^2)]=-\frac{\Lambda_V^2}{2(4\pi)^2m_{W0}^2}\frac{\ln(\Lambda_W^2/\Lambda_V^2)}{1-\Lambda_V^2/\Lambda_W^2}+\mathcal{O}(\Lambda^0) \tag{37}$$

and

$$-4i[\Delta_1(k^2)-\Delta_2(k^2)+\Delta_3(k^2)-\Delta_4(k^2)]=\frac{\Lambda_V^2}{8(4\pi)^2m_{W0}^4}\left(k^2-4m_{W0}^2\frac{\ln(\Lambda_W^2/\Lambda_V^2)}{1-\Lambda_V^2/\Lambda_W^2}+\frac{i\pi(k^2+4m_{W0}^2)(k^2-4m_{W0}^2)^{3/2}}{3\sqrt{k^2}\Lambda_V^2}\right)+\mathcal{O}(0,-2), \tag{38}$$

where the shorthand notation

$$\mathcal{O}(m,n)\equiv\mathrm{Re}\,\mathcal{O}(\Lambda^m)+\mathrm{Im}\,\mathcal{O}(\Lambda^n) \tag{39}$$

has been used.

### B. Tensor structure of $V^{abcd}_{\alpha\beta\gamma\delta}$ and $P^{abcd}_{\alpha\beta\gamma\delta}$

Now let us analyze the tensor structures that can appear in the bubble sum. The SU(2) tensors in Eqs. (16) and (24) are

$$\delta_{ab}\delta_{cd},\quad \delta_{ac}\delta_{bd},\quad \delta_{ad}\delta_{bc}, \tag{40}$$

which can be rewritten into the following linear combinations corresponding to isospin-0, -1, and -2 contributions $s_0$, $s_1$, and $s_2$, respectively, in the $(a,b)\leftrightarrow(c,d)$ channel, where we also indicate the parity under the symmetry transformations (18)–(20):

| | $a\leftrightarrow b$ | $c\leftrightarrow d$ | $(a,b)\leftrightarrow(c,d)$ |
|---|---|---|---|
| $s_0=\frac{1}{3}\,\delta_{ab}\delta_{cd}$ | + | + | + |
| $s_1=\frac{1}{2}\,(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc})$ | − | − | + |
| $s_2=-\frac{1}{3}\,\delta_{ab}\delta_{cd}+\frac{1}{2}\,(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})$ | + | + | + |

(41)

These relations can be inverted to give

$$\delta_{ab}\delta_{cd}=3s_0, \tag{42}$$

$$\delta_{ac}\delta_{bd}=s_0+s_1+s_2, \tag{43}$$

$$\delta_{ad}\delta_{bc}=s_0-s_1+s_2. \tag{44}$$

Define the product $s_ks_l$ by $(s_k)_{abmn}(s_l)_{mncd}$. This can be analyzed in terms of the $s_k$ again such that we have an algebra

$$s_ks_l=\sigma_{klm}s_m. \tag{45}$$

It is easy to see that the $\sigma_{klm}$ are given by

$$\sigma_{klm}=\begin{cases}1, & k=l=m,\\ 0, & \text{otherwise},\end{cases} \tag{46}$$

which merely means that the different isospin channels do not mix. This can be illustrated by defining the matrix $S$ with elements $S_{kl}=s_ks_l$,

$$S=\begin{pmatrix}s_0 & & \\ & s_1 & \\ & & s_2\end{pmatrix}, \tag{47}$$

where empty entries are vanishing, and observing the absence of nonzero off-diagonal elements.

Going to an isospin basis

$$|\pm\rangle\triangleq W_\mu^\pm=\frac{1}{\sqrt{2}}(W_\mu^1\mp iW_\mu^2), \tag{48}$$

$$|0\rangle\triangleq Z_\mu=iW_\mu^3, \tag{49}$$

we can use Clebsch-Gordan coefficients to write

$$s_0\triangleq|0,0\rangle\langle 0,0|, \tag{50}$$

$$s_1\triangleq\sum_{k=-1}^{1}|1,k\rangle\langle 1,k|, \tag{51}$$

$$s_2\triangleq\sum_{k=-2}^{2}|2,k\rangle\langle 2,k|, \tag{52}$$

where the first entry means the total isospin and the second its three-component.

The Lorentz tensors in Eqs. (16) and (24) are

$$g_{\alpha\beta}g_{\gamma\delta},\quad g_{\alpha\gamma}g_{\beta\delta},\quad g_{\alpha\delta}g_{\beta\gamma}, \tag{53}$$

$$g_{\alpha\beta}k_\gamma k_\delta,\quad g_{\alpha\gamma}k_\beta k_\delta,\quad g_{\alpha\delta}k_\beta k_\gamma,\quad g_{\gamma\delta}k_\alpha k_\beta,\quad g_{\beta\delta}k_\alpha k_\gamma,\quad g_{\beta\gamma}k_\alpha k_\delta, \tag{54}$$

$$k_\alpha k_\beta k_\gamma k_\delta . \tag{55}$$

Define

$$g^{\rm tr}_{\mu\nu}=g_{\mu\nu}-k_\mu k_\nu/k^2. \tag{56}$$

Then the tensors in Eqs. (53)–(55) can be expressed in terms of the linear combinations

$$t_1=\frac{1}{d-1}g^{\rm tr}_{\alpha\beta}g^{\rm tr}_{\gamma\delta}, \tag{57}$$

$$t_2=\frac{1}{\sqrt{d-1}k^2}g^{\rm tr}_{\alpha\beta}k_\gamma k_\delta, \tag{58}$$

$$t_3=\frac{1}{\sqrt{d-1}k^2}g^{\rm tr}_{\gamma\delta}k_\alpha k_\beta, \tag{59}$$

$$t_4=\frac{1}{k^4}k_\alpha k_\beta k_\gamma k_\delta, \tag{60}$$

$$t_5=-\frac{1}{d-1}g^{\rm tr}_{\alpha\beta}g^{\rm tr}_{\gamma\delta}+\frac{1}{2}(g^{\rm tr}_{\alpha\gamma}g^{\rm tr}_{\beta\delta}+g^{\rm tr}_{\alpha\delta}g^{\rm tr}_{\beta\gamma}), \tag{61}$$

$$t_6=\frac{1}{2}(g^{\rm tr}_{\alpha\gamma}g^{\rm tr}_{\beta\delta}-g^{\rm tr}_{\alpha\delta}g^{\rm tr}_{\beta\gamma}), \tag{62}$$

$$t_7=\frac{1}{2k^2}(g^{\rm tr}_{\alpha\gamma}k_\beta k_\delta+g^{\rm tr}_{\beta\delta}k_\alpha k_\gamma+g^{\rm tr}_{\alpha\delta}k_\beta k_\gamma+g^{\rm tr}_{\beta\gamma}k_\alpha k_\delta), \tag{63}$$

$$t_8=\frac{1}{2k^2}(g^{\rm tr}_{\alpha\gamma}k_\beta k_\delta-g^{\rm tr}_{\beta\delta}k_\alpha k_\gamma-g^{\rm tr}_{\alpha\delta}k_\beta k_\gamma-g^{\rm tr}_{\beta\gamma}k_\alpha k_\delta), \tag{64}$$

$$t_9=\frac{1}{2k^2}(g^{\rm tr}_{\alpha\gamma}k_\beta k_\delta-g^{\rm tr}_{\beta\delta}k_\alpha k_\gamma+g^{\rm tr}_{\alpha\delta}k_\beta k_\gamma-g^{\rm tr}_{\beta\gamma}k_\alpha k_\delta), \tag{65}$$

$$t_{10}=\frac{1}{2k^2}(g^{\rm tr}_{\alpha\gamma}k_\beta k_\delta+g^{\rm tr}_{\beta\delta}k_\alpha k_\gamma-g^{\rm tr}_{\alpha\delta}k_\beta k_\gamma-g^{\rm tr}_{\beta\gamma}k_\alpha k_\delta). \tag{66}$$

Define the product $t_k t_l$ by $(t_k)_{\alpha\beta}{}^{\mu\nu}(t_l)_{\mu\nu\gamma\delta}$ and get the algebra

$$t_k t_l=\tau_{klm}t_m . \tag{67}$$

The only nonzero $\tau_{klm}$ are

$$\begin{aligned}\tau_{1,1,1}&=\tau_{1,2,2}=\tau_{2,3,1}=\tau_{2,4,2}=\tau_{3,1,3}=\tau_{3,2,4}=\tau_{4,3,3}=\tau_{4,4,4}\\&=\tau_{5,5,5}=\tau_{6,6,6}=\tau_{7,7,7}=\tau_{7,8,8}=\tau_{6,9,7}=\tau_{8,10,8}=\tau_{9,7,9}\\&=\tau_{9,8,10}=\tau_{10,9,9}=\tau_{10,10,10}=1.\end{aligned} \tag{68}$$

To make the structure of this algebra more transparent, define the matrix $T$ with elements $T_{kl}=t_k t_l$,

$$T=\begin{pmatrix} t_1 & t_2 & & & & & & & & \\ & & t_1 & t_2 & & & & & & \\ t_3 & t_4 & & & & & & & & \\ & & t_3 & t_4 & & & & & & \\ & & & & t_5 & & & & & \\ & & & & & t_6 & & & & \\ & & & & & & t_7 & t_8 & & \\ & & & & & & & & t_7 & t_8 \\ & & & & & & t_9 & t_{10} & & \\ & & & & & & & & t_9 & t_{10} \end{pmatrix} \tag{69}$$

where all empty entries are vanishing and where the lines have been drawn to guide the eye.

Since the different isospin channels decouple from each other, let us decompose each relevant tensor $X^{abcd}_{\alpha\beta\gamma\delta}$ by writing

$$X^{abcd}_{\alpha\beta\gamma\delta}\equiv X=X_0 s_0+X_1 s_1+X_2 s_2, \tag{70}$$

e.g.,

$$V=V_0 s_0+V_1 s_1+V_2 s_2, \tag{71}$$

$$P=P_0 s_0+P_1 s_1+P_2 s_2, \tag{72}$$

$$B^{(L)}=B_0^{(L)}s_0+B_1^{(L)}s_1+B_2^{(L)}s_2. \tag{73}$$

The $X_I$ can be decomposed into the $t_k$ by writing

$$X_I=X^{(t)}_{Ik}t_k, \tag{74}$$

where $k$ runs from 1 to 10. For example, we have, for the anomalous couplings,

$$\begin{aligned}V_0=2i\{&[(d+3)g_4+(3d-1)g_5]t_1+\sqrt{d-1}(g_4+3g_5)\\&\times(t_2+t_3)+5(g_4+g_5)t_4+2(2g_4+g_5)(t_5+t_7)]\},\end{aligned} \tag{75}$$

$$V_1=-2i(g_4-2g_5)(t_6+t_{10}), \tag{76}$$

$$\begin{aligned}V_2=i[&2(dg_4+2g_5)t_1+2\sqrt{d-1}g_4(t_2+t_3)+4(g_4+g_5)t_4\\&+2(g_4+2g_5)(t_5+t_7)]\end{aligned} \tag{77}$$

and, for the integrated propagator part,

$$P_0 = P_2 = 2(\Delta_1 + d\Delta_2)t_1 + 2\sqrt{d-1}(\Delta_2+\Delta_4)(t_2+t_3) + 2(\Delta_1 + 2\Delta_2 + 2\Delta_3 + 4\Delta_4 + \Delta_5)t_4 + 2(\Delta_1+\Delta_2)t_5 + 2(\Delta_1+\Delta_2+\Delta_3+\Delta_4)t_7, \tag{78}$$

$$P_1 = 2(\Delta_1 - \Delta_2)t_6 + 2(\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4)t_{10}. \tag{79}$$

Now we can write

$$B_I^{(L+1)} = \frac{1}{2}(B_I^{(L)} P_I V_I + V_I P_I B_I^{(L)}). \tag{80}$$

Note that the symmetry (20) prevents $t_8$ and $t_9$ from appearing in the $V_I$ and $P_I$ and that therefore Eq. (69) tells us that they will not be generated at any point in our calculation. Equation (20) also restricts $t_2$ and $t_3$ to appear only in the combination $t_2+t_3$ in $V_I$, $P_I$, and $B_I^{(L)}$, which is not the case for the single terms on the right-hand side of Eq. (80).

Let us introduce a basis of Lorentz tensors which includes only those necessary to describe the $V_I$, $P_I$, and $B_I^{(L)}$:

| | $\alpha\leftrightarrow\beta$ | $\gamma\leftrightarrow\delta$ | $\binom{\alpha}{\beta}\leftrightarrow\binom{\gamma}{\delta}$ |
|---|---|---|---|
| $u_1 \equiv t_1 = \frac{1}{d-1} g^{\mathrm{tr}}_{\alpha\beta} g^{\mathrm{tr}}_{\gamma\delta}$ | + | + | + |
| $u_2 \equiv t_2+t_3 = \frac{1}{\sqrt{d-1}k^2}(g^{\mathrm{tr}}_{\alpha\beta}k_\gamma k_\delta + g^{\mathrm{tr}}_{\gamma\delta}k_\alpha k_\beta)$ | + | + | + |
| $u_3 \equiv t_4 = \frac{1}{k^4}k_\alpha k_\beta k_\gamma k_\delta$ | + | + | + |
| $u_4 \equiv t_5 = -\frac{1}{d-1}g^{\mathrm{tr}}_{\alpha\beta}g^{\mathrm{tr}}_{\gamma\delta} + \frac{1}{2}(g^{\mathrm{tr}}_{\alpha\gamma}g^{\mathrm{tr}}_{\beta\delta} + g^{\mathrm{tr}}_{\alpha\delta}g^{\mathrm{tr}}_{\beta\gamma})$ | + | + | + |
| $u_5 \equiv t_6 = \frac{1}{2}(g^{\mathrm{tr}}_{\alpha\gamma}g^{\mathrm{tr}}_{\beta\delta} - g^{\mathrm{tr}}_{\alpha\delta}g^{\mathrm{tr}}_{\beta\gamma})$ | − | − | + |
| $u_6 \equiv t_7 = \frac{1}{2k^2}(g^{\mathrm{tr}}_{\alpha\gamma}k_\beta k_\delta + g^{\mathrm{tr}}_{\beta\delta}k_\alpha k_\gamma + g^{\mathrm{tr}}_{\alpha\delta}k_\beta k_\gamma + g^{\mathrm{tr}}_{\beta\gamma}k_\alpha k_\delta)$ | + | + | + |
| $u_7 \equiv t_{10} = \frac{1}{2k^2}(g^{\mathrm{tr}}_{\alpha\gamma}k_\beta k_\delta + g^{\mathrm{tr}}_{\beta\delta}k_\alpha k_\gamma - g^{\mathrm{tr}}_{\alpha\delta}k_\beta k_\gamma - g^{\mathrm{tr}}_{\beta\gamma}k_\alpha k_\delta)$ | − | − | + |

(81)

As in Eq. (41), we have indicated the parity under the symmetry transformations (18)–(20).

To give the tensors $u_1 - u_7$ a physical interpretation, let us consider them in the rest system of $k_\mu$, i.e., where

$$\overline{k_\mu} \equiv \frac{k_\mu}{\sqrt{k^2}} = (1,0,0,0) \tag{82}$$

and

$$-g^{\mathrm{tr}}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{83}$$

Then the Lorentz index 0 refers to a spin-0 particle and the other three components to a spin-1 particle. In a general Lorentz frame, $W_\mu$ contains a spin-1 field with $k^\mu W_\mu = 0$ and a spin-0 field $k^\mu W_\mu$. Write the tensors in a bra and ket notation such that

$$\overline{k_\alpha} \triangleq |s\rangle_1, \tag{84}$$

$$\overline{k_\beta} \triangleq |s\rangle_2, \tag{85}$$

$$\overline{k_\gamma} \triangleq {}_1\langle s|, \tag{86}$$

$$\overline{k_\delta} \triangleq {}_2\langle s| \tag{87}$$

(the indices enumerate the particles), where $s$ refers to a ''scalar,'' and $|1)$, $|2)$, $|3)$ are states with definite spin-up components in the $x$, $y$, and $z$ directions, respectively. Going to a spin basis with $|s\rangle$ and

$$|\pm 1\rangle = \frac{1}{\sqrt{2}}[|1) \mp i|2)], \tag{88}$$

$$|0\rangle = i|3), \tag{89}$$

we can use Clebsch-Gordan coefficients to write

$$u_1 \triangleq |0,0\rangle\langle 0,0|, \tag{90}$$

$$u_2 \triangleq -(|0,0\rangle_1\langle s|_2\langle s| + |s\rangle_1|s\rangle_2\langle 0,0|), \tag{91}$$

$$u_3 \triangleq |s\rangle_1|s\rangle_2\ {}_1\langle s|_2\langle s|, \tag{92}$$

$$u_4 \triangleq \sum_{k=-2}^{2} |2,k\rangle\langle 2,k|, \tag{93}$$

$$u_5 \triangleq \sum_{k=-1}^{1} |1,k\rangle\langle 1,k|, \tag{94}$$

$$u_6 \triangleq -\frac{1}{2}\sum_{k=-1}^{1} (|k\rangle_1|s\rangle_2 + |s\rangle_1|k\rangle_2)({}_1\langle k|_2\langle s| + {}_1\langle s|_2\langle k|), \tag{95}$$

$$u_7 \triangleq -\frac{1}{2}\sum_{k=-1}^{1} (|k\rangle_1|s\rangle_2 - |s\rangle_1|k\rangle_2)({}_1\langle k|_2\langle s| - {}_1\langle s|_2\langle k|), \tag{96}$$

where the first entry means the total isospin and the second its three-component, when two entries are present. This allows for the interpretation of $u_1$, $u_5$, and $u_4$ as channels for spin-0, -1, and -2 combinations, respectively, from two spin-1 particles. Then $u_3$ is interpreted as a spin-0 combination of two scalar particles and $u_2$ as a mixing channel between the spin-0 combination of two spin-1 particles and the spin-0 combination of two scalar particles. $u_6$ and $u_7$ are different spin-1 combinations of a scalar and a spin-1 particle.

Now we can write, for the vertex,

$$V_0 = 2i[(d+3)g_4 + (3d-1)g_5]u_1 + \sqrt{d-1}(g_4+3g_5)u_2 + 5(g_4+g_5)u_3 + 2(2g_4+g_5)(u_4+u_6)], \tag{97}$$

$$V_1 = -2i(g_4-2g_5)(u_5+u_7), \tag{98}$$

$$V_2 = i[2(dg_4+2g_5)u_1 + 2\sqrt{d-1}g_4u_2 + 4(g_4+g_5)u_3 + 2(g_4+2g_5)(u_4+u_6)] \tag{99}$$

and, for the integrated propagator part,

$$P_0 = P_2 = 2(\Delta_1 + d\Delta_2)u_1 + 2\sqrt{d-1}(\Delta_2+\Delta_4)u_2 + 2(\Delta_1+2\Delta_2+2\Delta_3+4\Delta_4+\Delta_5)u_3 + 2(\Delta_1+\Delta_2)u_4 + 2(\Delta_1+\Delta_2+\Delta_3+\Delta_4)u_6, \tag{100}$$

$$P_1 = 2(\Delta_1-\Delta_2)u_5 + 2(\Delta_1-\Delta_2+\Delta_3+\Delta_4)u_7. \tag{101}$$

### C. Transfer matrices and their eigenvalues

Writing

$$B_I^{(L)} = B_{Ik}^{(L)} u_k, \tag{102}$$

we can define transfer matrices $M_I$, such that

$$B_{Ik}^{(L+1)} = M_{Ikl} B_{Il}^{(L)}. \tag{103}$$

Writing

$$V_I = V_{Ik} u_k \tag{104}$$

and using that

$$B_I^{(0)} = V_I, \tag{105}$$

we can write

$$B_{Ik}^{(L)} = (M_I^L)_{kl} V_{Il}. \tag{106}$$

This can be simplified further. From Eqs. (69) and (81) and the absence of $t_8$ and $t_9$ follows that $u_4$, $u_5$, $u_6$, and $u_7$ propagate independently. The symmetries (18) and (19) together with the parity properties noted in Eqs. (41) and (81) necessitate that $u_5$ and $u_7$ appear only in the isospin-1 channel, while the other $u_k$ appear only in the isospin-0 and isospin-2 channels, which we summarize in the following table:

| | | isospin | | |
|---|---|---|---|---|
| | | 0 | 1 | 2 |
| | 0 | $s_0u_1, s_0u_2, s_0u_3$ | | $s_2u_1, s_2u_2, s_2u_3$ |
| spin | 1 | $s_0u_6$ | $s_1u_5, s_1u_7$ | $s_2u_6$ |
| | 2 | $s_0u_4$ | | $s_2u_4$ |

(107)

Even though $s_1u_5$ and $s_1u_7$ carry the same spin and isospin assignments, they do not mix. We can write

$$V_0 = \sum_{k=1}^{3} V_{0k}^{(123)} u_k + V_0^{(4)} u_4 + V_0^{(6)} u_6, \tag{108}$$

$$V_1 = V_1^{(5)} u_5 + V_1^{(7)} u_7, \tag{109}$$

$$V_2 = \sum_{k=1}^{3} V_{2k}^{(123)} u_k + V_2^{(4)} u_4 + V_2^{(6)} u_6, \tag{110}$$

and then

$$B_{0k}^{(L)} = \sum_{l=1}^{3} (M_0^{(123)L})_{kl} V_{0l}^{(123)}, \quad k=1,2,3, \tag{111}$$

$$B_{04}^{(L)} = \lambda_0^{(4)L} V_0^{(4)}, \tag{112}$$

$$B_{06}^{(L)} = \lambda_0^{(6)L} V_0^{(6)}, \tag{113}$$

$$B_{15}^{(L)} = \lambda_1^{(5)L} V_1^{(5)}, \tag{114}$$

$$B_{17}^{(L)} = \lambda_1^{(7)L} V_1^{(7)}, \tag{115}$$

$$B_{2k}^{(L)} = \sum_{l=1}^{3} (M_2^{(123)L})_{kl} V_{2l}^{(123)}, \quad k=1,2,3, \tag{116}$$

$$B_{24}^{(L)}=\lambda_2^{(4)L}V_2^{(4)}, \tag{117}$$

$$B_{26}^{(L)}=\lambda_2^{(6)L}V_2^{(6)}, \tag{118}$$

with

$$\lambda_0^{(4)}=8i(2g_4+g_5)(\Delta_1+\Delta_2), \tag{119}$$

$$\lambda_0^{(6)}=8i(2g_4+g_5)(\Delta_1+\Delta_2+\Delta_3+\Delta_4), \tag{120}$$

$$\lambda_1^{(5)}=-4i(g_4-2g_5)(\Delta_1-\Delta_2), \tag{121}$$

$$\lambda_1^{(7)}=-4i(g_4-2g_5)(\Delta_1-\Delta_2+\Delta_3-\Delta_4), \tag{122}$$

$$\lambda_2^{(4)}=4i(g_4+2g_5)(\Delta_1+\Delta_2), \tag{123}$$

$$\lambda_2^{(6)}=4i(g_4+2g_5)(\Delta_1+\Delta_2+\Delta_3+\Delta_4), \tag{124}$$

$$M_0^{(123)}=2i\begin{pmatrix} 2a_0 & 2c_0 & 0\\ b_0 & a_0+d_0 & c_0\\ 0 & 2b_0 & 2d_0\end{pmatrix}, \tag{125}$$

$$a_0=[(d+3)g_4+(3d-1)g_5]\Delta_1+[(d^2+4d-1)g_4 +(3d^2+2d-3)g_5]\Delta_2+(d-1)(g_4+3g_5)\Delta_4, \tag{126}$$

$$b_0=\sqrt{d-1}\{(g_4+3g_5)\Delta_1+[(d+5)g_4+(3d+5)g_5]\Delta_2 +5(g_4+g_5)\Delta_4\}, \tag{127}$$

$$c_0=\sqrt{d-1}\{(g_4+3g_5)\Delta_1+[(d+5)g_4+(3d+5)g_5]\Delta_2 +2(g_4+3g_5)\Delta_3+[(d+7)g_4+(3d+11)g_5]\Delta_4 +(g_4+3g_5)\Delta_5\}, \tag{128}$$

$$d_0=5(g_4+g_5)\Delta_1+[(d+9)g_4+(3d+7)g_5]\Delta_2 +10(g_4+g_5)\Delta_3+[(d+19)g_4+(3d+17)g_5]\Delta_4 +5(g_4+g_5)\Delta_5, \tag{129}$$

$$M_2^{(123)}=2i\begin{pmatrix} 2a_2 & 2c_2 & 0\\ b_2 & a_2+d_2 & c_2\\ 0 & 2b_2 & 2d_2\end{pmatrix}, \tag{130}$$

$$a_2=(dg_4+2g_5)\Delta_1+[(d^2+d-1)g_4+2dg_5]\Delta_2 +(d-1)g_4\Delta_4, \tag{131}$$

$$b_2=\sqrt{d-1}\{g_4\Delta_1+[(d+2)g_4+2g_5]\Delta_2 +2(g_4+g_5)\Delta_4\}, \tag{132}$$

$$c_2=\sqrt{d-1}\{g_4\Delta_1+[(d+2)g_4+2g_5]\Delta_2 +2g_4\Delta_3+[(d+4)g_4+2g_5]\Delta_4+g_4\Delta_5\}, \tag{133}$$

$$d_2=2(g_4+g_5)(\Delta_1+2\Delta_3+\Delta_5)+[(d+3)g_4 +4g_5]\Delta_2+[(d+7)g_4+8g_5]\Delta_4. \tag{134}$$

All other $B_{Ik}^{(L)}$ are zero.

We still have to diagonalize $M_0^{(123)}$ and $M_2^{(123)}$. Before explicitly doing so, let us finish the formal development. Assume $M_0^{(123)}$ to have eigenvectors $v_{00}$, $v_{0+}$, $v_{0-}$ with corresponding eigenvalues $\lambda_{00}$, $\lambda_{0+}$, $\lambda_{0-}$. In the same way, assume $M_2^{(123)}$ to have eigenvectors $v_{20}$, $v_{2+}$, $v_{2-}$ with corresponding eigenvalues $\lambda_{20}$, $\lambda_{2+}$, $\lambda_{2-}$. The $v_{0k}$ and $v_{2k}$ are then linear combinations of the $u_1$, $u_2$, $u_3$.

If we write the $V_I$ as

$$V_0=\sum_{k=0,\pm} c_{0k}v_{0k}+c_0^{(4)}u_4+c_0^{(6)}u_6, \tag{135}$$

$$V_1=c_1^{(5)}u_5+c_1^{(7)}u_7, \tag{136}$$

$$V_2=\sum_{k=0,\pm} c_{2k}v_{2k}+c_2^{(4)}u_4+c_2^{(6)}u_6, \tag{137}$$

we can write

$$B_0^{(L)}=\sum_{k=0,\pm} c_{0k}\lambda_{0k}^L v_{0k}+c_0^{(4)}\lambda_0^{(4)L}u_4+c_0^{(6)}\lambda_0^{(6)L}u_6, \tag{138}$$

$$B_1^{(L)}=c_1^{(5)}\lambda_1^{(5)L}u_5+c_1^{(7)}\lambda_1^{(7)L}u_7, \tag{139}$$

$$B_2^{(L)}=\sum_{k=0,\pm} c_{2k}\lambda_{2k}^L v_{2k}+c_2^{(4)}\lambda_2^{(4)L}u_4+c_2^{(6)}\lambda_2^{(6)L}u_6. \tag{140}$$

The final result is then

$$\Sigma=\Sigma_0 s_0+\Sigma_1 s_1+\Sigma_2 s_2, \tag{141}$$

with

$$\Sigma_0=\sum_{L=0}^{\infty} B_0^{(L)}=\sum_{k=0,\pm}\frac{c_{0k}}{1-\lambda_{0k}}v_{0k}+\frac{c_0^{(4)}}{1-\lambda_0^{(4)}}u_4+\frac{c_0^{(6)}}{1-\lambda_0^{(6)}}u_6, \tag{142}$$

$$\Sigma_1=\sum_{L=0}^{\infty} B_1^{(L)}=\frac{c_1^{(5)}}{1-\lambda_1^{(5)}}u_5+\frac{c_1^{(7)}}{1-\lambda_1^{(7)}}u_7, \tag{143}$$

$$\Sigma_2=\sum_{L=0}^{\infty} B_2^{(L)}=\sum_{k=0,\pm}\frac{c_{2k}}{1-\lambda_{2k}}v_{2k}+\frac{c_2^{(4)}}{1-\lambda_2^{(4)}}u_4+\frac{c_2^{(6)}}{1-\lambda_2^{(6)}}u_6. \tag{144}$$

A resonance arises if for some $k^2$ an eigenvalue becomes unity.

From the decomposition of $V_0$, $V_1$, $V_2$ in terms of the $u_k$, Eqs. (97)–(99), we can read off

$$c_0^{(4)}=c_0^{(6)}=4i(2g_4+g_5), \tag{145}$$

$$c_1^{(5)}=c_1^{(7)}=-2i(g_4-2g_5), \tag{146}$$

$$c_2^{(4)}=c_2^{(6)}=2i(g_4+2g_5). \tag{147}$$

The eigenvectors of a matrix of the form

$$M=\begin{pmatrix} 2a & 2c & 0 \\ b & a+d & c \\ 0 & 2b & 2d \end{pmatrix} \tag{148}$$

are

$$v_0=\begin{pmatrix} -2c \\ a-d \\ 2b \end{pmatrix}, \tag{149}$$

$$v_\pm=\begin{pmatrix} c[a-d\pm\sqrt{(a-d)^2+4bc}] \\ 2bc \\ b[d-a\pm\sqrt{(a-d)^2+4bc}] \end{pmatrix}, \tag{150}$$

with respective eigenvalues

$$\lambda_0=a+d, \tag{151}$$

$$\lambda_\pm=a+d\pm\sqrt{(a-d)^2+4bc}. \tag{152}$$

Therefore, the eigenvalues of $M_0^{(123)}$ are

$$\lambda_{00}=2i(a_0+d_0), \tag{153}$$

$$\lambda_{0\pm}=2i[a_0+d_0 \pm\sqrt{(a_0-d_0)^2+4b_0c_0}]. \tag{154}$$

Keeping only quartically divergent terms, they become

$$\lambda_{00}=2i[(d^2+6d+16)g_4+(3d^2+8d+8)g_5]\Delta^{(4)}+\mathcal{O}(\Lambda^2), \tag{155}$$

$$\lambda_{0+}=4i(d+2)[(d+4)g_4+(3d+2)g_5]\Delta^{(4)}+\mathcal{O}(\Lambda^2), \tag{156}$$

$$\lambda_{0-}=16i(2g_4+g_5)\Delta^{(4)}+\mathcal{O}(\Lambda^2). \tag{157}$$

In the same way, the eigenvalues of $M_2^{(123)}$ are

$$\lambda_{20}=2i(a_2+d_2), \tag{158}$$

$$\lambda_{2\pm}=2i[a_2+d_2 \pm\sqrt{(a_2-d_2)^2+4b_2c_2}]. \tag{159}$$

Keeping again only quartically divergent terms, they become

$$\lambda_{20}=2i[(d^2+3d+4)g_4+2(d+4)g_5]\Delta^{(4)}+\mathcal{O}(\Lambda^2), \tag{160}$$

$$\lambda_{2+}=4i(d+2)[(d+1)g_4+2g_5]\Delta^{(4)}+\mathcal{O}(\Lambda^2), \tag{161}$$

$$\lambda_{2-}=8i(g_4+2g_5)\Delta^{(4)}+\mathcal{O}(\Lambda^2). \tag{162}$$

Now let us consider the rest of the eigenvalues $\lambda_I^{(k)}$. Keeping only quartically divergent contributions, we get

$$\lambda_0^{(4)}=16i(g_4+g_5)\Delta^{(4)}+\mathcal{O}(\Lambda^2), \tag{163}$$

$$\lambda_0^{(6)}=16i(g_4+g_5)\Delta^{(4)}+\mathcal{O}(\Lambda^2), \tag{164}$$

$$\lambda_2^{(4)}=8i(g_4+2g_5)\Delta^{(4)}+\mathcal{O}(\Lambda^2), \tag{165}$$

$$\lambda_2^{(6)}=8i(g_4+2g_5)\Delta^{(4)}+\mathcal{O}(\Lambda^2). \tag{166}$$

In $\lambda_1^{(5)}$ and $\lambda_1^{(7)}$ the quartically divergent contributions cancel. Combining Eq. (121) with Eq. (37) and Eq. (122) with Eq. (38) and keeping also quadratically divergent contributions and the leading imaginary part of $\lambda_1^{(7)}$ gives

$$\lambda_1^{(5)}=-(g_4-2g_5)\frac{\Lambda_V^2}{2(4\pi)^2m_{W0}^2}\frac{\ln(\Lambda_W^2/\Lambda_V^2)}{1-\Lambda_V^2/\Lambda_W^2}+\mathcal{O}(\Lambda^0), \tag{167}$$

$$\lambda_1^{(7)}=(g_4-2g_5)\frac{\Lambda_V^2}{8(4\pi)^2m_{W0}^4}\left(k^2-4m_{W0}^2\frac{\ln(\Lambda_W^2/\Lambda_V^2)}{1-\Lambda_V^2/\Lambda_W^2}+\frac{i\pi(k^2+4m_{W0}^2)(k^2-4m_{W0}^2)^{3/2}}{3\sqrt{k^2}\Lambda_V^2}\right)+\mathcal{O}(0,-2), \tag{168}$$

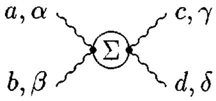
where we have used again the shorthand notation (39). Notice that the only eigenvalues that are not quartically divergent are $\lambda_1^{(5)}$ and $\lambda_1^{(7)}$. In fact, with finite $\Lambda_V$ and $\Lambda_W$ the corresponding integrals are convergent and dimensional regularization is used merely for convenience.

### D. Resonance

Combining these results with Eqs. (141)–(144) as well as Eq. (146) gives

$$\Sigma=\frac{-2i(g_4-2g_5)s_1u_5}{1+(g_4-2g_5)\dfrac{\Lambda_V^2}{2(4\pi)^2m_{W0}^2}\dfrac{\ln(\Lambda_W^2/\Lambda_V^2)}{1-\Lambda_V^2/\Lambda_W^2}+\mathcal{O}(\Lambda^0)} + \frac{-2i(g_4-2g_5)s_1u_7}{1-(g_4-2g_5)\dfrac{\Lambda_V^2}{8(4\pi)^2m_{W0}^4}\left(k^2-4m_{W0}^2\dfrac{\ln(\Lambda_W^2/\Lambda_V^2)}{1-\Lambda_V^2/\Lambda_W^2}+\dfrac{i\pi(k^2+4m_{W0}^2)(k^2-4m_{W0}^2)^{3/2}}{3\sqrt{k^2}\Lambda_V^2}\right)+\mathcal{O}(0,-2)}+\mathcal{O}(\Lambda^{-4}). \tag{169}$$

Under the assumption that the anomalous couplings dominate the gauge coupling and the additional assumption that $|g_4 - 2g_5| \gg 8(4\pi)^2 m_{W0}^4/(\Lambda_V^2 m_r^2)$ [$m_r$ is the resonance mass: see Eq. (173) below], the terms in Eq. (169) are in leading order independent of $g_4$ and $g_5$. Modeling the second term in Eq. (169) with a Breit-Wigner shape (and consequently neglecting the $k^2$ dependence of the width term) gives




$$= -iC_1 \left(\frac{\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}}{2}\right)\left(\frac{g^{tr}_{\alpha\gamma}g^{tr}_{\beta\delta} - g^{tr}_{\alpha\delta}g^{tr}_{\beta\gamma}}{2}\right)$$

$$+\frac{iC_2}{k^2 - m_r^2 + im_r\Gamma_r}\left(\frac{\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}}{2}\right)\left(\frac{g^{tr}_{\alpha\gamma}k_\beta k_\delta + g^{tr}_{\beta\delta}k_\alpha k_\gamma - g^{tr}_{\alpha\delta}k_\beta k_\gamma - g^{tr}_{\beta\gamma}k_\alpha k_\delta}{2k^2}\right) + \mathcal{O}(\Lambda^{-4}) \tag{170}$$

with

$$C_1 = \frac{4m_{W0}^2}{\Lambda_V^2}\left(\frac{\ln(\Lambda_W^2/\Lambda_V^2)}{1-\Lambda_V^2/\Lambda_W^2}\right)^{-1}, \tag{171}$$

$$C_2 = \frac{16(4\pi)^2 m_{W0}^4}{\Lambda_V^2}, \tag{172}$$

$$m_r = 2m_{W0}\sqrt{\frac{\ln(\Lambda_W^2/\Lambda_V^2)}{1-\Lambda_V^2/\Lambda_W^2}} + \mathcal{O}(\Lambda^{-2}), \tag{173}$$

and

$$\Gamma_r = \frac{\pi(m_r^2 + 4m_{W0}^2)(m_r^2 - 4m_{W0}^2)^{3/2}}{3m_r^2\Lambda_V^2} + \mathcal{O}(\Lambda^{-4}). \tag{174}$$

Note that we can trivially drop the superscript ''tr'' in the $C_2$ term in Eq. (170).

The direct calculation of the resonant part of the bubble sum (17) gives the mass (173) and the width (174) for the resonance. As a consistency check for the width, we have computed the decay rate of the resonance independently. For $k^2 \approx m_r^2$, the resonant part of the bubble sum (170) can be written as

$$\lambda\epsilon_{abm}[k_\alpha g_{\beta\mu} - k_\beta g_{\alpha\mu}](-i)\delta_{mn} \times \left(\frac{g^{\mu\nu}}{k^2 - m_r^2 + im_r\Gamma_r} + Xk^\mu k^\nu\right)\lambda\epsilon_{cdn} \times [(-k)_\gamma g_{\delta\nu} - (-k)_\delta g_{\gamma\nu}], \tag{175}$$

with

$$\lambda = \frac{2(4\pi)m_{W0}^2}{\Lambda_V m_r} \tag{176}$$

and arbitrary $X$. That is, the bubble sum can be decomposed into a spin-1 propagator part

$$-i\delta_{mn}\left(\frac{g_{\mu\nu}}{k^2 - m_r^2 + im_r\Gamma_r} + Xk_\mu k_\nu\right) \tag{177}$$

and two derivative couplings

$$\lambda\epsilon_{abc}(k_\alpha g_{\beta\gamma} - k_\beta g_{\alpha\gamma}). \tag{178}$$

This coupling can be used to compute the decay rate into two $W$ bosons. The result of this standard calculation coincides with Eq. (174).

## IV. DISCUSSION

The first thing one notices about the contributions to the bubble sum is that in most channels quartic divergences are present. In these cases we did not calculate the subleading terms. In case quartic divergences are present the subleading terms could be of the form $\Lambda^2 k^2$. If such terms are absent, the whole interaction is suppressed by $1/\Lambda^4$ and the channel is of no phenomenological interest. If such terms are present, resonances are present at a scale $\mathcal{O}(\Lambda^2)$. These are out of the reach of present colliders and possibly also of the LHC. Much more interesting is the $I=1$, $J=1$ channel where we find a low-lying resonance in the term $u_7$, which corresponds to longitudinal vector boson scattering. Depending on the ratio $\Lambda_W/\Lambda_V$, one finds the resonance below or above the vector boson threshold. On physical grounds we expect $\Lambda_V$ to be the smaller, as it is directly related to the Goldstone boson sector of the theory, where the strong interactions are supposed to take place. In that case the resonance always lies above the two-$W$ threshold. Because the interactions of the transversal vector bosons are suppressed by the gauge coupling, a reasonable assumption would be $\Lambda_V \approx g\Lambda_W$. This corresponds to $m_r \approx 200$ GeV. A recent comparison with the LEP-100 data gives a limit $\Lambda_V > 490$ GeV [7]. For $m_r = 200$ GeV, this gives $\Gamma_r < 12$ GeV. The fact that the resonance can be at such low energy is somewhat surprising, given our experience with chiral perturbation theory in pion physics.

To study the connection with pion physics we make the substitutions $g_4 = g^4\epsilon_4$, $g_5 = g^4\epsilon_5$, and $m_{W0} = gf_\pi/2$. In the resulting Lagrangian one takes $g \to 0$, with $\epsilon_4$, $\epsilon_5$, and $f_\pi$ fixed. This way one finds the standard nonlinear sigma model with two higher-derivative terms. We define $g_\rho = 4(\epsilon_4 - 2\epsilon_5)$. For didactical purposes we keep in the chiral perturbation theory here the tree-level terms, the imaginary part of the ordinary chiral loop, and the contribution of the first loop we calculated. One finds, for the $I=1$, $J=1$ amplitude,

$$a_{11}(s)=\frac{s}{96\pi f_\pi^2}\left[1+\frac{g_\rho s}{f_\pi^2}+\frac{g_\rho^2\Lambda_V^2}{32\pi^2 f_\pi^2}\frac{s^2}{f_\pi^4}+\frac{is}{96\pi f_\pi^2}\times\left(1+\frac{g_\rho^2 s^2}{f_\pi^4}\right)\right]. \tag{179}$$

After unitarizing the amplitude with the [1,1] Padé approximant one finds the $\rho$ resonance with a width given by

$$\Gamma_\rho=\frac{m_\rho^3}{96\pi f_\pi^2}\left(1-\frac{\Lambda_V^2}{32\pi^2 f_\pi^2}\right). \tag{180}$$

We had to ignore the $s^4$ term in the imaginary part of $a_{11}$ because it is of too high an order in chiral perturbation theory and other terms of this order have been ignored. The presence of the extra loops that we calculated results therefore in a correction to the Kawarabayashi-Suzuki-Riàzuddin-Fayyazuddin (KSRF) [9] relation $\Lambda_\rho=m_\rho^3/(96\pi f_\pi^2)$. Of course, this is not the full story for chiral perturbation theory, but it shows that the effects we calculated are only a small correction to pion physics. The reason we get a large effect in our calculation is that we assume that the interactions are dominated by the $g_\rho$ interaction. In the chiral limit this is not possible, because at low enough energy the lowest-order term in *s* always dominates. Since within the standard model there is the *W* threshold to consider, it makes sense to say that the anomalous term dominates. It means $|g_\rho|s^2/v^2 \gg s$ for $s>4m_{W0}^2$. Within the standard model this translates into $4|g_4-2g_5|\gg g^2$. This condition therefore quantifies what we mean by strong anomalous couplings.

The model we discussed fits in naturally in the class of models that give rise to resonances in *W* physics due to strong interactions. This class of models is collectively known as the BESS model, from breaking electroweak symmetry strongly. A recent review of this class of models is [10]. Nonetheless, as the discussion above points out, the model is subtly different from the models in the literature, due to the dominance of the anomalous couplings. This would make the model into a mere curiosity were it not for a very simple class of models giving rise to precisely the required structure. This is the class of the strongly interacting singlet Higgs (SISH) models [3]. These models, which contain beyond the standard model only extra scalar singlet particles, are actually the simplest possible renormalizable extensions of the standard model. Because the singlets couple only to the Higgs boson, they do not change the phenomenology at LEP-100 at the one-loop level. Two-loop effects are too small to be significant. In order to perform a precise phenomenology of the model, $SU_R(2)$ breaking effects should be taken into account. The model satisfies the LEP-100 limits on extra *Z* bosons, because both the coupling to leptons and the mixing with the *Z* boson are suppressed by an electroweak loop. The statistics at the Tevatron is too small to see the resonance. For the planned high-energy colliders the phenomenology should be straightforward, the resonances being produced via vector boson fusion. For LEP-200 the situation is somewhat subtle, as the resonance does not couple to the incoming electrons directly. Strong form factor effects could play a role. The precise phenomenology will be left to future work.

Given the fact that the couplings are strong, one can question how generic the results are. In principle, higher-order terms in the chiral perturbation theory could be important. For the formation of the resonance via a bubble sum, only four-point vertices contribute. The presence of higher-order terms will therefore not effect the structure of the calculation very much. Basically, we expect a form factor for $g_4-2g_5$ which, when inserted into the graphs, makes the explicit dependence on $\Lambda_V$ more complicated. However, the term $(g_4-2g_5)\Lambda_V^2$ in formula (169) is essentially determined by power counting; so one would expect corrections of the form $\Lambda^2\to\Lambda^2+\mathcal{O}(4m_W^2)$. Also, the precise formula for the resonance mass as a function of $\Lambda_W$, $\Lambda_V$ could become more complicated. However, that will not change the qualitative picture of $m_r\approx 200$ GeV, with a coupling of order $m_r/\Lambda_V$.

Finally, one can ask if the model is consistent with the LEP precision data. The limits derived on $g_4$, $g_5$ in [6,7] follow from simple perturbation theory and depend crucially on the behavior in the hypercharge sector of the theory, as they come from the limits on the $\rho$ parameter. For the strongly interacting case as described here they are not expected to be a good estimate and could possibly even be an order of magnitude wrong. The model is similar in appearance to the standard model in the large Higgs boson mass limit, which is disfavored by the LEP1 data. When one simply removes the Higgs boson from the standard model, the model becomes nonrenormalizable, but the radiative effects grow only logarithmically with the cutoff. The question is whether this scenario is ruled out by the LEP1 precision data. The LEP1 data appear to be in agreement with the standard model, with a preferred low Higgs boson mass. One is sensitive to the Higgs boson mass in three parameters, known as *S, T, U* or $\epsilon_1$, $\epsilon_2$, $\epsilon_3$. They receive corrections of the form $g^2[\ln(m_H/m_W)+\text{const}]$, where the constants are of order 1. The logarithmic enhancement is universal and would also appear in models without a Higgs boson as $\ln(\Lambda)$, where $\Lambda$ is the cutoff where new interactions should appear. Only when one can determine the three different constants can one say one has established the standard model. At present, the data do not suffice to do this to sufficiently high precision. In practice, one can compensate a change in the Higgs boson mass in the formulas with extra contributions to the *S, T,* and *U* parameters. As such terms are generated by the contributions of formula (8), there is enough freedom to fit the data; see [7] for a discussion. Whether a model with a low-lying resonance would actually improve the fit to the data depends on the couplings to the fermions and to the hypercharge.

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## APPENDIX: ONE-LOOP RESULTS

Here we list results for the necessary basic one-loop integrals and for the composite one-loop quantities $\Delta_1 - \Delta_5$.

Let $d$ be the spacetime dimension, $d = 4 - 2\epsilon$, let $\overline{\mu}$ be given by $\ln 4\pi\mu^2 - \gamma_E = \ln \overline{\mu}^2$ with the Euler-Mascheroni constant $\gamma_E$, and use the abbreviation $\int_p$ given in Eq. (23).

The only integrals we need are

$$I(m^2) \equiv \int_p \frac{1}{p^2 - m^2} = \frac{i m^2}{(4\pi)^2}\left(\frac{1}{\epsilon} + 1 - \ln\frac{m^2}{\overline{\mu}^2}\right) + \mathcal{O}(\epsilon) \tag{A1}$$

and

$$I(k^2; m_a^2, m_b^2) \equiv \int_p \frac{1}{[(p+k)^2 - m_a^2 + i\varepsilon](p^2 - m_b^2 + i\varepsilon)}. \tag{A2}$$

Define

$$D \equiv k^4 + m_a^4 + m_b^4 - 2k^2 m_a^2 - 2k^2 m_b^2 - 2 m_a^2 m_b^2. \tag{A3}$$

For $k^2 \geqslant 0$ we get the following results in a straightforward calculation.

### 1. $I(k^2; m_a^2, m_b^2)$ for $D \leqslant 0$

$D \leqslant 0$ is equivalent to the statement that none of $k^2$, $m_a^2$, $m_b^2$ is larger than the sum of the other two:

$$I(k^2; m_a^2, m_b^2) = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon} + 2 - \frac{k^2 + m_a^2 - m_b^2}{2k^2}\ln\frac{m_a^2}{\overline{\mu}^2} - \frac{k^2 + m_b^2 - m_a^2}{2k^2}\ln\frac{m_b^2}{\overline{\mu}^2} - \frac{\sqrt{-D}}{k^2}\left(\arctan\frac{k^2 + m_a^2 - m_b^2}{\sqrt{-D}} + \arctan\frac{k^2 + m_b^2 - m_a^2}{\sqrt{-D}}\right)\right] + \mathcal{O}(\epsilon). \tag{A4}$$

### 2. $I(k^2; m_a^2, m_b^2)$ for $D \geqslant 0$ with $k^2 \leqslant |m_a^2 - m_b^2|$

Here

$$I(k^2; m_a^2, m_b^2) = \frac{i}{(4\pi)^2}\left\{\frac{1}{\epsilon} + 2 + \frac{m_b^2 - m_a^2 - k^2}{2k^2}\ln\frac{m_a^2}{\overline{\mu}^2} + \frac{m_a^2 - m_b^2 - k^2}{2k^2}\ln\frac{m_b^2}{\overline{\mu}^2} + \frac{\sqrt{D}}{k^2}\ln\frac{m_a^2 + m_b^2 - k^2 + \sqrt{D}}{m_a^2 + m_b^2 - k^2 - \sqrt{D}}\right\} + \mathcal{O}(\epsilon). \tag{A5}$$

Equation (A5) can also be written as

$$I(k^2; m_a^2, m_b^2) = \frac{i}{(4\pi)^2}\left\{\frac{1}{\epsilon} + 2 + \frac{m_b^2 - m_a^2 - k^2 + \sqrt{D}}{2k^2}\ln\frac{m_a^2}{\overline{\mu}^2} + \frac{m_a^2 - m_b^2 - k^2 + \sqrt{D}}{2k^2}\ln\frac{m_b^2}{\overline{\mu}^2} + \frac{\sqrt{D}}{k^2}\ln\frac{m_a^2 + m_b^2 - k^2 + \sqrt{D}}{2\overline{\mu}^2}\right\} + \mathcal{O}(\epsilon) \tag{A6}$$

or

$$I(k^2; m_a^2, m_b^2) = \frac{i}{(4\pi)^2}\left\{\frac{1}{\epsilon} + 2 + \frac{m_b^2 - m_a^2 - k^2 - \sqrt{D}}{2k^2}\ln\frac{m_a^2}{\overline{\mu}^2} + \frac{m_a^2 - m_b^2 - k^2 - \sqrt{D}}{2k^2}\ln\frac{m_b^2}{\overline{\mu}^2} - \frac{\sqrt{D}}{k^2}\ln\frac{m_a^2 + m_b^2 - k^2 - \sqrt{D}}{2\overline{\mu}^2}\right\} + \mathcal{O}(\epsilon). \tag{A7}$$

### 3. $I(k^2; m_a^2, m_b^2)$ for $D \geqslant 0$ with $k^2 \geqslant |m_a^2 - m_b^2|$

Here

$$I(k^2; m_a^2, m_b^2) = \frac{i}{(4\pi)^2}\left\{\frac{1}{\epsilon} + 2 + \frac{m_b^2 - m_a^2 - k^2}{2k^2}\ln\frac{m_a^2}{\overline{\mu}^2} + \frac{m_a^2 - m_b^2 - k^2}{2k^2}\ln\frac{m_b^2}{\overline{\mu}^2} + \frac{\sqrt{D}}{k^2}\ln\frac{k^2 - m_a^2 - m_b^2 - \sqrt{D}}{k^2 - m_a^2 - m_b^2 + \sqrt{D}}\right\} - \frac{\sqrt{D}}{16\pi k^2} + \mathcal{O}(\epsilon). \tag{A8}$$

### 4. $I(k^2; m^2, m^2)$

If $D = k^2(k^2 - 4m^2) \leqslant 0$, i.e., for $k^2 \leqslant 4m^2$, we get, from Eq. (A4),

$$I(k^2; m^2, m^2) = \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon} + 2 - \ln\frac{m^2}{\overline{\mu}^2} - 2\sqrt{\frac{4m^2}{k^2} - 1}\,\arctan\frac{1}{\sqrt{\frac{4m^2}{k^2} - 1}}\right) + \mathcal{O}(\epsilon). \tag{A9}$$

For $k^2 \ll m^2$, we can expand in powers of $k^2/m^2$ to get

$$I(k^2;m^2,m^2) = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon} - \ln\frac{m^2}{\overline{\mu}^2} + \frac{1}{6}\left(\frac{k^2}{m^2}\right) + \frac{1}{60}\left(\frac{k^2}{m^2}\right)^2\right] + \mathcal{O}(\epsilon,(k^2/m^2)^3). \tag{A10}$$

If $D = k^2(k^2 - 4m^2) \geqslant 0$, i.e., for $k^2 \geqslant 4m^2$, Eq. (A9) becomes

$$I(k^2;m^2,m^2) = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon} + 2 - \ln\frac{m^2}{\overline{\mu}^2} + \sqrt{1-\frac{4m^2}{k^2}}\ln\frac{1-\frac{2m^2}{k^2} - \sqrt{1-\frac{4m^2}{k^2}}}{1-\frac{2m^2}{k^2} + \sqrt{1-\frac{4m^2}{k^2}}}\right] - \frac{\sqrt{1-\frac{4m^2}{k^2}}}{16\pi} + \mathcal{O}(\epsilon). \tag{A11}$$

### 5. $I(k^2;m^2,0)$ for $k^2 \leqslant m^2$

Now $D = |k^2 - m^2| \geqslant 0$. Let us assume $k^2 \leqslant m^2$. Then we get, from Eq. (A6) or (A7),

$$\begin{aligned} I(k^2;0,m^2) = I(k^2;m^2,0) &= \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon} + 2 + \frac{m^2-k^2}{k^2}\ln\frac{m^2-k^2}{\overline{\mu}^2} - \frac{m^2}{k^2}\ln\frac{m^2}{\overline{\mu}^2}\right] + \mathcal{O}(\epsilon) \\ &= \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon} + 2 + \frac{m^2}{k^2}\ln\left(1-\frac{k^2}{m^2}\right) - \ln\frac{m^2-k^2}{\overline{\mu}^2}\right] + \mathcal{O}(\epsilon). \end{aligned} \tag{A12}$$

For $k^2 \ll m^2$, we can expand in powers of $k^2/m^2$ to get

$$\begin{aligned} I(k^2;0,m^2) &= I(k^2;m^2,0) \\ &= \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon} + 1 - \ln\frac{m^2}{\overline{\mu}^2} + \frac{1}{2}\left(\frac{k^2}{m^2}\right) + \frac{1}{6}\left(\frac{k^2}{m^2}\right)^2 + \frac{1}{12}\left(\frac{k^2}{m^2}\right)^3 + \frac{1}{20}\left(\frac{k^2}{m^2}\right)^4\right] \\ &\quad + \mathcal{O}(\epsilon,(k^2/m^2)^5). \end{aligned} \tag{A13}$$

### 6. $I(k^2;m_a^2,m_b^2)$ for $k^2,m_a^2 \ll m_b^2$

If $k^2, m_a^2 \ll m_b^2$, we can expand Eq. (A6) or (A7) in negative powers of $m_b^2$ to get

$$\begin{aligned} I(k^2;m_a^2,m_b^2) = \frac{i}{(4\pi)^2}\Bigg[&\frac{1}{\epsilon} + 1 - \ln\frac{m_b^2}{\overline{\mu}^2} + \frac{\frac{1}{2}k^2 + m_a^2\ln\frac{m_a^2}{m_b^2}}{m_b^2} + \frac{k^2\left(\frac{1}{6}k^2 + \frac{3}{2}m_a^2\right) + m_a^2(k^2+m_a^2)\ln\frac{m_a^2}{m_b^2}}{m_b^4} \\ &+ \frac{k^2\left(\frac{1}{12}k^4 + \frac{7}{3}k^2m_a^2 + \frac{5}{2}m_a^4\right) + m_a^2(k^4 + 3k^2m_a^2 + m_a^4)\ln\frac{m_a^2}{m_b^2}}{m_b^6} \\ &+ \frac{k^2\left(\frac{1}{20}k^6 + \frac{35}{12}k^4m_a^2 + \frac{17}{2}k^2m_a^4 + \frac{7}{2}m_a^6\right) + m_a^2(k^6 + 6k^4m_a^2 + 6k^2m_a^4 + m_a^6)\ln\frac{m_a^2}{m_b^2}}{m_b^8}\Bigg] + \mathcal{O}(\epsilon, m_b^{-10}\ln m_b^2). \end{aligned} \tag{A14}$$

### 7. $I(k^2;m_a^2,m_b^2)$ for $k^2 \ll m_a^2,m_b^2$

Here

$$I(k^2;m_a^2,m_b^2)=\frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon}+1-\frac{m_a^2\ln\frac{m_a^2}{\overline{\mu}^2}-m_b^2\ln\frac{m_b^2}{\overline{\mu}^2}}{m_a^2-m_b^2}+\left(\frac{m_a^2+m_b^2}{2(m_a^2-m_b^2)^2}-\frac{m_a^2m_b^2}{(m_a^2-m_b^2)^3}\ln\frac{m_a^2}{m_b^2}\right)k^2\right.$$
$$+\left(\frac{m_a^4+10m_a^2m_b^2+m_b^4}{6(m_a^2-m_b^2)^4}-\frac{m_a^2m_b^2(m_a^2+m_b^2)}{(m_a^2-m_b^2)^5}\ln\frac{m_a^2}{m_b^2}\right)k^4+\left(\frac{m_a^6+29m_a^4m_b^2+29m_a^2m_b^4+m_b^6}{12(m_a^2-m_b^2)^6}\right.$$
$$\left.\left.-\frac{m_a^2m_b^2(m_a^4+3m_a^2m_b^2+m_b^4)}{(m_a^2-m_b^2)^7}\ln\frac{m_a^2}{m_b^2}\right)k^6\right]+\mathcal{O}(k^8,\epsilon). \tag{A15}$$

**8. Calculation of $\Delta_1$, $\Delta_2$, $\Delta_3$, $\Delta_4$, $\Delta_5$**

To compute the quantities $\Delta_1-\Delta_5$ defined in Eq. (24), we first need some preliminaries. Let $f$ in the following be some function of $p^2$ and $(p+k)^2$.

Contracting

$$\int_p p_\alpha f=A\frac{k_\alpha}{k^2} \tag{A16}$$

with $k_\alpha$ gives

$$A=\int_p (p\cdot k)f. \tag{A17}$$

Contracting

$$\int_p p_\alpha p_\beta f=A' g_{\alpha\beta}+B'\frac{k_\alpha k_\beta}{k^2} \tag{A18}$$

with $g_{\alpha\beta}$ and $k_\alpha k_\beta$ and solving the resulting equations for $A'$ and $B'$, we get

$$A'=\frac{1}{d-1}\int_p\left(p^2-\frac{(p\cdot k)^2}{k^2}\right)f, \tag{A19}$$

$$B'=\frac{1}{d-1}\int_p\left(-p^2+d\frac{(p\cdot k)^2}{k^2}\right)f. \tag{A20}$$

Contracting

$$\int_p p_\alpha p_\beta p_\gamma f=A''\frac{k_\alpha g_{\beta\gamma}+k_\beta g_{\gamma\alpha}+k_\gamma g_{\alpha\beta}}{k^2}+B''\frac{k_\alpha k_\beta k_\gamma}{k^4} \tag{A21}$$

with $k_\alpha g_{\beta\gamma}$ and $k_\alpha k_\beta k_\gamma$ and solving the resulting equations for $A''$ and $B''$, we get

$$A''=\frac{1}{d-1}\int_p\left(p^2(p\cdot k)-\frac{(p\cdot k)^3}{k^2}\right)f, \tag{A22}$$

$$B''=\frac{1}{d-1}\int_p\left(-3p^2(p\cdot k)+(d+2)\frac{(p\cdot k)^3}{k^2}\right)f. \tag{A23}$$

Contracting

$$\int_p p_\alpha p_\beta p_\gamma p_\delta f=A'''(g_{\alpha\beta}g_{\gamma\delta}+g_{\alpha\gamma}g_{\beta\delta}+g_{\alpha\delta}g_{\beta\gamma})$$
$$+B'''\frac{g_{\alpha\beta}k_\gamma k_\delta+g_{\alpha\gamma}k_\beta k_\delta+g_{\alpha\delta}k_\beta k_\gamma+g_{\gamma\delta}k_\alpha k_\beta+g_{\beta\delta}k_\alpha k_\gamma+g_{\beta\gamma}k_\alpha k_\delta}{k^2}+C'''\frac{k_\alpha k_\beta k_\gamma k_\delta}{k^4} \tag{A24}$$

with $g_{\alpha\beta}g_{\gamma\delta}$, $k_\alpha k_\beta g_{\gamma\delta}$ and $k_\alpha k_\beta k_\gamma k_\delta$ and solving the resulting equations for $A'''$, $B'''$, and $C'''$, we get

$$A'''=\frac{1}{(d-1)(d+1)}\int_p\left(p^4-2\frac{p^2(p\cdot k)^2}{k^2}+\frac{(p\cdot k)^4}{k^4}\right)f, \tag{A25}$$

$$B'''=\frac{1}{(d-1)(d+1)}\int_p\left(-p^4+(d+3)\frac{p^2(p\cdot k)^2}{k^2}-(d+2)\frac{(p\cdot k)^4}{k^4}\right)f, \tag{A26}$$

$$C''' = \frac{1}{(d-1)(d+1)} \int_p \left( 3p^4 - 6(d+2)\frac{p^2(p\cdot k)^2}{k^2} + (d+2)(d+4)\frac{(p\cdot k)^4}{k^4} \right) f. \qquad \text{(A27)}$$

Using the above results, some trivial but lengthy algebra on the integrand, the properties of dimensional regularization, the results for one-loop integrals in the preceding parts of the Appendix, as well as

$$\int_p (p\cdot k)^{2n} f(p^2) = c_n (k^2)^n \int_p (p^2)^n f(p^2), \qquad \text{(A28)}$$

with

$$c_0 = 1, \quad c_n = \frac{2n-1}{d+2n-2} c_{n-1}, \qquad \text{(A29)}$$

i.e.,

$$c_n = \frac{\Gamma(n+1/2)\Gamma(d/2)}{\Gamma(n+d/2)\Gamma(1/2)}, \qquad \text{(A30)}$$

we can compute $\Delta_1 - \Delta_5$. The results are given in the main text in Eqs. (26)–(35).


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