Representations of fermionic correlators at finite temperatures

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The symmetry group of the staggered fermion transfer matrix in a spatial direction is constructed at finite temperature. Hadron-like operators carrying irreducible representations of this group are written down from the breaking of the zero temperature group. Analysis of the correlators in free field theory suggests new measurements which can test current interpretations. [S0556-2821(99)05519-8]

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I. INTRODUCTION

Lattice simulations of field theories in equilibrium at finite temperature (T) use a discretization of the Euclidean formulation for partition functions:

$$Z(\beta)=\int \mathcal{D}\phi \exp\left[-\int_0^\beta dt \int d^3x \mathcal{L}(\phi)\right], \tag{1}$$

where ϕ is a generic field, $\mathcal{L}$ the Lagrangian density, and the Euclidean ''time'' runs from 0 to $\beta=1/T$. The path integral is over bosonic (fermionic) field configurations which are periodic (anti-periodic) in Euclidean time. Because of the lack of symmetry between the space and Euclidean time directions in Eq. (1), this problem has only a subgroup of the full 4-dimensional rotational symmetry of the $T=0$ Euclidean theory. In this paper we focus on the lattice discretized problem, where all continuum symmetries break to a discrete subgroup.

It is possible to write the partition function of Eq. (1) as the trace of the transfer matrix in one of the spatial directions. The symmetry groups we examine leave such a transfer matrix invariant. The eigenvectors of the matrix carry irreducible representations (irreps) of these symmetry groups.

Thermodynamics depends only on the leading eigenvalue, which always belongs to a scalar representation of the symmetry group. Hence the group theory is not crucial for the study of properties such as the phase structure, transition temperature, T_c, and other thermodynamic quantities. In fact, extensive measurements have been made of T_c for pure gauge theories, and those with massless fermions [1], and our group theoretical analysis adds very little to this.

However, the symmetry properties are crucial to the study of screening correlation functions and the determination of screening masses. These can be written in terms of the ratio of the largest and an appropriate other eigenvalue of the transfer matrix. The significance of the equality (or otherwise) of two screening masses will depend on whether or not the correlation function lies in the same irrep of the symmetry group of the transfer matrix. In the gauge sector of the theory this analysis has been carried out and applied to the study of screening masses [2], to demonstrate dimensional reduction in a fully non-perturbative manner.

Screening masses obtained from correlation functions built out of staggered fermion field operators have also been extensively studied in the past [3–5]. Screening masses in the high temperature phase of QCD seem to approach those expected from free field theory as $T\to 2T_c$ [4–6]. Some other measurements which seem to indicate that the picture may be more complicated [6], also turn out to be explained in terms of weakly interacting quarks [7]. All these studies have relied entirely on the $T=0$ analysis of the lattice symmetry group of staggered fermions.

In this paper we present the first analysis of the symmetries of the corresponding finite temperature problem. We find that all the screening masses measured until now see only one of the representations of the symmetry group. Many other masses can be studied, and are likely to yield further information about the theory. The free field theory of these other representations is worked out.

One observation arising from the application of these group theoretical results to previous simulations is worth mentioning in the introduction. Since the $T=0$ scalar and pseudo-scalar mesons, and the symmetric linear combinations of the three components of the vector and pseudovector, lie in the same irreducible representation (irrep) of the point group of a $T>0$ spatial slice through the lattice, they must have degenerate masses in a free fermion theory. When interactions switch on, the relevant symmetry becomes that of an enveloping group, and the four degenerate masses split into two pairs of degenerate masses. Observation of such a splitting for $T\leqslant 2T_c$ [3–6] must then be interpreted as evidence for interactions [5]. Nothing further can be said purely from the study of these correlators. Whether the spectrum of screening masses comes from a weakly interacting effective theory, or whether it is very similar to the spectrum at zero temperature, are questions which can only be answered by measuring the masses in the other representations which we write down explicitly.

In Sec. II we present a brief review of the symmetries of staggered fermions at $T=0$. This serves to set up the notation, and indicates what changes to expect at finite temperature. Section III contains our main results on the characterization of the group of symmetries of the spatial transfer matrix at $T>0$ and its irreducible representations (irreps). Free field theory results for the screening masses is discussed in Sec. IV, where presently available data are also discussed.

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TABLE I. Symmetry operations on staggered fermions. The upper (lower) signs in $\mathcal{R}$ are used when $\kappa>\lambda$ ($\kappa<\lambda$). Here $\epsilon(x)=(-1)^{x_1+x_2+x_3+x_4}$, $\eta_i(x)=\Pi_{k<i}(-1)^{x_k}$ and $\zeta_i(x)=\Pi_{k>i}(-1)^{x_k}$.

Operation	Action
$R_{\kappa\lambda}$	$\chi(x)\to\mathcal{R}(R_{\kappa\lambda}^{-1}x)\,\chi(R_{\kappa\lambda}^{-1}x)$ $\mathcal{R}(x)=\frac{1}{2}[1\pm\eta_\kappa(x)\eta_\lambda(x)\mp\zeta_\kappa(x)\zeta_\lambda(x)$ $+\eta_\kappa(x)\eta_\lambda(x)\zeta_\kappa(x)\zeta_\lambda(x)]$
I	$\chi(x)\to\eta_4(x)\chi(Ix)$
S_μ	$\chi(x)\to\zeta_\mu(x)\chi(x+a_\mu)$
C	$\chi(x)\to\epsilon(x)\chi(x)$
$U_B(1)$	$\chi(x)\to e^{i\Theta_B}\chi(x)$

Two Appendices contain the technical details of induced representations and character tables for the irreps of mesons.

II. SYMMETRIES OF STAGGERED FERMIONS

In this section we review the breaking of continuum spin-flavor symmetries for lattice staggered fermions [8] at zero temperature, and identify how this pattern changes at finite temperature. The continuum symmetry for four flavors of fermions is $SU_r(2)\otimes U_f(4)$, where the first factor is the rotational symmetry, and the second is the flavor symmetry. We follow the notational conventions of [9,10].

At $T=0$ we are interested in the symmetries of fermion operators which have zero momentum in the directions orthogonal to the Euclidean time t:

$$\chi_A=\sum_{\mathbf{m}} T_z^{-m_3}T_y^{-m_2}T_x^{-m_1}\chi(r)T_x^{m_1}T_y^{m_2}T_z^{m_3}=\sum_{\mathbf{m}} \chi(\mathbf{x}+2\mathbf{m}a). \tag{2}$$

Here the index A denotes the corners of the hypercube on which the appropriate component of the quark field resides, a is the lattice spacing, T_i are the generators of translations in the ith direction, and we have assumed that there are periodic boundary conditions in all directions on the slice. In writing Eq. (2), we have chosen to study correlation functions of operators separated in the time direction. Because of the 4-dimensional discrete rotational symmetry of the $T=0$ theory, we could have chosen to study propagation in any other direction with the same result.

The symmetry elements of the theory are listed in Table I. For staggered fermions, the shifts by one lattice spacing, S_μ, are mixed flavor and translation operations. Pure translations are $T_\mu=S_\mu^2$. We have chosen the transfer matrix $\mathbf{T}$ to be T_t. Nothing would have changed, at $T=0$, if we had instead chosen $\mathbf{T}$ to be T_z.

Discrete flavor operations, $\Xi_\mu=S_\mu T_\mu^{-1/2}$, are vectors under rotations, generated by $R_{\kappa\lambda}$ and transform as

$$R_{ij}^{-1}\Xi_k R_{ij}=\delta_{ik}\Xi_j+\delta_{jk}\Xi_i+|\epsilon_{ijk}|\Xi_k. \tag{3}$$

Here and elsewhere, Greek indices run from 1 to 4; Latin indices over the three directions summed in Eq. (2) or its analogue. A subgroup $U_B(1)$ of the continuum flavor group remains unbroken on the lattice; this charge corresponds to the fermion number q. The representations of Ξ_μ in an irrep with fermion number q, $\mathcal{D}_q(\Xi_\mu)$, obey the relation

$$\mathcal{D}_q(\Xi_\mu)\mathcal{D}_q(\Xi_\nu)=e^{i\pi q}\mathcal{D}_q(\Xi_\nu)\mathcal{D}_q(\Xi_\mu). \tag{4}$$

Inversion, I, commutes with Ξ_4, and anti-commutes (commutes) with the other Ξ_k in representations with odd (even) values of q. Parity is defined by $P=\Xi_4 I$. The remaining discrete symmetry is that of charge-conjugation, C.

The symmetries of $\mathbf{T}$ are the rest-frame group

$$RF(\Xi_\mu,R_{kl},I,C)\otimes U_B(1). \tag{5}$$

We have used the notation $G(X)$ to mean the group G generated by the operation(s) X. A subgroup of RF is the group of isometries of the lattice, called the geometric rest frame group, $GRF(\Xi_\mu,R_{kl},I)$. In turn, GRF contains the time slice group, which is the point group of the lattice:

$$TS(R_{kl},I)=O_h(R_{kl},I)=O(R_{kl})\otimes Z_2(I). \tag{6}$$

This chain of groups builds up to the continuum symmetry group:

$$TS\subset GRF\subset RF\subset SU_d(2)\otimes U_B(1), \tag{7}$$

where $SU_d(2)$ is the diagonal subgroup of the direct product $SU_r(2)\otimes SU_f(2)$ of rotations and flavor. The breaking of $SU_f(4)$ to $SU_f(2)$ is specified by requiring that the fundamental of $SU_f(4)$ break into the irrep $(\frac{1}{2},\frac{1}{2})$ of $SU_f(2)$.

All correlation functions block diagonalize into irreps of TS. This group, O_h, is the group of symmetries of a cube. It has 48 elements in 10 conjugacy classes [13]. It has four one-dimensional irreps $A_1^\pm$ and $A_2^\pm$, two two-dimensional irreps $E^\pm$ and four three-dimensional irreps $F_1^\pm$ and $F_2^\pm$. The physical interpretation of each mass is obtained by tracing the descent of the irrep of TS through the whole chain in Eq. (7) from the irreps of the continuum symmetry, $SO(4)\otimes SU(4)$. This is done in [9,10]. See also [11] for some details of the treatment of correlation functions.

For the study of equilibrium finite temperature, $T>0$, physics we are interested in screening masses and screening correlation functions, i.e., in the eigenvalues of the transfer matrix in spatial directions. Two distinctions from the $T=0$ theory should be borne in mind.

The first is that there are anti-periodic boundary conditions in the Euclidean time direction on fermions. As a result the lowest Fourier component has a non-vanishing momentum in this direction:

$$T_t^{-N_t/2}\chi(x)T_t^{N_t/2}=e^{i\pi}\chi(x),$$
$$T_t^{-N_t/2}\overline{\chi}(x)T_t^{N_t/2}=e^{-i\pi}\overline{\chi}(x), \tag{8}$$

where N_t is the number of lattice points in the time direction. This is a trivial change. For fermion bilinear operators it makes no difference. Operators with an odd number of fermion fields are treated slightly differently. For example, the projection on the lowest momentum state of a fermion field is not written as in Eq. (2), but as

$$\chi_A=\sum_{\mathbf{m}} e^{2i\pi m_t/N_t}\chi(\mathbf{x}+2\mathbf{m}a), \tag{9}$$

where **m** runs over the coordinates in a spatial slice, i.e., over two spatial directions and the temporal direction [4]. The phase factor in the sum is just the statement that the lowest Matsubara frequency for fermions is πT.

In this paper we shall concern ourselves with the second, and more important, difference—the isometries of a slice of the lattice. Since we are interested in screening masses, we consider slices through the lattice orthogonal to one of the spatial directions, say the *z*-direction [as in Eq. (9) above]. Then the isometries of the *z*-slice generate

$$\underline{TS}=D_4^h=D_4(R_{xy},R_{xt}^2)\otimes Z_2(I). \tag{10}$$

The identification of this group is easy, because it differs from O_h [Eq. (6)] by the fact that rotations of $\pi/2$ in the xt and yt planes is not allowed. The spectrum of the screening masses requires a classification by the irreps of D_4^h. The continuum symmetry will be built up by the group chain

$$\underline{TS}\subset\underline{GRF}\subset\underline{RF}\subset\mathcal{C}\otimes U_B(1)\subset SU_d(2)\otimes U_B(1), \tag{11}$$

where $\mathcal{C}=O(2)\otimes Z_2(I)$ is the invariance group of a cylinder. To generate each of the lattice groups in the chain, we use the construction at $T=0$, only leaving out odd powers of the rotations R_{kt}.

D_4^h has 16 elements in six conjugacy classes [13]. There are eight one-dimensional irreps labeled $A_1^\pm$, $A_2^\pm$, $B_1^\pm$ and $B_2^\pm$, and two two-dimensional irreps $E^\pm$. The reductions of the irreps of O_h to D_4^h is as

$$A_1^P\to A_1^P,\quad A_2^P\to B_1^P,$$

$$F_1^P\to A_2^P\oplus E^P,\quad F_2^P\to B_2^P\oplus E^P,$$

$$E^P\to A_1^P\oplus B_1^P. \tag{12}$$

More details can be found in [2].

In the rest of this paper we shall give these decompositions of meson and hadron operators using the language of the $T=0$ theory. This calls for some care in the interpretation of results—although we shall talk of charge conjugation, *C*, and parity, *P*, and the operators will have the same structure and algebra as in the $T=0$ theory, they may represent quite different physical quantities [12].

III. THE SYMMETRY GROUP AT $T>0$

In this section the symmetry groups are written down. The representation theory of these groups in the meson (quark-antiquark) sector is examined in detail. The symmetries of the quark fields are also examined briefly, and the representation theory in the baryon sector is dealt with in less detail.

A. Meson operators

In a meson representation, the quark number $q=0$. As a result, the representants X_k of the flavor generators Ξ_k commute. Consequently *C* and *I* commute with X_k. At $T=0$, it has been shown that [10]

$$RF=GRF(X_\mu,R_{kl},I)\otimes Z_2(C), \tag{13}$$

where

$$GRF(X_\mu,R_{kl},I)=G(\tilde{X}_k,R_{kl})\otimes Z_2(I)\otimes Z_2(X_1X_2X_3)\otimes Z_2(X_4), \tag{14}$$

with $\tilde{X}_k=X_kX_1X_2X_3$. The irreps of GRF are denoted $\mathbf{r}^{\sigma_4\sigma_{123}}$, where **r** denotes an irrep of *G*, and σ_4 and σ_{123} are signs which denote the irreps of the Z_2 factor groups generated by X_4 and $X_1X_2X_3$ respectively.

Next we identify the group *G*. The $\tilde{X}_k$ generate a 4 element Abelian group called the Viergruppe, $V=Z_2\otimes Z_2$. This is a normal subgroup of *G*. The transformation properties of $\tilde{X}_k$ under rotations, Eq. (3), show that G is the semi-direct product $G=V(\tilde{X}_k)\bowtie O$. Now, the cubic group $O=V(R_{kl}^2)\bowtie S_3$, where the normal subgroup, $V(R_{kl}^2)$ is generated by the three rotations by angle π [14], and the other factor is the permutation group of 3 elements. From Eq. (3) it is clear that $V(R_{kl}^2)$ has trivial action on $V(\tilde{X}_k)$, and we can write

$$G(\tilde{X}_k,R_{kl})=(V(\tilde{X}_k)\otimes V(R_{kl}^2))\bowtie S_3. \tag{15}$$

Since the normal subgroup is Abelian, the irreps of *G* can be efficiently generated by the method of induced representations. Details are given in Appendix A, where we recover the results of [10].

This method makes it easy to construct the $T>0$ group,

$$\underline{GRF}=\check{G}\otimes Z_2(I)\otimes Z_2(X_1X_2X_3)\otimes Z_2(X_4). \tag{16}$$

TABLE II. Irreps of *G*, defined in Eq. (14), and their reduction at finite temperature to irreps of $\check{G}$, defined in Eq. (16). The irreps which are realized for mesons are marked. Meson states do not exhaust all the irreps of *G*, *O* or $\check{G}$, but do exhaust all the irreps of D_4.

G	O	$\check{G}$	D_4	Meson
$\mathbf{1}$	A_1	$\mathbf{1}_0$	A_1	Yes
$\mathbf{1}'$	A_2	$\mathbf{1}_1$	B_1	
$\mathbf{2}$	E	$\mathbf{1}_0+\mathbf{1}_1$	A_1+B_1	
$\mathbf{3}$	F_1	$\mathbf{2}_2+\mathbf{1}_6$	A_2+E	Yes
$\mathbf{3}'$	F_2	$\mathbf{2}_2+\mathbf{1}_7$	B_2+E	
$\mathbf{3}''$	F_1	$\mathbf{2}_0+\mathbf{1}_2$	A_2+E	Yes
$\mathbf{3}'''$	F_2	$\mathbf{2}_0+\mathbf{1}_3$	B_2+E	
$\mathbf{3}''''$	A_1+E	$\mathbf{2}_4+\mathbf{1}_4$	$2A_1+B_1$	Yes
$\mathbf{3}'''''$	A_2+E	$\mathbf{2}_4+\mathbf{1}_5$	A_1+2B_1	
$\mathbf{6}$	F_1+F_2	$\mathbf{2}_1+\mathbf{2}_3+\mathbf{2}_5$	A_2+B_2+2E	Yes

TABLE III. Representations of local staggered mesons. Only the A_1^+ operators have been used in simulations until now. Reduction of three-link separated mesons follows an identical pattern and generates the opposite parity irreps of D_4^h.

GRF	$\check{G}$	D_4^h	Operator
$\mathbf{1}^{++}$	$\mathbf{1}_0$	A_1^+	$\Sigma_x \overline{\chi(x)}\chi(x)$
$\mathbf{1}^{+-}$	$\mathbf{1}_0$	A_1^+	$\Sigma_x \eta_4(x)\zeta_4(x)\overline{\chi(x)}\chi(x)$
$\mathbf{3}''''^{+-}$	$\mathbf{1}_4$	A_1^+	$\Sigma_x \epsilon(x)\eta_3(x)\zeta_3(x)\overline{\chi(x)}\chi(x)$
	$\mathbf{2}_4$	A_1^+	$\Sigma_x \epsilon(x)[\eta_1(x)\zeta_1(x)+\eta_2(x)\zeta_2(x)]\overline{\chi(x)}\chi(x)$
		B_1^+	$\Sigma_x \epsilon(x)[\eta_1(x)\zeta_1(x)-\eta_2(x)\zeta_2(x)]\overline{\chi(x)}\chi(x)$
$\mathbf{3}''''^{++}$	$\mathbf{1}_4$	A_1^+	$\Sigma_x \epsilon(x)\eta_4(x)\zeta_4(x)\eta_3(x)\zeta_3(x)\overline{\chi(x)}\chi(x)$
	$\mathbf{2}_4$	A_1^+	$\Sigma_x \epsilon(x)\eta_4(x)\zeta_4(x)[\eta_1(x)\zeta_1(x)$ $+\eta_2(x)\zeta_2(x)]\overline{\chi(x)}\chi(x)$
		B_1^+	$\Sigma_x \epsilon(x)\eta_4(x)\zeta_4(x)[\eta_1(x)\zeta_1(x)$ $-\eta_2(x)\zeta_2(x)]\overline{\chi(x)}\chi(x)$

The rotation generators in $\check{G}$ are R_{12} and R_{13}^2, and they generate the group D_4. Since $D_4 = V(R_{kl}^2)\bowtie Z_2(R_{12})$, we have

$$\check{G} = (V(\tilde{X}_k)\otimes V(R_{kl}^2))\bowtie Z_2(R_{12}) \qquad (17)$$

$\check{G}$ has 32 elements in 14 conjugacy classes. The irreps can be constructed by the method of induced representations (see Appendix A). There are 8 one-dimensional and 6 two-dimensional irreps of $\check{G}$.

The content of the various GRF irreps is shown in Table II. The reduction of irreps of G to those of O, $\check{G}$ and D_4 are performed using the character tables in Appendix B. These reductions are consistent with those given in Eq. (12). The irreps obtained for mesonic (quark-antiquark) operators can be identified through the Clebsch-Gordan series for the GRF of fermionic representations.

We demonstrate the reduction of the $\mathbf{1}^{+\pm}$ and $\mathbf{3}''''^{+\pm}$ irreps of GRF to the irreps of D_4^h in Table III, using the full set of local meson operators. All the operators which have been used to date for computing screening masses belong to the D_4^h irrep A_1^+, and conversely, all the A_1^+ operators in Table III have been used in measurements. Notice, however, that this one irrep of $\underline{TS}$ descends from different irreps of $\underline{GRF}$. The two A_1^+ irreps descending from $\mathbf{1}_0$ must give degenerate masses,[1] as must the pairs descending from the $\mathbf{1}_4$ and the $\mathbf{2}_4$. However, there is no group theoretical necessity for the three pairs to have the same mass.

The reduction of $\mathbf{3}^{-\pm}$, $\mathbf{3}''^{-\pm}$ and $\mathbf{6}^{-\pm}$ irreps obtained for one-link separated meson operators is given in Table IV. Reductions of two-link separated meson operators can also be read off from the structure of these reductions. The latter give positive I parity irreps of D_4^h. Combining these two sets we have a set of $A_2^\pm$, $B_2^\pm$ and $E^\pm$ irreps. Three link separated operators reduce in the same way as the local meson operators but give the opposite I parity. These two sets together give us the remaining irreps of D_4^h, i.e., $A_1^\pm$ and $B_1^\pm$.

[1]This is the phenomenon of ''parity doubling'' at high temperature.

B. Quark and baryon operators

Quark and baryon operators carry odd fermion charge. The representants of Ξ_μ anti-commute and generate the 32 element Clifford group $CL(4)$. Its commutator subgroup is isomorphic to Z_2 and its Abelization, $CL(4)/Z_2$ is precisely the group $V(\tilde{X}_k)\otimes Z_2(X_4)\otimes Z_2(X_1X_2X_3)$ encountered as the normal subgroup of the mesonic rotation-shift group [14]. Apart from the 16 one-dimensional irreps of this group, $CL(4)$ also has a four-dimensional quaternionic irrep familiar to us from the algebra of the Dirac matrices.

The rotation-shift group for fermionic operators at $T=0$ is

$$G_F = CL(4)\bowtie O = (CL(4)\otimes V(R_{kl}^2))\bowtie S_3. \qquad (18)$$

This group has 45 conjugacy classes, and hence 45 irreps. Forty of these have been identified as the irreps $\mathbf{r}^{\sigma_4\sigma_{123}}$ of the mesonic rotation-shift group. The remaining 5 are obtained by inducing with the remaining non-trivial irrep of $CL(4)$. This gives the five real irreps **8**, **8′**, **16**, **24** and **24′**.

The defining representation of G_F, **8**, is given by zero momentum staggered fermions on a time slice Eq. (2). Under O_h the octet breaks [9]

$$\mathbf{8}\to A_1^+ + A_1^- + F_1^+ + F_1^-. \qquad (19)$$

The Clebsch-Gordan coefficients for $\mathbf{8}\times\mathbf{8}\times\mathbf{8}$ show that only the **8**, **8′** and **16** are found as irreps of baryons [9]. The local baryon operators built from staggered fermions transform as the A_1^+ component of the **8**.

The rotation-shift group for fermionic operators at $T>0$ is

$$\underline{G}_F = CL(4)\bowtie D_4 = (CL(4)\otimes V(R_{kl}^2))\bowtie Z_2. \qquad (20)$$

This group has 61 conjugacy classes, and hence 61 irreps. Fifty-six of these have been identified as the irreps $\mathbf{r}^{\sigma_4\sigma_{123}}$ of the $T>0$ mesonic rotation-shift group. The remaining 5 are obtained by inducing with the remaining non-trivial irrep of

TABLE IV. Representations of one-link separated staggered mesons. Here $D_\mu\phi(\mathbf{x})=\phi(\mathbf{x}+\hat{\mu})+\phi(\mathbf{x}-\hat{\mu})$. Reduction of two-link separated mesons follows an identical pattern and generates the opposite parity irreps of D_4^h.

GRF	$\check{G}$	D_4^h	Operator
$\mathbf{3}^{-+}$	$\mathbf{1}_6$	A_2^-	$\Sigma_x\eta_3(x)\overline{\chi}(x)D_3\chi(x)$
	$\mathbf{2}_2$	E^-	$\Sigma_x\eta_{1,2}(x)\overline{\chi}(x)D_{1,2}\chi(x)$
$\mathbf{3}^{--}$	$\mathbf{1}_6$	A_2^-	$\Sigma_x\eta_4(x)\zeta_4(x)\eta_3(x)\overline{\chi}(x)D_3\chi(x)$
	$\mathbf{2}_2$	E^-	$\Sigma_x\eta_4(x)\zeta_4(x)\eta_{1,2}(x)\overline{\chi}(x)D_{1,2}\chi(x)$
$\mathbf{3}''^{--}$	$\mathbf{1}_2$	A_2^-	$\Sigma_x\epsilon(x)\zeta_3(x)\overline{\chi}(x)D_3\chi(x)$
	$\mathbf{2}_0$	E^-	$\Sigma_x\epsilon(x)\zeta_{1,2}(x)\overline{\chi}(x)D_{1,2}\chi(x)$
$\mathbf{3}''^{-+}$	$\mathbf{1}_2$	A_2^-	$\Sigma_x\eta_4(x)\zeta_4(x)\epsilon(x)\zeta_3(x)\overline{\chi}(x)D_3\chi(x)$
	$\mathbf{2}_0$	E^-	$\Sigma_x\eta_4(x)\zeta_4(x)\epsilon(x)\zeta_{1,2}(x)\overline{\chi}(x)D_{1,2}\chi(x)$
$\mathbf{6}^{--}$	$\mathbf{2}_5$	A_2^-	$\Sigma_x\epsilon(x)[\eta_1(x)+\eta_2(x)]\eta_3(x)\overline{\chi}(x)D_3\chi(x)$
		B_2^-	$\Sigma_x\epsilon(x)[\eta_1(x)-\eta_2(x)]\eta_3(x)\overline{\chi}(x)D_3\chi(x)$
	$\mathbf{2}_1$	E^-	$\Sigma_x\epsilon(x)\eta_1(x)\eta_2(x)\overline{\chi}(x)D_2\chi(x),\ 1\leftrightarrow 2$
	$\mathbf{2}_3$	E^-	$\Sigma_x\epsilon(x)\eta_3(x)\eta_1(x)\overline{\chi}(x)D_1\chi(x),\ 1\rightarrow 2$
$\mathbf{6}^{-+}$	$\mathbf{2}_5$	A_2^-	$\Sigma_x\epsilon(x)\eta_4(x)\zeta_4(x)[\eta_1(x)+\eta_2(x)]\eta_3(x)\overline{\chi}(x)D_3\chi(x)$
		B_2^-	$\Sigma_x\epsilon(x)\eta_4(x)\zeta_4(x)[\eta_1(x)-\eta_2(x)]\eta_3(x)\overline{\chi}(x)D_3\chi(x)$
	$\mathbf{2}_1$	E^-	$\Sigma_x\epsilon(x)\eta_4(x)\zeta_4(x)\eta_1(x)\eta_2(x)\overline{\chi}(x)D_2\chi(x),\ 1\leftrightarrow 2$
	$\mathbf{2}_3$	E^-	$\Sigma_x\epsilon(x)\eta_4(x)\zeta_4(x)\eta_3(x)\eta_1(x)\overline{\chi}(x)D_1\chi(x),\ 1\rightarrow 2$

$CL(4)$. This gives the five real irreps $\mathbf{8}_0$, $\mathbf{8}_1$, $\mathbf{8}_2$, $\mathbf{8}_3$, $\mathbf{16}$. The reductions of the $T=0$ baryon irreps at $T>0$ are

$$\mathbf{8}\rightarrow\mathbf{8}_0,\quad \mathbf{8}'\rightarrow\mathbf{8}_2,\quad \mathbf{16}\rightarrow\mathbf{8}_0+\mathbf{8}_2. \tag{21}$$

Under D_4^h, we find that

$$\mathbf{8}_0\rightarrow A_1^+ + A_1^- + A_2^+ + A_2^- + E^+ + E^-. \tag{22}$$

The components of the quark field which carry different representations of TS and $\underline{TS}$ are shown in Table V.

Given the classification of baryon operators in [9] it is a simple matter to construct the D_4^h irreps from them. We do not present a detailed table, because the state of the art in measurements with baryons has not progressed far beyond the measurement of the purely local operators even at $T=0$.

IV. FREE FIELD THEORY AND BEYOND

In a free field theory of staggered fermions, the symmetries of the Hamiltonian become symmetries of the field configuration. As a result, there are many more degeneracies among the screening masses than in the general problem. The only relevant group turns out to be $\underline{TS}$, in the sense that all correlators in the same irrep of $\underline{TS}$ have degenerate masses, even if they descend through different irreps of $\underline{RF}$ and $\underline{GRF}$. However, the degeneracies are even higher than would be predicted by the application of the group theory of $\underline{TS}$.

The local "mesons" in the A_1^+ irrep of D_4^h have been analyzed extensively in free field theory (FFT) [4,5]. It is known that the correlation functions show the typical even-odd structure of staggered fermion correlators. The screening mass, μ, in this channel is

$$\mu a = 2\sqrt{m^2a^2+\sin^2\left(\frac{\pi}{N_\tau}\right)}\rightarrow\frac{2\pi}{N_\tau}=2\pi Ta. \tag{23}$$

Here a is the lattice spacing, N_τ is the lattice size in the Euclidean time direction, and m is the quark mass. The limit is taken for small m and large N_τ, and is equal to twice the minimum Matsubara frequency, $2\pi T$ [3,4,15]. Finite size effects are clearly strong, even in FFT, and have been analyzed before [4,5].

The remaining local "mesons" are in the B_1^+ irrep of D_4^h. In FFT these correlation functions vanish. This is easy to understand. In FFT the x and y direction propagators are exactly equivalent, and hence their difference (see Table III) cancels.

TABLE V. Representations of the quark field χ_A. The "zero-momentum" projection is performed as shown in Eq. (9). The $\mathbf{8}$ of G_F is the same as the $\mathbf{8}_0$ of the $\underline{G_F}$.

GRF	O_h	D_4^h	Component
$\mathbf{8}$	A_1^+	A_1^+	$\chi(0)$
	A_1^-	A_1^-	$\chi(\hat{x}+\hat{y}+\hat{z})$
	F_1^+	A_2^+	$\chi(\hat{x}+\hat{y})$
		E^+	$\{\chi(\hat{x}+\hat{z}),-\chi(\hat{y}+\hat{z})\}$
	F_1^-	A_2^-	$\chi(\hat{z})$
		E^-	$\{\chi(\hat{x}),\chi(\hat{y})\}$

TABLE VI. Induced representations of $G=(V\otimes V)\bowtie S_3$. A representative member of each orbit is underlined. The trivial isotropy group is denoted by $\{E\}$. This construction reproduces the result of [10].

Orbit	Isotropy	Dimension	Multiplicity
<u>(0,0)</u>	S_3	1	2
		2	1
<u>(0,1)</u>,(0,2),(0,3)	$Z_2(R_{12})$	3	2
<u>(1,0)</u>,(2,0),(3,0)	$Z_2(R_{12})$	3	2
<u>(1,1)</u>,(2,2),(3,3)	$Z_2(R_{12})$	3	2
<u>(1,2)</u>,(1,3),(2,1),(2,3),(3,1),(3,2)	$\{E\}$	6	1

The non-local mesons also divide into two groups. The $A_1^{\pm}$, $A_2^{\pm}$ and $E^{\pm}$ irreps of D_4^h give rise to screening masses according to Eq. (23). In contrast, the $B_1^{\pm}$ and $B_2^{\pm}$ irreps vanish in FFT. Deviations from any of these free field theory results can be used as a measure of the interaction strength between fermions.

It is interesting to recall past measurements [3–5]. Screening masses have been measured with four correlators—

the D_4^h irrep A_1^+ descending from the $\mathbf{1}_0^{+\pm}$ of the <u>GRF</u>, and

a linear combination of the two pairs of D_4^h irreps A_1^+ descending from the $\mathbf{1}_4^{+\pm}$ and $\mathbf{2}_4^{+\pm}$ of the <u>GRF</u> (see Table III).

Since they all belong to the same irrep of <u>TS</u>, we might expect them to give the same screening mass in FFT. Extensive numerical work was performed in the 4-flavor [4] and quenched [5] $SU(3)$ theories at $T\approx T_c$, $T=3T_c/2$ and $T=2T_c$. It was found that the masses within each group become degenerate already quite close to T_c, but the masses of the two groups differed by about 10% even at $2T_c$. In the light of the preceding calculations, the natural explanation for this observation is that there are residual interactions.

A minor controversy persists in the interpretation of these observations. The two viewpoints can be summarized as

(1) Fermions in QCD at $T>T_c$ are weakly coupled, since the A_1^+ screening masses coming from the $\mathbf{1}_4^{+\pm}$ and the $\mathbf{2}_4^{+\pm}$ agree very well with Eq. (23). This picture becomes a better approximation with increasing T.

(2) QCD at $T_c<T<2T_c$ is not very different from that at $T<T_c$, since the A_1^+ from the $\mathbf{1}_0^{+\pm}$ is quite different from Eq. (23). Furthermore, the A_1^+ from the $\mathbf{1}_4^{+\pm}$ and the $\mathbf{2}_4^{+\pm}$ are just the same as the vector/pseudo-vector at zero temperature.

To resolve this controversy one needs first to measure the B_1^+ coming from the $\mathbf{2}_4^{+\pm}$ and check whether it agrees with either of the models above. Beyond this, one needs to measure screening masses in other irreps of the <u>GRF</u> which are in the same irreps of the continuum group as the measured local fermion bilinears. This would check whether the second model is viable or not.

TABLE VII. Induced representations of $\check{G}=(V\otimes V)\bowtie Z_2$. A representative member of each orbit is underlined. The trivial isotropy group is denoted by $\{E\}$.

Orbit	Isotropy	Dimension	Multiplicity
<u>(0,0)</u>	Z_2	1	2
<u>(0,3)</u>	Z_2	1	2
<u>(3,0)</u>	Z_2	1	2
<u>(3,3)</u>	Z_2	1	2
<u>(0,1)</u>,(0,2)	$\{E\}$	2	1
<u>(1,0)</u>,(2,0)	$\{E\}$	2	1
<u>(1,1)</u>,(2,2)	$\{E\}$	2	1
<u>(1,2)</u>,(2,1)	$\{E\}$	2	1
<u>(1,3)</u>,(2,3)	$\{E\}$	2	1
<u>(3,1)</u>,(3,2)	$\{E\}$	2	1

V. SUMMARY

In this paper we have studied the symmetries of the transfer matrix for staggered fermions. By using the method of induced representations we have reproduced old, known results for the spin-flavor symmetry group and its irreps at zero temperature [9,10]. This method allows a simple generalization to the analysis of the spatial direction transfer matrix at finite temperature. Our main result is the identification of the irreps of the finite temperature symmetry group given in Table IX.

Using this, we have decomposed the $T=0$ irreps into the $T>0$ irreps. This reduction is shown for ''mesons'' in Tables III and IV. For quarks the reduction is given in Table V. The reduction for local ''baryons'' may be read off from the same table.

The phenomenon of ''parity doubling'' at high temperatures is allowed by the group theory. In a free fermion theory, in fact, the degeneracies are much higher—the screening masses are classified by the point group of the spatial slice of the lattice. In an interacting theory this is not true; the descent of each irrep through the chain of enveloping groups [see Eq. (11)] is important. This has been seen in the A_1^+ irrep of the point group. Whether the physics at $T>T_c$ is a small perturbation around the free theory or the zero temperature theory (or something else altogether) can be explored by studying other irreps. The B_1^+ is a good candidate because it is also built from local fermion bilinear operators.

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APPENDIX A: INDUCED REPRESENTATIONS

The method of induced representations for a semi-direct product group $G=N\bowtie H$, for Abelian N, can be found in [14]. Here we quote the results required in this paper. The

TABLE VIII. The character table for $G=(V(u)\otimes V(v))\bowtie S_3$. The second line of the table gives the number of operators in each class.

G	O	E	u	v	$u\cdot v$	uv	$\mathcal{C}$	R	uR	vR	uvR
		1	3	3	3	6	32	12	12	12	12
$\mathbf{1}$	A_1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}'$	A_2	1	1	1	1	1	1	−1	−1	−1	−1
$\mathbf{2}$	E	2	2	2	2	2	−1	0	0	0	0
$\mathbf{3}$	F_1	3	3	−1	−1	−1	0	1	1	−1	−1
$\mathbf{3}'$	F_2	3	3	−1	−1	−1	0	−1	−1	1	1
$\mathbf{3}''$	F_1	3	−1	−1	3	−1	0	1	−1	−1	1
$\mathbf{3}'''$	F_1	3	−1	−1	3	−1	0	−1	1	1	−1
$\mathbf{3}''''$	$E+A_1$	3	−1	3	−1	−1	0	1	−1	1	−1
$\mathbf{3}'''''$	$E+A_2$	3	−1	3	−1	−1	0	−1	1	−1	1
$\mathbf{6}$	F_1+F_2	6	−2	−2	−2	2	0	0	0	0	0

dual, $\hat{N}$ (set of equivalence classes of irreps of N), is isomorphic to N, and the action of H on $\hat{N}$ is isomorphic to its action on N.

Under the action of H, the dual breaks up into disjoint orbits O_i, i.e., $\hat{N}=\oplus_i O_i$. Examine the isotropy group, $H_i \subset H$ of one representative $\chi_i \in O_i$. We need to know two cases—

When the orbit O_i has only one element, i.e., $H_i=H$, then the induced representations are precisely the irreps of H.

For other χ_i when H_i is Abelian, the dimension of the induced representation is the number of elements in the coset H/H_i, and the multiplicity of such irreps is given by the number of classes in H_i.

All 4 irreps of the Viergruppe, V, are one-dimensional. We label them by the numbers 0, 1, 2 and 3. The trivial irrep is called 0. The irrep labeled k $(\neq 0)$ has $\chi(E)=\chi(X_k)=1$ and the other two characters -1. Irreps of the direct product $V\otimes V$ are labeled by the ordered pair (k,l) where k is an irrep of the first factor and l of the second. S_3 has two one-dimensional irreps (the trivial and the sign) and one two-dimensional irrep. Z_2 has two one-dimensional irreps, the trivial and the sign. These are the only inputs into the construction of the irreps we require. The construction of the irreps of G and $\check{G}$ [see Eqs. (15) and (17)] follow from the rules above, and are given in Tables VI and VII respectively.

APPENDIX B: CHARACTER TABLES

In this appendix we define the irreps of G and $\check{G}$ [see Eqs. (15),(17)] by writing down the character tables.

In Appendix A we showed that the group G of Eq. (15) has 10 irreps. Hence the 96 elements of the group fall into 10 conjugacy classes. For every $g\in G$, we can write uniquely $g=uvs$, where $u\in V(\tilde{X}_k)$, $v\in V(R^2_{kl})$, and $s\in S_3$. We use the notation R_{kl} and $\mathcal{C}$ respectively for the operators in S_3 which permute the pair kl and make a cyclic shift. In terms

TABLE IX. The character table for $\check{G}=(V(u)\otimes V(v))\bowtie Z_2$. The second line of the table gives the number of operators in each class.

$\check{G}$	D_4	E	u	v	uv	$\mathbf{u}$	$\mathbf{v}$	$u\mathbf{v}$	$v\mathbf{u}$	$\mathbf{u}\cdot\mathbf{v}$	$\mathbf{uv}$	R	$\mathbf{u}R$	$\mathbf{v}R$	$\mathbf{uv}R$
		1	1	1	1	2	2	2	2	2	2	4	4	4	4
$\mathbf{1}_0$	A_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_1$	B_1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1
$\mathbf{1}_2$	A_2	1	1	1	1	−1	−1	−1	−1	1	1	1	−1	−1	1
$\mathbf{1}_3$	B_2	1	1	1	1	−1	−1	−1	−1	1	1	−1	1	1	−1
$\mathbf{1}_4$	A_1	1	1	1	1	−1	1	1	−1	−1	−1	1	−1	1	−1
$\mathbf{1}_5$	B_1	1	1	1	1	−1	1	1	−1	−1	−1	−1	1	−1	1
$\mathbf{1}_6$	A_2	1	1	1	1	1	−1	−1	1	−1	−1	1	1	−1	−1
$\mathbf{1}_7$	B_2	1	1	1	1	1	−1	−1	1	−1	−1	−1	−1	1	1
$\mathbf{2}_0$	E	2	−2	−2	2	0	0	0	0	2	−2	0	0	0	0
$\mathbf{2}_1$	E	2	−2	−2	2	0	0	0	0	−2	2	0	0	0	0
$\mathbf{2}_2$	E	2	2	−2	−2	2	0	0	−2	0	0	0	0	0	0
$\mathbf{2}_3$	E	2	2	−2	−2	−2	0	0	2	0	0	0	0	0	0
$\mathbf{2}_4$	A_1+B_1	2	−2	2	−2	0	2	−2	0	0	0	0	0	0	0
$\mathbf{2}_5$	A_2+B_2	2	−2	2	−2	0	−2	2	0	0	0	0	0	0	0

of this decomposition, the 10 conjugacy classes are—the identity E, u_i (with $i=1,2,3$), v_i, $u\cdot v$ (meaning by $u_i v_i$), $u_i v_j$ (with $j\neq i$, and denoted uv), $\mathcal{C}$, R_{kl}, $u_i R_{ij}$ (denoted uR), $v_i R_{ij}$ (denoted vR), $u_i v_j R_{ij}$ (denoted uvR). The character table is constructed by standard methods, and given in Table VIII. The reduction of irreps of G to that of the subgroup of cubic rotations O is performed by inspecting the characters[2] of the conjugacy classes of O. These are the rotations by $\pi/2$ (R), rotations by π (v), rotations by $2\pi/3$ ($\mathcal{C}$) and the remaining 2-fold rotations (vR).

The construction, in Appendix A, of the irreps of the group $\check{G}$ of Eq. (17) shows that there are 14 conjugacy classes. For every $g\in\check{G}$ we have the unique decomposition $g=uvs$, where $u\in V(\tilde{X}_k)$, $v\in V(R_{kl}^2)$, and $s\in Z_2$. Using the notation R for the nontrivial element of Z_2, we can write the conjugacy classes as E, $u=u_3$, $v=v_3$, u_i (with $i=1,2$ and denoted $\mathbf{u}$), $\mathbf{v}$ (in the same notation), uv, $u_i v_i$ (with $i=1,2$ and denoted $\mathbf{u}\cdot\mathbf{v}$), $u\mathbf{v}$, $v\mathbf{u}$, $\mathbf{uv}$ (meaning $u_i v_j$ with $i\neq j=1,2$), R (which is equivalent to uvR, uR and vR), $\mathbf{u}R$ (also equivalent to $v\mathbf{u}R$), $\mathbf{v}R$ (also equivalent to $u\mathbf{v}R$), and $\mathbf{uv}R$. The character table is given in Table IX. The decomposition to irreps of D_4 needs the identification $R_{12}^2=v$, the π rotations about the x and y axes are $\mathbf{v}$, $R_{12}=R$ and the remaining 2-fold symmetries are $\mathbf{v}R$.

[2]Character tables for O_h and D_4^h may be found in [13].

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