Cosmological solutions of Hořava-Witten theory

André Lukas and Burt A. Ovrut
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104-6396

Daniel Waldram
Department of Physics, Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544
(Received 21 September 1998; published 24 September 1999)

We discuss cosmological solutions of Hořava-Witten theory describing the strongly coupled heterotic string. At energies below the grand-unified scale, the effective theory is five, not four, dimensional, where the additional coordinate parametrizes an S^1/Z_2 orbifold. Furthermore, it admits no homogeneous solutions. Rather, the static vacuum state, appropriate for a reduction to four-dimensional $N=1$ supersymmetric models, is a BPS domain wall pair. Relevant cosmological solutions are those associated with this BPS state. In particular, such solutions must be inhomogeneous, depending on the orbifold coordinate as well as on time. We present two examples of this new type of cosmological solution, obtained by separation of variables rather than by exchange of the time and radius coordinates of a brane solution, as in previous work. The first example represents the analogue of a rolling radii solution with the radii specifying the geometry of the domain wall pair. This is generalized in the second example to include a nontrivial ''Ramond-Ramond'' scalar.
[S0556-2821(99)04818-3]

PACS number(s): 11.25.Mj, 11.27.+d, 98.80.Cq

I. INTRODUCTION

Hořava and Witten have shown that the strongly coupled $E_8\times E_8$ heterotic string can be identified as the 11-dimensional limit of M theory compactified on an S^1/Z_2 orbifold with a set of E_8 gauge fields at each ten-dimensional orbifold fixed plane [1,2]. Furthermore, Witten has demonstrated that there exists a consistent compactification of this M-theory limit on a ''deformed'' Calabi-Yau threefold, leading to a supersymmetric $N=1$ theory in four dimensions [3,5]. Matching at the tree level to the phenomenological gravitational and grand-unified-theory (GUT) couplings [3,4], one finds the orbifold must be larger than the Calabi-Yau radius, by a factor of 10 or so. Since the GUT scale (about 10^{16} GeV) is set by the size of the Calabi-Yau threefold, this implies that at energies below the unification scale there is a regime where the universe appears five dimensional. This five-dimensional regime represents a new setting for early universe cosmology, which has been traditionally studied in the framework of the four-dimensional effective action.

In a previous paper [6], the effective five-dimensional Hořava-Witten theory was derived for the universal fields, which are independent of the particular form of the Calabi-Yau manifold. In this derivation the standard embedding of the spin connection in one of the E_8 gauge groups has been used. This five-dimensional theory has a number of interesting and unusual features. The theory resides in a five-dimensional space which is a product of a smooth four-dimensional manifold times the orbifold S^1/Z_2. As a result, it splits into a bulk $N=1$, $d=5$ supersymmetric theory with the gravity supermultiplet and the universal hypermultiplet, and two four-dimensional ''boundary'' theories which reside on the two orbifold fixed hyperplanes. The additional fields of the boundary theories are $N=1$, $d=4$ gauge multiplets and chiral multiplets. More specifically, due to the standard embedding there is an E_6 gauge field and gauge matter on one side whereas the other side carries an E_8 gauge field. The reduction from 11 to 5 dimensions requires the inclusion of non-zero values of the four-form field strength in the internal Calabi-Yau directions. This non-zero form field arises because, even for the standard embedding, gauge field and gravitational sources in the form field Bianchi identity do not cancel. As a result one obtains a gauged version of five-dimensional supergravity with a potential term that had not previously been constructed. In addition, the theory has boundary potentials for the projection of the bulk scalar field onto the orbifold planes.

These potentials lead to a particularly interesting effect: the boundary sources mean that the theory has no solutions homogeneous in the orbifold direction. In particular, flat space is not a solution. Instead, the ''vacuum'' which leads to a supersymmetric flat four-dimensional space is a three-brane domain wall solution. The three-brane couples to the bulk potential and is supported by the sources provided by the two boundary potentials. More precisely, it is a double domain wall solution with the two $(3+1)$-dimensional world volumes each covering an orbifold plane and the orbifold itself as the transverse coordinate. It is Bogomol'nyi-Prasad-Sommerfield (BPS), preserving half of the $d=5$ supersymmetries, and so is the appropriate background for a further reduction to four-dimensional $N=1$ supergravity theories. In such a reduction, four-dimensional space-time becomes identified with the three-brane world volume.

Thus we have the interesting possibility of a five-dimensional early universe in Hořava-Witten theory. Furthermore, as a result of the presence of boundary potentials, such five-dimensional cosmologies should be inhomogeneous in the orbifold coordinate. What should realistic models look like? In the ideal case, one would have a situation in which the internal six-dimensional Calabi-Yau space and the orbifold evolve in time for a short period and then settle down to their ''phenomenological'' values while the three non-compact dimensions continue to expand. Then, for late time, when all physical scales are much larger than the orbifold size, the theory is effectively four dimensional and should, in the ''static'' limit, provide a realistic supergravity model of particle physics. As we have argued above, such realistic supergravity models originate from a reduction of the five-dimensional theory in its domain wall vacuum state. Hence, in the ''static'' limit at late time, realistic cosmological solutions should reduce to the domain wall or perhaps a modification thereof that incorporates breaking of the remaining four-dimensional $N=1$ supersymmetry. Consequently, one is forced to look for solutions which depend on the orbifold coordinate as well as on time. The main goal of this paper is to present simple examples of such cosmological solutions in five-dimensional heterotic M theory to illustrate some of the characteristic cosmological features of the theory.

In earlier work [7,8], we showed how a general class of cosmological solutions, that is, time-dependent solutions of the equations of motion that are homogeneous and isotropic in our physical $d=3$ subspace, can be obtained in both superstring theories and M theory defined in spacetimes *without a boundary*. Loosely speaking, we showed that a cosmological solution could be obtained from any p-brane or D-brane by inverting the roles of the time and ''radial'' spatial coordinate. This method will clearly continue to work in Hořava-Witten theory as long as one exchanges time with a radial coordinate not aligned in the orbifold direction. An example of this in 11-dimensions, based on the solution of [9], has been given in [10]. It can, however, not be applied to the fundamental domain wall since its radial direction coincides with the orbifold coordinate. This coordinate is bounded and cannot be turned into time. Also, as argued above, exchanging radius and time in the domain wall solution would not be desirable since it should be viewed as the vacuum state and hence should not be modified in such a way. Instead, the domain wall itself should be made time dependent, thereby leading to solutions that depend on both time and the orbifold coordinate. As a result, we have to deal with coupled partial differential equations, but under certain constraints, these can be solved by separation of variables, though the equations remain non-linear. Essentially, we are allowing the moduli describing the geometry of the domain wall and the excitations of other five-dimensional fields to become time dependent. Technically, we will simply take the usual *Ansätze* for the five-dimensional fields, but now allow the functions to depend on *both* the time and radial coordinates. We will further demand that these functions each factor into a purely time dependent piece and a purely radial dependent piece. This is not, in general, sufficient to separate the equations of motion. However, we will show that subject to certain constraints a separation of variables is achieved. We can solve these separated equations and find new, cosmologically relevant solutions. In this paper, we will restrict our attention to two examples representing cosmological extensions of the pure BPS three-brane.

The first example is simply the domain wall itself with two of its three moduli made time dependent. We show that a separation of variables occurs in this case. It turns out that these moduli behave like ''rolling radii'' [11] which constitute fundamental cosmological solutions in weakly coupled string theory. Unlike those rolling radii which represent scale factors of homogeneous, isotropic spaces, here they measure the separation of the two walls of the three-brane and its world volume size (which, at the same time, is the size of ''our'' three-dimensional universe). All in all, we therefore have a time-dependent domain wall pair with its shape staying rigid but its size and separation evolving like rolling radii.

For the second example, we consider a similar setting as for the first but, in addition, we allow a nonvanishing ''Ramond-Ramond'' scalar. This terminology is perhaps a little misleading, but relates to the fact that the scalar would be a type II Ramond-Ramond field in the case where the orbifold was replaced by a circle. This makes connection with type II cosmologies with non-trivial Ramond-Ramond fields discussed in [7,8]. Separation of variables occurs for a specific time-independent form of this scalar. The orbifold-dependent part then coincides with the domain wall with, however, the addition of the Ramond-Ramond scalar. This non-vanishing value of the scalar breaks supersymmetry even in the static limit. We find that the time-dependent part of the equations fits into the general scheme of M-theory cosmological solutions with form fields as presented in Refs. [7,8]. Applying the results of these papers, the domain wall moduli are found to behave like rolling radii asymptotically for early and late times. The evolution rates in these asymptotic regions are different and the transitions between them can be attributed to the nontrivial Ramond-Ramond scalar.

Let us now summarize our conventions. We use coordinates x^α with indices $\alpha,\beta,\gamma,\ldots=0,\ldots,3,11$ to parametrize the five-dimensional space M_5. Throughout this paper, when we refer to the orbifold, we will work in the ''upstairs'' picture with the orbifold S^1/Z_2 in the x^{11}-direction. We choose the range $x^{11}\in[-\pi\rho,\pi\rho]$ with the end points being identified. The Z_2 orbifold symmetry acts as $x^{11}\rightarrow -x^{11}$. Then there exist two four-dimensional hyperplanes fixed under the Z_2 symmetry which we denote by $M_4^{(i)}$, $i=1,2$. Locally, they are specified by the conditions $x^{11}=0,\pi\rho$. The indices $\mu,\nu,\rho,\ldots=0,\ldots,3$ are used for the four-dimensional space orthogonal to the orbifold. Fields will be required to have a definite behavior under the Z_2 orbifold symmetry, so that a general field Φ is either even or odd, with $\Phi(x^{11})=\pm\Phi(-x^{11})$.

II. FIVE-DIMENSIONAL EFFECTIVE ACTION

The five-dimensional effective action for Hořava-Witten theory, obtained from the 11-dimensional theory by compactifying on a Calabi-Yau three-fold with standard embedding, was derived in [6] for the universal zero modes, that is, the five-dimensional graviton supermultiplet and the breathing mode of the Calabi-Yau space, along with its superpartners. These last fields form a hypermultiplet in five dimensions. Furthermore, the theory contains four-dimensional $N=1$ gauge multiplets and chiral gauge matter fields on the orbifold planes. To keep the discussion as simple as possible we will omit the gauge matter fields in the effective five-dimensional action since they are of no relevance to the cosmological solutions considered in this paper. The general Lagrangian will be presented elsewhere [12,13].

In detail, we have the five-dimensional gravity supermultiplet with the metric $g_{\alpha\beta}$ and an Abelian gauge field $\mathcal{A}_\alpha$ as the bosonic fields. The bosonic fields in the universal hypermultiplet are the real scalar field V (the dilaton, measuring the volume of the internal Calabi-Yau space), the three-form $C_{\alpha\beta\gamma}$ and the complex Ramond-Ramond scalar ξ. Note that the three-form $C_{\alpha\beta\gamma}$ can be dualized to a scalar field σ. Hence the hypermultiplet contains four real scalar fields. As explained in the Introduction, all bulk fields should be even or odd under the Z_2 orbifold symmetry. One finds that the fields $g_{\mu\nu}$, $g_{11,11}$, $\mathcal{A}_{11}$, σ must be even whereas $g_{\mu 11}$, $\mathcal{A}_\mu$, ξ must be odd. If one studies cosmological solutions of the theory, these transformation properties are important as they restrict the set of allowed solutions to those with the correct Z_2 symmetry. Now consider the boundary theories. In the five-dimensional space M_5, the orbifold fixed planes constitute the four-dimensional hypersurfaces $M_4^{(i)}$, $i=1,2$. Since the standard embedding has been used in the reduction from 11 to 5 dimensions, there is an E_6 gauge field $A_\mu^{(1)}$ and gauge matter fields on the orbifold plane $M_4^{(1)}$. For simplicity, we will set these gauge matter fields to zero in the following. This will not effect our solutions. On the orbifold plane $M_4^{(2)}$ there is an E_8 gauge field $A_\mu^{(2)}$.

The five-dimensional effective action of Hořava-Witten theory is then given by

$$S_5 = S_{\text{bulk}} + S_{\text{bound}} \tag{1}$$

where

$$\begin{aligned} S_{\text{bulk}} = -\frac{1}{2\kappa_5^2}\int_{M_5}\sqrt{-g}\bigg\{ & R + \frac{3}{2}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} \\ & + \frac{1}{\sqrt{2}}\epsilon^{\alpha\beta\gamma\delta\epsilon}\mathcal{A}_\alpha\mathcal{F}_{\beta\gamma}\mathcal{F}_{\delta\epsilon} + \frac{1}{2V^2}\partial_\alpha V\partial^\alpha V \\ & + \frac{1}{2V^2}[\partial_\alpha\sigma - i(\xi\partial_\alpha\bar{\xi} - \bar{\xi}\partial_\alpha\xi) - 2\alpha_0\epsilon(x^{11})\mathcal{A}_\alpha] \\ & \times[\partial^\alpha\sigma - i(\xi\partial^\alpha\bar{\xi} - \bar{\xi}\partial^\alpha\xi) - 2\alpha_0\epsilon(x^{11})\mathcal{A}^\alpha] \\ & + \frac{2}{V}\partial_\alpha\xi\partial^\alpha\bar{\xi} + \frac{1}{3V^2}\alpha_0^2\bigg\} \end{aligned} \tag{2}$$

$$\begin{aligned} S_{\text{bound}} = & \frac{\sqrt{2}}{\kappa_5^2}\int_{M_4^{(1)}}\sqrt{-g}V^{-1}\alpha_0 - \frac{\sqrt{2}}{\kappa_5^2}\int_{M_4^{(2)}}\sqrt{-g}V^{-1}\alpha_0 \\ & - \frac{1}{16\pi\alpha_{\text{GUT}}}\sum_{i=1}^{2}\int_{M_4^{(i)}}\sqrt{-g}\{V\text{tr}F_{\mu\nu}^{(i)}F^{(i)\mu\nu} \\ & - \sigma\text{tr}F_{\mu\nu}^{(i)}\tilde{F}^{(i)\mu\nu}\}. \end{aligned} \tag{3}$$

where $\mathcal{F}_{\alpha\beta} = \partial_\alpha\mathcal{A}_\beta - \partial_\beta\mathcal{A}_\alpha$ and the $F_{\mu\nu}^{(i)}$ are the field strengths of the boundary gauge fields, while $\tilde{F}^{(i)\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}^{(i)}$. Furthermore, κ_5 and α_{GUT} are the five-dimensional Newton constant and the gauge coupling respectively. The constant α_0 in the above action is given by [3,6]

$$\alpha_0 = -\frac{1}{8\sqrt{2}\pi v}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_X \omega\wedge\text{tr}\,R^{(\Omega)}\wedge R^{(\Omega)}, \quad v = \int_X\sqrt{\Omega}. \tag{4}$$

Here $\Omega_{a\bar{b}}$ is the metric of the Calabi-Yau space X, $R^{(\Omega)}$ is the corresponding curvature two-form and $\omega_{a\bar{b}} = i\Omega_{a\bar{b}}$ is the Kähler form. Furthermore, κ is the 11-dimensional Newton constant. We remark that α_0 is related to the presence of internal gravity and gauge field instantons. It can be expressed solely in terms of the curvature since the standard embedding relates those two types of instantons. In the above action, we have dropped higher-derivative terms. The sigma model for the scalar fields is the well-known coset $\mathcal{M}_Q = SU(2,1)/SU(2)\times U(1)$ of the universal hypermultiplet. The coupling of σ to $\mathcal{A}_\alpha$ implies that a $U(1)$ symmetry on $\mathcal{M}_Q$ has been gauged. This gauging also induces the α_0-dependent potential term in Eq. (3). It has been demonstrated [6,13] that the above action is indeed the bosonic part of a minimal $N=1$ gauged supergravity theory in five dimensions coupled to chiral boundary theories.

The most striking features of this action from the viewpoint of cosmology (and otherwise) are the bulk and boundary potentials for the dilaton V in S_{bulk} and S_{bound}. These potential terms are proportional to the parameter α_0 and their origin is directly related to the nonzero internal four-form that had to be included in the dimensional reduction from 11 dimensions. We stress that this non-zero four-form results from the source terms in the 11 dimensional Bianchi identity which are non-vanishing even for the standard embedding which we consider here. For a very similar reason we have non-vanishing boundary potentials. They arise from the internal part of the 10-dimensional boundary action and include the contributions from the gauge field kinetic terms as well as from the curvature R^2 terms. Observe that these potentials are equal but have opposite signs. Therefore, although they cancel in a four-dimensional limit, they do not cancel separately on each boundary. These potentials lead to sources in the Einstein equation and the equation of motion for V and σ that are proportional to $\delta(x^{11})$ or $\delta(x^{11}-\pi\rho)$. Hence, as long as V is finite (the internal Calabi-Yau space is compact) purely time-dependent solutions of the theory do not exist as they could never cancel these delta-function sources. One is therefore led to always consider dependence on time and the orbifold coordinate x^{11}. The presence of a bulk potential proportional to V^{-2} seems to indicate that the dilaton has a runaway behavior and the internal space decompactifies at late time. This picture, however, is too naive in that it ignores the boundary potentials and the Z_2 symmetries of the fields. In fact, as we will show, the correct static domain wall vacuum of the theory depends on the orbifold direction in a way so as to exactly cancel these potentials. Consequently, it is important to note that for cosmological solutions based on the domain wall the time-dependent scale factors do not feel the potential terms.

III. DOMAIN-WALL VACUUM SOLUTION

In this section, we would like to review the static ''vacuum'' solution of the five-dimensional Hořava-Witten theory, as given in [6]. As argued in the Introduction, this solution is the basis for physically relevant cosmological solutions. It is clear from the five-dimensional action given in the previous section that flat spacetime is not a solution of the equations of motion. It is precluded from being a solution by the potential terms, both in the bulk and on the boundaries. If not flat space, what is the natural vacuum solution? To answer this, notice that the theory (1) has all of the prerequisites necessary for a three-brane solution to exist. Generally, in order to have a $(D-2)$-brane in a D-dimensional theory, one needs to have a $(D-1)$-form field or, equivalently, a cosmological constant. This is familiar from the eight-brane [14] in the massive type IIA supergravity in ten dimensions [15], and has been systematically studied for theories in arbitrary dimension obtained by generalized dimensional reduction [16] using the method of Scherk and Schwarz [17]. In our case, this cosmological term is provided by the bulk potential term in the action (1), precisely the term that disallowed flat space as a solution. From the viewpoint of the bulk theory, we could have multi three-brane solutions with an arbitrary number of parallel branes located at various places in the x^{11} direction. However, elementary brane solutions have singularities at the location of the branes, needing to be supported by source terms. The natural candidates for those source terms, in our case, are the boundary actions. This restricts the possible solutions to those representing a pair of parallel three-branes corresponding to the orbifold planes. This pair of domain walls can be viewed as the ''vacuum'' of the five-dimensional theory, in the sense that it provides the appropriate background for a reduction to the $d=4$, $N=1$ effective theory.

From the above discussion, it is clear that in order to find a three-brane solution, we should start with the *Ansatz*

$$ds_5^2=a(y)^2dx^\mu dx^\nu\eta_{\mu\nu}+b(y)^2dy^2$$

$$V=V(y) \tag{5}$$

where a and b are functions of $y=x^{11}$ and all other fields vanish. The general solution for this *Ansatz*, satisfying the equations of motion derived from action (1), is given by

$$a=a_0H^{1/2}$$

$$b=b_0H^2, \quad H=\frac{\sqrt{2}}{3}\alpha_0|y|+h_0 \tag{6}$$

$$V=b_0H^3$$

where a_0, b_0 and h_0 are constants. We note that the boundary source terms have fixed the form of the harmonic function H in the above solution. Without specific information about the sources, the function H would generically be glued together from an arbitrary number of linear pieces with slopes $\pm\sqrt{2}\alpha_0/3$. The edges of each piece would then indicate the location of the source terms. The necessity of matching the boundary sources at $y=0$ and $\pi\rho$, however, has forced us to consider only two such linear pieces, namely $y\in[0,\pi\rho]$ and $y\in[-\pi\rho,0]$. These pieces are glued together at $y=0$ and $\pi\rho$ (recall here that we have identified $\pi\rho$ and $-\pi\rho$). To see this explicitly, let us consider one of the equations of motion, specifically, the equation derived from the variation of $g_{\mu\nu}$. For the *Ansatz* in Eq. (5), this is given by

$$\frac{a''}{a}+\frac{a'^2}{a^2}-\frac{a'}{a}\frac{b'}{b}+\frac{1}{12}\frac{V'^2}{V^2}+\frac{\alpha_0^2}{18}\frac{b^2}{V^2}$$

$$=\frac{\sqrt{2}\alpha_0}{3}\frac{b}{V}[\delta(y)-\delta(y-\pi\rho)] \tag{7}$$

where the prime denotes differentiation with respect to y. The term involving the delta functions arises from the stress energy on the boundary planes. Inserting the solution (6) in this equation, we have

$$\partial_y^2H=\frac{2\sqrt{2}}{3}\alpha_0[\delta(y)-\delta(y-\pi\rho)] \tag{8}$$

which shows that the solution represents two parallel three-branes located at the orbifold planes. Using the five-dimensional supersymmetry transformations presented in Ref. [6], one can check that this solution indeed preserves four of the eight supersymmetries of the theory.

Let us discuss the meaning of this solution. As is apparent from the *Ansatz* (5), it has $(3+1)$-dimensional Poincaré invariance and, as just stated, it preserves four supercharges. Therefore, a dimensional reduction to four dimensions in this solution leads to an $N=1$ supergravity theory. In fact, this is just the same ''physical'' four-dimensional effective theory that one obtains by reducing Hořava-Witten theory directly from 11 to 4 dimensions using the background of Ref. [3]. This has been explicitly demonstrated in Refs. [12,13]. This effective four-dimensional theory is the starting point of low energy particle phenomenology. Indeed, the linearized version of the five-dimensional domain wall solution is nothing else but the zero mode part of the 11 dimensional solution of Ref. [3] ''pulled'' down to five dimensions [13]. The two parallel three-branes of the solution, separated by the bulk, are oriented in the four uncompactified space-time dimensions, and carry the physical low-energy gauge and matter fields. Therefore, from the low-energy point of view where the orbifold is not resolved, the three-brane world volume is identified with four-dimensional space-time. In this sense the Universe resides on the world volume of a three-brane. It is the purpose of the following sections to put this picture into the context of cosmology, that is, to make it dynamical. Consequently, we are looking for time dependent solutions based on the static domain wall which we have just presented.

IV. DOMAIN-WALL COSMOLOGICAL SOLUTION

In this section, we will present a cosmological solution related to the static domain wall vacuum of the previous section. As discussed in Refs. [7,8], a convenient way to find such a solution is to use *Ansatz* (5) where the $y=x^{11}$ coordinate in the functions a, b and V is replaced by the time coordinate τ. However, in Hořava-Witten theory the boundary planes preclude this from being a solution of the equations of motion, since it does not admit homogeneous solutions. To see this explicitly, let us consider the g_{00} equation of motion, where we replace $a(y)\to\alpha(\tau)$, $b(y)\to\beta(\tau)$ and $V\to\gamma(\tau)$. We find that

$$\frac{\dot{\alpha}^2}{\alpha^2}+\frac{\dot{\alpha}}{\alpha}\frac{\dot{\beta}}{\beta}-\frac{1}{12}\frac{\dot{\gamma}^2}{\gamma^2}-\frac{\alpha_0^2}{18}\frac{1}{\gamma^2} =-\frac{\sqrt{2}\alpha_0}{3}\frac{1}{\beta\gamma}[\delta(y)-\delta(y-\pi\rho)], \tag{9}$$

where the overdot denotes differentiation with respect to τ. Again, the term containing the delta functions arises from the boundary planes. It is clear that, because of the y dependence introduced by the delta functions, this equation has no globally defined solution. The structure of Eq. (9) suggests that a solution might be found if one were to let functions a, b and V depend on both τ and y coordinates. This would be acceptable from the point of view of cosmology, since any such solution would be homogeneous and isotropic in the spatial coordinates x^m where $m,n,r,\ldots=1,2,3$. In fact, the previous *Ansatz* was too homogeneous, being independent of the y coordinate as well. Instead, we are interested in solutions where the inhomogeneous vacuum domain wall evolves in time.

We now construct a cosmological solution where all functions depend on both τ and y. We start with the *Ansatz*

$$ds_5^2=-N(\tau,y)^2d\tau^2+a(\tau,y)^2dx^mdx^n\eta_{mn}+b(\tau,y)^2dy^2$$
$$V=V(\tau,y). \tag{10}$$

Note that we have introduced a separate function N into the purely temporal part of the metric. This *Ansatz* leads to equations of motion that mix the τ and y variables in a complicated non-linear way. In order to solve this system of equations, we will try to separate the two variables. That is, we let

$$\begin{aligned} N(\tau,y)&=n(\tau)a(y)\\ a(\tau,y)&=\alpha(\tau)a(y)\\ b(\tau,y)&=\beta(\tau)b(y)\\ V(\tau,y)&=\gamma(\tau)V(y). \end{aligned} \tag{11}$$

There are two properties of this *Ansatz* that we wish to point out. The first is that for $n=\alpha=\beta=\gamma=1$ it becomes identical to Eq. (5). Second, we note that n can be chosen to be any function by performing a redefinition of the τ variable. That is, we can think of n as being subject to a gauge transformation. There is no *a priori* reason to believe that separation of variables will lead to a solution of the equations of motion derived from the action (1). However, as we now show, there is indeed such a solution. It is instructive to present one of the equations of motion. With the above *Ansatz*, the g_{00} equation of motion is given by[1]

$$\frac{a^2}{b^2}\left(\frac{a''}{a}+\frac{a'^2}{a^2}-\frac{a'}{a}\frac{b'}{b}+\frac{1}{12}\frac{V'^2}{V^2}+\frac{\alpha_0^2}{18}\frac{b^2}{V^2}\frac{\beta^2}{\gamma^2} -\frac{\sqrt{2}}{3}\alpha_0\frac{b}{V}[\delta(y)-\delta(y-\pi\rho)]\frac{\beta}{\gamma}\right) =\frac{\beta^2}{n^2}\left(\frac{\dot{\alpha}^2}{\alpha^2}+\frac{\dot{\alpha}}{\alpha}\frac{\dot{\beta}}{\beta}-\frac{1}{12}\frac{\dot{\gamma}^2}{\gamma^2}\right). \tag{12}$$

Note that if we set $n=\alpha=\beta=\gamma=1$, this equation becomes identical to Eq. (7). Similarly, if we set $a=b=V=1$ and take the gauge $n=1$, this equation becomes the same as Eq. (9). As is, the above equation does not separate. However, the obstruction to a separation of variables is the two terms proportional to α_0. Note that both of these terms would be strictly functions of y only if we demanded that $\beta\propto\gamma$. Without loss of generality, one can take

$$\beta=\gamma. \tag{13}$$

We will, henceforth, assume that this is the case. Note that this result is already indicated by the structure of integration constants (moduli) in the static domain wall solution (6). With this condition, the left hand side of Eq. (12) is purely y dependent, whereas the right hand side is purely τ dependent. Both sides must now equal the same constant which, for simplicity, we take to be zero. The equation obtained by setting the left hand side to zero is identical to the pure y equation (7). The equation for the pure τ dependent functions is

$$\frac{\dot{\alpha}^2}{\alpha^2}+\frac{\dot{\alpha}}{\alpha}\frac{\dot{\beta}}{\beta}-\frac{1}{12}\frac{\dot{\gamma}^2}{\gamma^2}=0. \tag{14}$$

Hence, separation of variables can be achieved for the g_{00} equation by demanding that Eq. (13) be true. What is more remarkable is that, subject to the constraint that $\beta=\gamma$, all the equations of motion separate. The pure y equations are identical to those of the previous section and, hence, the domain wall solution (6) remains valid as the y-dependent part of the solution.

The full set of τ equations is found to be

$$\frac{\dot{\alpha}^2}{\alpha^2}+\frac{\dot{\alpha}}{\alpha}\frac{\dot{\beta}}{\beta}-\frac{1}{12}\frac{\dot{\gamma}^2}{\gamma^2}=0 \tag{15}$$

$$2\frac{\ddot{\alpha}}{\alpha}-2\frac{\dot{\alpha}}{\alpha}\frac{\dot{n}}{n}+\frac{\ddot{\beta}}{\beta}-\frac{\dot{\beta}}{\beta}\frac{\dot{n}}{n}+\frac{\dot{\alpha}^2}{\alpha^2}+2\frac{\dot{\alpha}}{\alpha}\frac{\dot{\beta}}{\beta}+\frac{1}{4}\frac{\dot{\gamma}^2}{\gamma^2}=0 \tag{16}$$

[1]From now on, we denote by a, b, V the y-dependent part of the *Ansatz* (11).

$$\frac{\ddot{\alpha}}{\alpha}-\frac{\dot{\alpha}}{\alpha}\frac{\dot{n}}{n}+\frac{\dot{\alpha}^2}{\alpha^2}+\frac{1}{12}\frac{\dot{\gamma}^2}{\gamma^2}=0 \tag{17}$$

$$\frac{\ddot{\gamma}}{\gamma}+3\frac{\dot{\alpha}\dot{\gamma}}{\alpha\gamma}+\frac{\dot{\beta}\dot{\gamma}}{\beta\gamma}-\frac{\dot{\gamma}^2}{\gamma^2}-\frac{\dot{n}\dot{\gamma}}{n\gamma}=0. \tag{18}$$

In these equations we have displayed β and γ independently, for reasons to become apparent shortly. Of course, one must solve these equations subject to the condition that $\beta=\gamma$. As a first attempt to solve these equations, it is most convenient to choose a gauge for which

$$n=\text{const} \tag{19}$$

so that τ becomes proportional to the comoving time t, since $dt=n(\tau)d\tau$. In such a gauge, the equations simplify considerably and we obtain the solution

$$\begin{aligned}\alpha&=A|t-t_0|^p\\ \beta&=\gamma=B|t-t_0|^q\end{aligned} \tag{20}$$

where

$$p=\frac{3}{11}\left(1\mp\frac{4}{3\sqrt{3}}\right),\quad q=\frac{2}{11}(1\pm2\sqrt{3}) \tag{21}$$

and A, B and t_0 are arbitrary constants. We have therefore found a cosmological solution, based on the separation *Ansatz* (11), with the y-dependent part being identical to the domain wall solution (6) and the scale factors α, β, γ evolving according to the power laws (20). This means that the shape of the domain wall pair stays rigid while its size and the separation between the walls evolve in time. Specifically, α measures the size of the spatial domain wall world volume (the size of the three-dimensional universe), while β specifies the separation of the two walls (the size of the orbifold). Because of the separation constraint $\gamma=\beta$, the time evolution of the Calabi-Yau volume, specified by γ, is always tracking the orbifold. From this point of view, we are allowing two of the three moduli in Eq. (6), namely a_0 and b_0, to become time dependent. Since these moduli multiply the harmonic function H, it is then easy to see why a solution by separation of variables was appropriate.

To understand the structure of the above solution, it is useful to rewrite its time dependent part in a more systematic way using the formalism developed in Refs. [7,8]. First, let us define new functions $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ by

$$\alpha=e^{\hat{\alpha}},\quad \beta=e^{\hat{\beta}},\quad \gamma=e^{6\hat{\gamma}} \tag{22}$$

and introduce the vector notation

$$\vec{\alpha}=(\alpha^i)=\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\gamma}\end{pmatrix},\quad \vec{d}=(d_i)=\begin{pmatrix}3\\1\\0\end{pmatrix}. \tag{23}$$

Note that the vector $\vec{d}$ specifies the dimensions of the various subspaces, where the entry $d_1=3$ is the spatial world volume dimensions, $d_2=1$ is the orbifold dimension and we insert 0 for the dilaton. On the ''moduli space'' spanned by $\vec{\alpha}$ we introduce the metric

$$\begin{aligned}G_{ij}&=2(d_i\delta_{ij}-d_id_j)\\ G_{in}&=G_{ni}=0\\ G_{nn}&=36,\end{aligned} \tag{24}$$

which in our case explicitly reads

$$G=-12\begin{pmatrix}1&\frac{1}{2}&0\\ \frac{1}{2}&0&0\\ 0&0&-3\end{pmatrix}. \tag{25}$$

Furthermore, we define E by

$$E=\frac{e^{\vec{d}\cdot\vec{\alpha}}}{n}=\frac{e^{3\hat{\alpha}+\hat{\beta}}}{n}. \tag{26}$$

The equations of motion (15)–(18) can then be rewritten as

$$\frac{1}{2}E\dot{\vec{\alpha}}^TG\dot{\vec{\alpha}}=0,\quad \frac{d}{d\tau}(EG\dot{\vec{\alpha}})=0. \tag{27}$$

It is straightforward to show that if we choose a gauge $n=\text{const}$, these two equations exactly reproduce the solution given in Eqs. (20) and (21). The importance of this reformulation of the equations of motion lies, however, in the fact that we now get solutions more easily by exploiting the gauge choice for n. For example, let us now choose the gauge

$$n=e^{\vec{d}\cdot\vec{\alpha}}. \tag{28}$$

Note that in this gauge $E=1$. The reader can verify that this gauge choice greatly simplifies solving the equations. The result is that

$$\begin{aligned}\hat{\alpha}&=6\hat{\gamma}=C\tau+k_1\\ \hat{\beta}&=(6\pm4\sqrt{3})C\tau+k_2\end{aligned} \tag{29}$$

where C, k_1 and k_2 are arbitrary constants. Of course this solution is completely equivalent to the previous one, Eq. (20), but written in a different gauge. We will exploit this gauge freedom to effect in the next section.

To discuss cosmological properties we define the Hubble parameters

$$\vec{H}=\frac{d}{dt}\vec{\alpha} \tag{30}$$

where t is the comoving time. From Eqs. (20) and (22) we easily find

$$\vec{H}=\frac{\vec{p}}{t-t_0}, \quad \vec{p}=\begin{pmatrix} p \\ q \\ \frac{1}{6}q \end{pmatrix}. \tag{31}$$

Note that the powers $\vec{p}$ satisfy the constraints

$$\vec{p}^T G\vec{p}=0, \quad \vec{d}\cdot\vec{p}=1. \tag{32}$$

These relations are characteristic for rolling radii solutions [11] which are fundamental cosmological solutions of weakly coupled heterotic string theory. Comparison of the equations of motion (27) indeed shows that the scale factors $\vec{\alpha}$ behave like rolling radii. The original rolling radii solutions describe freely evolving scale factors of a product of homogeneous, isotropic spaces. In our case, the scale factors also evolve freely (since the time-dependent part of the equations of motion, obtained after separating variables, does not contain a potential) but they describe the time evolution of the domain wall. This also proves our earlier claim that the potential terms in the five-dimensional action (1) do not directly influence the time dependence but are canceled by the static domain wall part of the solution.

Let us now be more specific about the cosmological properties of our solution. First note from Eq. (31) that there exist two different types of time ranges, namely $t<t_0$ and $t>t_0$. In the first case, which we call the $(-)$ branch, the evolution starts at $t\to-\infty$ and runs into a future curvature singularity [7,8] at $t=t_0$. In the second case, called the $(+)$ branch, we start out in a past curvature singularity at $t=t_0$ and evolve toward $t\to\infty$. In summary, we therefore have the branches

$$t\in\begin{cases}[-\infty,t_0] & (-)\ \text{branch},\\ [t_0,+\infty] & (+)\ \text{branch}.\end{cases} \tag{33}$$

For both of these branches we have two options for the powers $\vec{p}$, defined in Eq. (31), corresponding to the two different signs in Eq. (21). Numerically, we find

$$\vec{p}_\uparrow\simeq\begin{pmatrix} +.06 \\ +.81 \\ +.14 \end{pmatrix}, \quad \vec{p}_\downarrow\simeq\begin{pmatrix} +.48 \\ -.45 \\ -.08 \end{pmatrix} \tag{34}$$

for the upper and lower sign in Eq. (21) respectively. We recall that the three entries in these vectors specify the evolution powers for the spatial world volume of the three-brane, the domain wall separation and the Calabi-Yau volume. The expansion of the domain wall world volume has so far been measured in terms of the five-dimensional Einstein frame metric $g^{(5)}_{\mu\nu}$. This is also what the above numbers p_1 reflect. Alternatively, one could measure this expansion with the four-dimensional Einstein frame metric $g^{(4)}_{\mu\nu}$ so that the curvature scalar on the world volume is canonically normalized. From the relation

$$g^{(4)}_{\mu\nu}=(g_{11,11})^{1/2}g^{(5)}_{\mu\nu} \tag{35}$$

we find that this modifies p_1 to

$$\tilde{p}_1=p_1+\frac{p_2}{2}. \tag{36}$$

In the following, we will discuss both frames. We recall that the separation condition $\beta=\gamma$ implies that the internal Calabi-Yau space always tracks the orbifold. In the discussion we can, therefore, concentrate on the spatial world volume and the orbifold, corresponding to the first and second entries in Eq. (34). Let us first consider the $(-)$ branch. In this branch $t\in[-\infty,t_0]$ and, hence, $t-t_0$ is always negative. It follows from Eq. (31) that a subspace will expand if its $\vec{p}$ component is negative and contract if it is positive. For the first set of powers $\vec{p}_\uparrow$ in Eq. (34) both the world volume and the orbifold contract in the five-dimensional Einstein frame. The same conclusion holds in the four-dimensional Einstein frame. For the second set, $\vec{p}_\downarrow$, in both frames the world volume contracts while the orbifold expands. Furthermore, since the Hubble parameter of the orbifold increases in time, the orbifold undergoes superinflation.

Now we turn to the $(+)$ branch. In this branch $t\in[t_0,\infty]$ and, hence, $t-t_0$ is always positive. Consequently, a subspace expands for a positive component of $\vec{p}$ and contracts otherwise. In addition, since the absolute values of all powers $\vec{p}$ are smaller than 1, an expansion is always subluminal. For the vector $\vec{p}_\uparrow$ the world volume and the orbifold expand in both frames. On the other hand, the vector $\vec{p}_\downarrow$ describes an expanding world volume and a contracting orbifold in both frames. This last solution perhaps corresponds most closely to our notion of the early universe.

V. COSMOLOGICAL SOLUTIONS WITH RAMOND FORMS

Thus far, we have looked for both static and cosmological solutions where the form fields ξ, $\mathcal{A}_\alpha$ and σ have been set to zero. As discussed in previous papers [7,8], turning on one or several such fields can drastically alter the solutions and their cosmological properties. Hence, we would like to explore cosmological solutions with such non-trivial fields. For clarity, in this paper we will restrict the discussion to turning on the Ramond-Ramond scalar ξ only, postponing the general discussion to another publication.

The *Ansatz* we will use is the following. For the metric and dilaton field, we choose

$$ds_5^2=-N(\tau,y)^2d\tau^2+a(\tau,y)^2dx^m dx^n\eta_{mn}+b(\tau,y)^2dy^2$$

$$V=V(\tau,y). \tag{37}$$

For the ξ field, we assume that $\xi=\xi(\tau,y)$ and, hence, the field strength $F_\alpha=\partial_\alpha\xi$ is given by

$$F_0=Y(\tau,y), \quad F_5=X(\tau,y). \tag{38}$$

All other components of F_α vanish. Note that since ξ is complex, both X and Y are complex. Once again, we will solve the equations of motion by separation of variables. That is, we let

$$N(\tau,y)=n(\tau)N(y)$$

$$a(\tau,y)=\alpha(\tau)a(y)$$

$$b(\tau,y)=\beta(\tau)b(y) \tag{39}$$

$$V(\tau,y)=\gamma(\tau)V(y)$$

and

$$X(\tau,y)=\chi(\tau)X(y) \tag{40}$$

$$Y(\tau,y)=\phi(\tau)Y(y). \tag{41}$$

Note that, in addition to the ξ field, we have also allowed for the possibility that $N(y)\neq a(y)$. Again, there is no *a priori* reason to believe that a solution can be found by separation of variables. However, as above, there is indeed such a solution, although the constraints required to separate variables are more subtle. It is instructive to present one of the equations of motion. With the above *Ansatz*, the g_{00} equation of motion becomes[2]

$$\frac{N^2}{b^2}\left(\frac{a''}{a}+\frac{a'^2}{a^2}-\frac{a'}{a}\frac{b'}{b}+\frac{1}{12}\frac{V'^2}{V^2}+\frac{\alpha_0^2}{18}\frac{b^2}{V^2}\frac{\beta^2}{\gamma^2}-\frac{\sqrt{2}\alpha_0}{3}\frac{b}{V}[\delta(y)-\delta(y-\pi\rho)]\frac{\beta}{\gamma}\right)$$
$$=\frac{\beta^2}{n^2}\left(\frac{\dot\alpha^2}{\alpha^2}+\frac{\dot\alpha}{\alpha}\frac{\dot\beta}{\beta}-\frac{1}{12}\frac{\dot\gamma^2}{\gamma^2}\right)-\frac{N^2}{3b^2}\frac{|X|^2}{V}\frac{|\chi|^2}{\gamma}-\frac{\beta^2}{3n^2}\frac{|Y|^2}{V}\frac{|\phi|^2}{\gamma}. \tag{42}$$

Note that if we set $X=Y=0$ and $N=a$, this equation becomes identical to Eq. (12). We now see that there are two different types of obstructions to the separation of variables. The first type, which we encountered in the previous section, is in the two terms proportional to α_0. Clearly, we can separate variables only if we demand that

$$\beta=\gamma \tag{43}$$

as we did previously. However, for non-vanishing X and Y this is not sufficient. The problem, of course, comes from the last two terms in Eq. (42). There are a number of options one could try in order to separate variables in these terms. It is important to note that X and Y are not completely independent, but are related to each other by the integrability condition $\partial_\tau X(\tau,y)=\partial_y Y(\tau,y)$. We find that, because of this condition, it is impossible to obtain a solution by separation of variables that has both $X(\tau,y)$ and $Y(\tau,y)$ non-vanishing. Now $X(\tau,y)$, but not $Y(\tau,y)$, can be made to vanish by taking $\xi=\xi(\tau)$; that is, ξ is a function of τ only. However, we can find no solution by separation of variables under this circumstance. Thus, we are finally led to the choice $\xi=\xi(y)$. In this case $Y(\tau,y)=0$ and we can, without loss of generality, choose

$$\chi=1. \tag{44}$$

At this point, the only obstruction to separation of variables in Eq. (42) is the next to last term, $N^2|X|^2/3b^2V\gamma$. Setting $\gamma=$const is too restrictive, so we must demand that

$$X=\frac{bV^{1/2}}{N}c_0e^{i\theta(y)} \tag{45}$$

where c_0 is a non-zero but otherwise arbitrary real constant and $\theta(y)$ is an, as yet, undetermined phase. Putting this condition into the ξ equation of motion

$$\partial_y\left(\frac{a^3N}{bV}X\right)=0 \tag{46}$$

we find that θ is a constant θ_0 and $a\propto V^{1/6}$ with an arbitrary coefficient. Note that the last condition is consistent with the static vacuum solution (6). Inserting this result into the g_{05} equation of motion

$$\frac{\dot\alpha}{\alpha}\left(\frac{a'}{a}-\frac{N'}{N}\right)=\frac{\dot\beta}{\beta}\left(\frac{a'}{a}-\frac{1}{6}\frac{V'}{V}\right) \tag{47}$$

we learn that $N\propto a$ with an arbitrary coefficient. Henceforth, we choose $N=a$ which is consistent with the static vacuum solution (6). Inserting all of these results, the g_{00} equation of motion now becomes

$$\frac{a^2}{b^2}\left(\frac{a''}{a}+\frac{a'^2}{a^2}-\frac{a'}{a}\frac{b'}{b}+\frac{1}{12}\frac{V'^2}{V^2}+\frac{\alpha_0^2}{18}\frac{b^2}{V^2}-\frac{\sqrt{2}}{3}\alpha_0\frac{b}{V}[\delta(y)-\delta(y-\pi\rho)]\right)$$
$$=\frac{\beta^2}{n^2}\left(\frac{\dot\alpha^2}{\alpha^2}+\frac{\dot\alpha}{\alpha}\frac{\dot\beta}{\beta}-\frac{1}{12}\frac{\dot\gamma^2}{\gamma^2}\right)-\frac{c_0^2}{3}\frac{1}{\gamma}. \tag{48}$$

Note that the left-hand side is of the same form as the static vacuum equation (7). The effect of turning on the ξ background is to add a purely τ dependent piece to the right hand side. Putting these results into the remaining four equations of motion, we find that they too separate, with the left hand side being purely y dependent and the right hand side purely τ dependent. Again, we find that in these equations the left hand sides are identical to those in the static vacuum equations and the effect of turning on ξ is to add extra τ dependent terms to the right hand sides. In each equation, both

[2]In the following, N, a, b, V denote the y-dependent part of the *Ansatz* (39).

sides must now equal the same constant which, for simplicity, we take to be zero. The y equations for a, b and V thus obtained by setting the left hand side to zero are identical to the static vacuum equations. Hence, we have shown that

$$\begin{aligned} N&=a=a_0H^{1/2}\\ b&=b_0H^2, \quad H=\frac{\sqrt{2}}{3}\alpha_0|y|+h_0\\ V&=b_0H^3\\ X&=x_0H^3 \end{aligned} \tag{49}$$

where $x_0=c_0e^{i\theta_0}a_0^{-1}b_0^{3/2}$ is an arbitrary constant.

The τ equations obtained by setting the right hand side to zero are the following:

$$\frac{\dot{\alpha}^2}{\alpha^2}+\frac{\dot{\alpha}}{\alpha}\frac{\dot{\beta}}{\beta}-\frac{1}{12}\frac{\dot{\gamma}^2}{\gamma^2}-\frac{c_0^2}{3}\frac{n^2}{\beta^2\gamma}=0 \tag{50}$$

$$2\frac{\ddot{\alpha}}{\alpha}-2\frac{\dot{\alpha}}{\alpha}\frac{\dot{n}}{n}+\frac{\ddot{\beta}}{\beta}-\frac{\dot{\beta}}{\beta}\frac{\dot{n}}{n}+\frac{\dot{\alpha}^2}{\alpha^2}+2\frac{\dot{\alpha}}{\alpha}\frac{\dot{\beta}}{\beta}+\frac{1}{4}\frac{\dot{\gamma}^2}{\gamma^2}-c_0^2\frac{n^2}{\beta^2\gamma}=0 \tag{51}$$

$$\frac{\ddot{\alpha}}{\alpha}-\frac{\dot{\alpha}}{\alpha}\frac{\dot{n}}{n}+\frac{\dot{\alpha}^2}{\alpha^2}+\frac{1}{12}\frac{\dot{\gamma}^2}{\gamma^2}+\frac{c_0^2}{3}\frac{n^2}{\beta^2\gamma}=0 \tag{52}$$

$$\frac{\ddot{\gamma}}{\gamma}+3\frac{\dot{\alpha}\dot{\gamma}}{\alpha\gamma}+\frac{\dot{\beta}\dot{\gamma}}{\beta\gamma}-\frac{\dot{\gamma}^2}{\gamma^2}-\frac{\dot{n}\dot{\gamma}}{n\gamma}-2c_0^2\frac{n^2}{\beta^2\dot{\gamma}}=0. \tag{53}$$

In these equations we have, once again, displayed β and γ independently, although they should be solved subject to the condition $\beta=\gamma$. Note that the above equations are similar to the τ equations in the previous section, but each now has an additional term proportional to c_0^2. These extra terms considerably complicate finding a solution of the τ equations. Here, however, is where the formalism introduced in the previous section becomes important. Defining $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ as in Eq. (22), and $\vec{\alpha}$, E and G as in Eqs. (23), (26) and (25) respectively, Eqs. (50)–(53) can be written in the form

$$\frac{1}{2}E\dot{\vec{\alpha}}^TG\dot{\vec{\alpha}}+E^{-1}U=0, \quad \frac{d}{d\tau}(EG\dot{\vec{\alpha}})+E^{-1}\frac{\partial U}{\partial\vec{\alpha}}=0 \tag{54}$$

where the potential U is defined as

$$U=2c_0^2e^{\vec{q}\cdot\vec{\alpha}} \tag{55}$$

with

$$\vec{q}=\begin{pmatrix}6\\0\\-6\end{pmatrix}. \tag{56}$$

We can now exploit the gauge freedom of n to simplify these equations. Choose the gauge

$$n=e^{(\vec{d}-\vec{q})\cdot\vec{\alpha}} \tag{57}$$

where $\vec{d}$ is defined in Eq. (23). Then E becomes proportional to the potential U so that the potential terms in Eq. (54) turn into constants. Thanks to this simplification, the equations of motion can be integrated which leads to the general solution [7,8]

$$\vec{\alpha}=\vec{c}\ln|\tau_1-\tau|+\vec{w}\ln\left(\frac{s\tau}{\tau_1-\tau}\right)+\vec{k} \tag{58}$$

where τ_1 is an arbitrary parameter which we take, without loss of generality, to be positive and

$$\vec{c}=2\frac{G^{-1}\vec{q}}{\langle\vec{q},\vec{q}\rangle}, \quad s=\mathrm{sgn}(\langle\vec{q},\vec{q}\rangle). \tag{59}$$

The scalar product is defined as $\langle\vec{q},\vec{q}\rangle=\vec{q}^TG^{-1}\vec{q}$. The vectors $\vec{w}$ and $\vec{k}$ are integration constants subject to the constraints

$$\begin{aligned} \vec{q}\cdot\vec{w}&=1\\ \vec{w}^TG\vec{w}&=0\\ \vec{q}\cdot\vec{k}&=\ln(c_0^2|\langle\vec{q},\vec{q}\rangle|). \end{aligned} \tag{60}$$

This solution is quite general in that it describes an arbitrary number of scale factors with equations of motion given by Eq. (54). Let us now specify to our example. For G and $\vec{q}$ as given in Eqs. (25) and (56) we find that

$$\langle\vec{q},\vec{q}\rangle=1; \tag{61}$$

hence $s=1$, and

$$\vec{c}=\begin{pmatrix}0\\-2\\-\frac{1}{3}\end{pmatrix}. \tag{62}$$

Recall that we must, in addition, demand that $\beta=\gamma$. Note that the last two components of $\vec{c}$ are consistent with this equality. We can also solve the constraints (60) subject to the condition $\beta=\gamma$. The result is

$$\vec{w}=\begin{pmatrix}w_3+\frac{1}{6}\\6w_3\\w_3\end{pmatrix}, \quad \vec{k}=\begin{pmatrix}k_3+\frac{1}{6}\ln c_0^2\\6k_3\\k_3\end{pmatrix} \tag{63}$$

where

$$w_3 = -\frac{1}{6} \pm \frac{\sqrt{3}}{12} \tag{64}$$

and k_3 is arbitrary. We conclude that in the gauge specified by Eq. (57), the solution is given by

$$\hat{\alpha} = \left(w_3 + \frac{1}{6}\right)\ln\left(\frac{\tau}{\tau_1 - \tau}\right) + k_3 + \frac{1}{6}\ln c_0^2$$

$$\hat{\beta} = -2\ln|\tau_1 - \tau| + 6w_3\ln\left(\frac{\tau}{\tau_1 - \tau}\right) + 6k_3$$

$$\hat{\gamma} = -\frac{1}{3}\ln|\tau_1 - \tau| + w_3\ln\left(\frac{\tau}{\tau_1 - \tau}\right) + k_3 \tag{65}$$

with w_3 as above. As a consequence of $s=1$, the range for τ is restricted to

$$0 < \tau < \tau_1 \tag{66}$$

in this solution. Let us now summarize our result. We have found a cosmological solution with a nontrivial Ramond-Ramond scalar ξ starting with the separation *Ansatz* (39). To achieve a separation of variables we had to demand that $\beta = \gamma$, as previously, and that the Ramond-Ramond scalar depend on the orbifold coordinate but not on time. Then the orbifold dependent part of the solution is given by Eq. (49) and is identical to the static domain wall solution with the addition of the Ramond-Ramond scalar. The time dependent part, in the gauge (57), is specified by Eq. (65). Furthermore, we have found that the time-dependent part of the equations of motion can be cast in a form familiar from cosmological solutions studied previously [7,8]. Those solutions describe the evolution for scale factors of homogeneous, isotropic subspaces in the presence of antisymmetric tensor fields and are, therefore, natural generalizations of the rolling radii solutions. Each antisymmetric tensor field introduces an exponential type potential similar to the one in Eq. (55). For the case with only one nontrivial form field, the general solution could be found and is given by Eq. (58). We have, therefore, constructed a strong coupling version of these generalized rolling radii solutions with a one-form field strength, where the radii now specify the domain wall geometry rather that the size of maximally symmetric subspaces. We stress that the potential U in the time-dependent equations of motion does not originate from the potentials in the action (1) but from the nontrivial Ramond-Ramond scalar. The potentials in the action are canceled by the static domain wall part of the solution, as in the previous example.

From the similarity to the known generalized rolling radii solutions, we can also directly infer some of the basic cosmological properties of our solution, using the results of Refs. [7,8]. We expect the integration constants to split into two disjunct sets which lead to solutions in the $(-)$ branch, comoving time range $t \in [-\infty, t_0]$, and the $(+)$ branch, comoving time range $t \in [t_0, \infty]$, respectively. The $(-)$ branch ends in a future curvature singularity and the $(+)$ branch starts in a past curvature singularity. In both branches the solutions behave like rolling radii solutions asymptotically, that is, at $t \to -\infty, t_0$ in the $(-)$ branch and at $t \to t_0, \infty$ in the $(+)$ branch. The two asymptotic regions in both branches have different expansion properties in general and the transition between them can be attributed to the nontrivial form field.

Let us now analyze this in more detail for our solution, following the method presented in Refs. [7,8]. First we should express our solution in terms of the comoving time t by integrating $dt = n(\tau)d\tau$. The gauge parameter $n(\tau)$ is explicitly given by

$$n = e^{(\vec{d} - \vec{q}) \cdot \vec{k}} |\tau_1 - \tau|^{-x + \Delta - 1} |\tau|^{x-1} \tag{67}$$

where

$$x = \vec{d} \cdot \vec{w}, \quad \Delta = \vec{d} \cdot \vec{c}. \tag{68}$$

Given this expression, the integration cannot easily be performed in general except in the asymptotic regions $\tau \to 0, \tau_1$. These regions will turn out to be precisely the asymptotic rolling-radius limits. Therefore, for our purpose, it suffices to concentrate on those regions. Eq. (67) shows that the resulting range for the comoving time depends on the magnitude of Δ and x (note that Δ is a fixed number, for a given model, whereas x depends on the integration constants). It turns out that for all values of the integration constants we have either $x < \Delta$ or $x > 0 > \Delta$. This splits the space of integration constant into two disjunct sets corresponding to the $(-)$ and the $(+)$ branch as explained before. More precisely, we have the mapping

$$\tau \to t \in \begin{cases} [-\infty, t_0] & \text{for } x < \Delta < 0, \quad (-) \text{ branch}, \\ [t_0, +\infty] & \text{for } x > 0 > \Delta, \quad (+) \text{ branch}, \end{cases} \tag{69}$$

where t_0 is a finite arbitrary time (which can be different for the two branches). We recall that the range of τ is $0 < \tau < \tau_1$. The above result can be easily read off from the expression (67) for the gauge parameter. Performing the integration in the asymptotic region we can express τ in terms of the comoving time and find the Hubble parameters, defined by Eq. (30), and the powers $\vec{p}$. Generally, we have

$$\vec{p} = \begin{cases} \dfrac{\vec{w}}{x} & \text{at} \quad \tau \simeq 0, \\ \dfrac{\vec{w} - \vec{c}}{x - \Delta} & \text{at} \quad \tau \simeq \tau_1. \end{cases} \tag{70}$$

Note that, from the mapping (69), the expression at $\tau \simeq 0$ describes the evolution powers at $t \to -\infty$ in the $(-)$ branch and at $t \simeq t_0$ in the $(+)$ branch, that is, the evolution powers in the early asymptotic region. Correspondingly, the expression for $\tau \simeq \tau_1$ applies to the late asymptotic regions, that is, to $t \simeq t_0$ in the $(-)$ branch and to $t \to \infty$ in the $(+)$ branch. As before, these powers satisfy the rolling radius constraints (32).

Let us now insert the explicit expression for $\vec{d}$, $\vec{w}$ and $\vec{c}$, Eqs. (23), (63) and (62), which specify our example into those formulas. First, from Eq. (68) we find that

$$x=-1\pm 3\frac{\sqrt{3}}{4}, \quad \Delta=-2. \tag{71}$$

Note that the space of integration constants just consists of two points in our case, represented by the two signs in the expression for x above. Clearly, from the criterion (69) the upper sign leads to a solution in the $(+)$ branch and the lower sign to a solution in the $(-)$ branch. In each branch we therefore have a uniquely determined solution. Using Eq. (70) we can calculate the asymptotic evolution powers in the $(-)$ branch:

$$\vec{p}_{-,t\to-\infty}=\begin{pmatrix}+.06\\+.81\\+.13\end{pmatrix}, \quad \vec{p}_{-,t\to t_0}=\begin{pmatrix}+.48\\-.45\\-.08\end{pmatrix}. \tag{72}$$

Correspondingly, for the $(+)$ branch we have

$$\vec{p}_{+,t\to t_0}=\begin{pmatrix}+.48\\-.45\\-.08\end{pmatrix}, \quad \vec{p}_{+,t\to\infty}=\begin{pmatrix}+.06\\+.81\\+.13\end{pmatrix}. \tag{73}$$

Note that these vectors are in fact the same as in the $(-)$ branch, with the time order being reversed. This happens because they are three conditions on the powers $\vec{p}$ that hold in both branches, namely the two rolling radii constraints (32) and the separation constraint $\beta=\gamma$, Eq. (43), which implies that $p_3=6p_2$. Since two of these conditions are linear and one is quadratic, we expect at most two different solutions for $\vec{p}$. As in the previous solution, the time variation of the Calabi-Yau volume (third entry) is tracking the orbifold variation (second entry) as a consequence of the separation condition and, hence, needs not to be discussed separately. The first entry gives the evolution power for the spatial world volume in the five-dimensional Einstein frame. For a conversion to the four-dimensional Einstein frame one should again apply Eq. (36). It is clear from the above numbers, however, that this conversion does not change the qualitative behavior of the world volume evolution in any of the cases. Having said this, let us first discuss the $(-)$ branch. At $t\to-\infty$ the powers are positive and, hence, the world volume and the orbifold are contracting. The solution then undergoes the transition induced by the Ramond-Ramond scalar. Then at $t\simeq t_0$ the world volume is still contracting while the orbifold has turned into superinflating expansion. In the $(+)$ branch we start out with a subluminally expanding world volume and a contracting orbifold at $t\simeq t_0$. After the transition both subspaces have turned into subluminal expansion.

VI. CONCLUSION

In this paper we have presented the first examples of cosmological solutions in five-dimensional Hořava-Witten theory. They are physically relevant in that they are related to the exact BPS three-brane pair in five dimensions, whose $D=4$ world volume theory exhibits $N=1$ supersymmetry. A wider class of such cosmological solutions can be obtained and will be presented elsewhere. We expect solutions of this type to provide the fundamental scaffolding for theories of the early universe derived from Hořava-Witten theory, but they are clearly not sufficient as they stand. The most notable deficiency is the fact that they are vacuum solutions, devoid of any matter, radiation or potential stress energy. Inclusion of such stress energy is essential to understand the behavior of early universe cosmology. A study of its effect on the cosmology of Hořava-Witten theory is presently underway [18].

ACKNOWLEDGMENTS

A.L. is supported in part by the Deutsche Forschungsgemeinschaft DFG. A.L. and B.A.O. are supported in part by the DOE under contract No. DE-AC02-76-ER-03071. D.W. is supported in part by the DOE under contract No. DE-FG02-91ER40671.

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