Light-cone fluctuations in flat spacetimes with nontrivial topology

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The quantum light-cone fluctuations in flat spacetimes with compactified spatial dimensions or with boundaries are examined. The discussion is based upon a model in which the source of the underlying metric fluctuations is taken to be quantized linear perturbations of the gravitational field. General expressions are derived, in the transverse trace-free gauge, for the summation of graviton polarization tensors, and for vacuum graviton two-point functions. Because of the fluctuating light cone, the flight time of photons between a source and a detector may be either longer or shorter than the light propagation time in the background classical spacetime. We calculate the mean deviations from the classical propagation time of photons due to the changes in the topology of the flat spacetime. These deviations are in general larger in the directions in which topology changes occur and are typically of the order of the Planck time, but they can get larger as the travel distance increases. [S0556-2821(99)10118-8]

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I. INTRODUCTION

The existence of fixed light-cone structures is one of the characteristics of classical gravitational theory. Light cones are basically hypersurfaces which distinguish timelike separation from spacelike separation and divide spacetime into causally distinct regions. However, if gravity is to be quantized, it is natural to expect that the quantum metric fluctuations would smear out the light cone, and the concept of a fixed light-cone structure has to be abandoned. Based upon the observation that the ultraviolet divergences of quantum field theory arise from the light-cone singularities of two-point functions, and that quantum fluctuations of the spacetime metric ought to smear out the light cone, thus possibly removing these singularities, Pauli [1] conjectured many years ago that the ultraviolet divergences of quantum field theory might be removed if gravity is quantized. This idea was further explored by several other authors [2–4]. At the present time, this conjecture remains unproven. If light cones fluctuate, so do horizons, which are, of course, light cones. The horizon fluctuations could then presumably lead to information leakage across the black hole in a way that is not allowed by classical physics. Bekenstein and Mukhanov [5] have suggested that horizon fluctuations could result in discreetness of the spectrum of black holes. Since the existence of black hole horizons is the origin of the so-called black hole information paradox, which has been widely discussed in the literature but still remains to be resolved, the study of light-cone fluctuations might help us better understand the problem.

Recently the problem of light-cone fluctuations has been investigated [6,7] in a model of quantum linearized theory of gravity, where the fluctuations are produced by gravitons propagating on a background spacetime. The light cone is smeared out if the linearized gravitational perturbations are quantized. It has been demonstrated that gravitons in a quantum state, such as a squeezed vacuum state, or a thermal state, can produce light-cone fluctuations, thus smearing out the light cone. Because of the fluctuating light cone, the propagation time of a classical light pulse over distance r is no longer precisely r, but undergoes fluctuations around a mean value of r. The fluctuations in the photon arrival time can also be understood as fluctuations in the velocity of light. This model has been applied to study the quantum cosmological and black hole horizon fluctuations [8]. It is interesting to note that recently, the quantum gravitational metric fluctuations have also been discussed within a different context, i.e., a Liouville string formulation of quantum gravity [9,10]. In this paper we shall examine light-cone fluctuations in flat spacetime with nontrivial topology based upon the model proposed in Ref. [6]. In Sec. II, we review the basic formalism and examine its gauge invariance, then derive general expressions for the vacuum graviton two-point functions in the transverse trace-free gauge. In Sec. III we study the light cone fluctuations in flat spacetimes with a compactified spatial dimension, and with a single plane boundary. Our results are summarized and discussed in Sec. VI.

II. BASIC FORMALISM AND GRAVITON TWO-POINT FUNCTION IN TRANSVERSE TRACE-FREE GAUGE

Let us consider a flat background spacetime with a linearized perturbation $h_{\mu\nu}$ propagating upon it, so the spacetime metric may be written as

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=(\eta_{\mu\nu}+h_{\mu\nu})dx^\mu dx^\nu$$
$$=dt^2-d\mathbf{x}^2+h_{\mu\nu}dx^\mu dx^\nu. \tag{1}$$

Let $\sigma(x,x')$ be one half of the squared geodesic separation for any pair of spacetime points x and x', and $\sigma_0(x,x')$ be the corresponding quantity in the flat background. We can expand, in the presence of the perturbation, $\sigma(x,x')$ in powers of $h_{\mu\nu}$ as

$$\sigma=\sigma_0+\sigma_1+\sigma_2+\cdots, \tag{2}$$

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where σ_1 is first order in $h_{\mu\nu}$, etc. We now suppose that the linearized perturbation $h_{\mu\nu}$ is quantized, and that the quantum state $|\psi\rangle$ is a "vacuum" state in the sense that we can decompose $h_{\mu\nu}$ into positive and negative frequency parts $h^+_{\mu\nu}$ and $h^-_{\mu\nu}$, respectively, such that

$$h^+_{\mu\nu}|\psi\rangle = 0, \quad \langle\psi|h^-_{\mu\nu} = 0. \tag{3}$$

It follows immediately that

$$\langle h_{\mu\nu}\rangle = 0 \tag{4}$$

in state $|\psi\rangle$. In general, however, $\langle (h_{\mu\nu})^2\rangle_R \neq 0$, where the expectation value is understood to be suitably renormalized. This reflects the quantum metric fluctuations.

A. Basic formalism and gauge invariance

If we average the retarded Green's function, $G_{ret}(x,x')$, for a massless scalar field, over quantized metric fluctuations, we get [6]

$$\langle G_{ret}(x,x')\rangle = \frac{\theta(t-t')}{8\pi^2}\sqrt{\frac{\pi}{2\langle\sigma_1^2\rangle}}\exp\left(-\frac{\sigma_0^2}{2\langle\sigma_1^2\rangle}\right). \tag{5}$$

This form is valid for the case in which $\langle\sigma_1^2\rangle > 0$. It reveals that the delta-function behavior of the classical Green's function, G_{ret}, has been smeared out into a Gaussian function peaked around the classical light cone. This smearing can be understood as due to the fact that photons may be either slowed down or speeded up by the light-cone fluctuations. Photon propagation now becomes a statistical phenomenon, with some photons traveling slower than the light on the classical spacetime, and others traveling faster. Note that the Gaussian function in Eq. (5) is symmetrical about the classical light cone, $\sigma_0 = 0$, and so the quantum fluctuations are equally likely to produce a time advance as a time delay.

Light-cone fluctuations are in principle observable. It has been shown, by considering light pulses between a source and a detector separated by a distance r, that the mean deviation from the classical propagation time is related to $\langle\sigma_1^2\rangle$ by [6]

$$\Delta t = \frac{\sqrt{\langle\sigma_1^2\rangle}}{r}. \tag{6}$$

Note, however, that Δt is the ensemble averaged deviation, not necessarily the expected variation in flight time of two photons emitted close together in time. The latter can be much smaller than Δt due to the fact that the gravitational field may not fluctuate significantly in the interval between the two photons. This point is discussed in detail in Ref. [7]. In order to find Δt in a particular situation, we need to calculate the quantum expectation value $\langle\sigma_1^2\rangle$ in a chosen quantum state. For this purpose, we first have to compute σ_1 for a given classical perturbation along a certain geodesic, then average σ_1^2 over the quantized metric perturbation. If we consider a null geodesic specified by

$$dt^2 = d\mathbf{x}^2 - h_{\mu\nu}dx^\mu dx^\nu, \tag{7}$$

then by following the same steps as those of Ref. [6], we can show that in a general gauge

$$\sigma_1 = \frac{1}{2}\Delta r\int_{r_0}^{r_1} h_{\mu\nu}n^\mu n^\nu\, dr, \tag{8}$$

and

$$\langle\sigma_1^2\rangle = \frac{1}{4}(\Delta r)^2\int_{r_0}^{r_1}dr\int_{r_0}^{r_1}dr'\, n^\mu n^\nu n^\rho n^\sigma \times\langle h_{\mu\nu}(x)h_{\rho\sigma}(x')\rangle_R. \tag{9}$$

Here $dr = |d\mathbf{x}|$, $\Delta r = r_1 - r_0$ and $n^\mu = dx^\mu/dr$. The graviton two-point function, $\langle h_{\mu\nu}(x)h_{\rho\sigma}(x')\rangle_R$, is understood to be renormalized, so that it is finite when $x = x'$ and vanishes when the quantum state of the gravitons is the Minkowski vacuum state.

A few comments on the derivation of Eq. (5) are in order here. It is obtained by averaging the Fourier representation of a δ-function. It may come as a surprise that although we started with an analytic expansion of σ in powers of $h_{\mu\nu}$, the result is not analytic as $\langle\sigma_1^2\rangle \to 0$. This arises because we use the first order expansion of σ in the argument of an exponential function, but afterwards retain all powers of $h_{\mu\nu}$. One can reasonably ask whether this is a valid procedure. A test of the self-consistency is to retain the σ_2 term and then follow the same procedure. The result is Eq. (5) with σ_0^2 replaced by $\sigma_0^2 + \sigma_2$. This has the same physical interpretation as before; the only effect of the σ_2 part is to shift the location of the mean light cone. Thus in this order we encounter the backreaction of the gravitons in perturbing the original classical geometry to a new classical geometry. Although this is less than a complete demonstration of the validity of Eq. (5), it does indicate that it arises from a self-consistent calculation. In any case, the only result that we really need in the remainder of this paper is Eq. (6), which

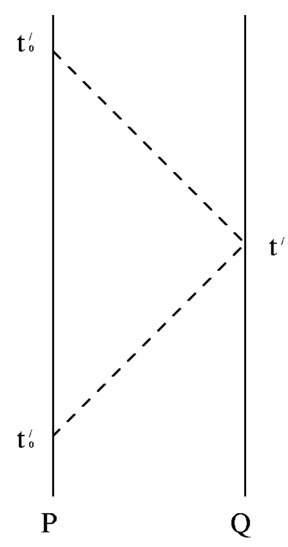

FIG. 1. A light ray (dashed line) makes a round trip travel between two points, P and Q, in space.

may be derived either from Eq. (5), or else more directly by averaging the square of Eq. (2).

Let us now turn to the question of the gauge invariance of the formalism. Under a gauge transformation specified by

$$x'^{\mu}=x^{\mu}+\xi^{\mu}(x), \tag{10}$$

$$h'_{\mu\nu}(x')=h_{\mu\nu}(x)-\xi_{(\mu,\nu)}(x), \tag{11}$$

where $\xi^{\mu}(x)$ is of order $h_{\mu\nu}$, the quantities σ_1 and $\langle\sigma_1^2\rangle$ are not in general invariant. However, we can show that this is due to the fact that Δt is a coordinate time interval rather than a proper time interval. To better understand the gauge invariance, let us examine a situation in which a light signal travels between two points in space labeled by P and Q, with a classical metric perturbation $h_{\mu\nu}$ in the intervening region, as illustrated in Fig. 1.

For simplicity, let us assume that the propagation is in the x-direction. We shall look at the travel time in two different gauges, or coordinate systems, primed and unprimed. For light rays traveling in x direction, we have

$$\frac{dt}{dx}=\pm\sqrt{1-h_{\mu\nu}(x)\frac{dx^{\mu}}{dx}\frac{dx^{\nu}}{dx}}\approx\pm1\mp\frac{1}{2}h_{\mu\nu}(x)\frac{dx^{\mu}}{dx}\frac{dx^{\nu}}{dx}. \tag{12}$$

Here the upper sign is used for outgoing light rays and the lower sign for incoming rays. So, one way travel time δt in the unprimed gauge is

$$\begin{aligned}\delta t_{P\to Q}&=\int_{x_P}^{x_Q}dx-\frac{1}{2}\int_{x_P}^{x_Q}h_{\mu\nu}(x)\frac{dx^{\mu}}{dx}\frac{dx^{\nu}}{dx}dx\\&=\int_{x_P}^{x_Q}dx-\frac{1}{2}\int_{x_P}^{x_Q}h'_{\mu\nu}(x')\frac{dx^{\mu}}{dx}\frac{dx^{\nu}}{dx}dx\\&\quad-\frac{1}{2}\int_{x_P}^{x_Q}\xi_{(\mu,\nu)}(x)\frac{dx^{\mu}}{dx}\frac{dx^{\nu}}{dx}dx,\end{aligned} \tag{13}$$

which, within the linearized theory, can approximated as

$$\begin{aligned}\delta t_{P\to Q}&=\int_{x_P}^{x_Q}dx-\frac{1}{2}\int_{x'_P}^{x'_Q}h'_{\mu\nu}(x')\frac{dx'^{\mu}}{dx'}\frac{dx'^{\nu}}{dx'}dx'-\frac{1}{2}\int_{x_P}^{x_Q}\xi_{(\mu,\nu)}(x)\frac{dx^{\mu}}{dx}\frac{dx^{\nu}}{dx}dx,\\&=\int_{x_P}^{x_Q}dx-\frac{1}{2}\int_{x'_P}^{x'_Q}h'_{\mu\nu}(x')\frac{dx'^{\mu}}{dx'}\frac{dx'^{\nu}}{dx'}dx'-\int_{x_P}^{x_Q}\frac{d\xi_x}{dx}dx-\int_{x_P}^{x_Q}\frac{d\xi_t}{dx}dx\\&=x_Q(t')-\xi_x(Q,t')-(x_P(t_0)-\xi_x(P,t_0))-\xi_t(Q,t')+\xi_t(P,t_0)\\&\quad-\frac{1}{2}\int_{x'_P}^{x'_Q}h'_{\mu\nu}(x')\frac{dx'^{\mu}}{dx'}\frac{dx'^{\nu}}{dx'}dx',\end{aligned} \tag{14}$$

where we have used the fact $dt/dx=1$ for outgoing light rays within our approximation. Similarly, we have

$$\begin{aligned}\delta t_{Q\to P}&=-\int_{x_Q}^{x_P}dx+\frac{1}{2}\int_{x_Q}^{x_P}h_{\mu\nu}(x)\frac{dx^{\mu}}{dx}\frac{dx^{\nu}}{dx}dx\\&=\int_{x_P}^{x_Q}dx-\frac{1}{2}\int_{x_P}^{x_Q}h'_{\mu\nu}(x')\frac{dx^{\mu}}{dx}\frac{dx^{\nu}}{dx}dx+\frac{1}{2}\int_{x_Q}^{x_P}\xi_{(\mu,\nu)}(x)\frac{dx^{\mu}}{dx}\frac{dx^{\nu}}{dx}dx,\\&=\int_{x_P}^{x_Q}dx-\frac{1}{2}\int_{x'_P}^{x'_Q}h'_{\mu\nu}(x')\frac{dx'^{\mu}}{dx'}\frac{dx'^{\nu}}{dx'}dx'+\int_{x_Q}^{x_P}\frac{d\xi_x}{dx}dx-\int_{x_Q}^{x_P}\frac{d\xi_t}{dx}dx\\&=x_Q(t')-\xi_x(Q,t')-(x_P(t'_0)-\xi_x(P,t'_0))+\xi_t(Q,t')-\xi_t(P,t'_0)\\&\quad-\frac{1}{2}\int_{x'_P}^{x'_Q}h'_{\mu\nu}(x')\frac{dx'^{\mu}}{dx'}\frac{dx'^{\nu}}{dx'}dx',\end{aligned} \tag{15}$$

using the fact that for incoming light rays, $dt/dx=-1$. Note that

$$x'_Q(t)=x_Q(t)-\xi_x(Q,t), \tag{16}$$

$$x'_P(t)=x_P(t)-\xi_x(P,t), \tag{17}$$

so

$$\delta t_{P\to Q}=\delta t'_{P\to Q}-\xi_t(Q,t')+\xi_t(P,t_0), \tag{18}$$

and

$$\delta t_{Q\to P}=\delta t'_{Q\to P}+\xi_t(Q,t')-\xi_t(P,t'_0). \tag{19}$$

It follows that the round trip travel time is

$$\Delta t=\delta t_{P\to Q}+\delta t_{Q\to P}=\Delta t'+\xi_t(P,t_0)-\xi_t(P,t'_0). \tag{20}$$

Therefore, the one way travel times, $\delta t_{P\to Q}$ and $\delta t_{Q\to P}$ are, in general, not invariant unless both the source and the detector are outside the regions where gravitational perturbations $h_{\mu\nu}$ are non-zero. In that case, it is physically reasonable to set $\xi(P,t)$ and $\xi(Q,t)$ to zero. Similarly, the round trip time Δt is invariant only if the source (it also acts as a detector in this case) is outside of the gravitational perturbations.

However, it is interesting to note that the round trip proper time interval for the source, $\Delta\tau$, is gauge invariant. Denote the proper time intervals in two different gauges by $\Delta\tau$ and $\Delta\tau'$, and keep in mind the fact that on the world line of the source, generally, $dx^i/dt \ll 1$. We then have

$$\begin{aligned}\Delta\tau'&=\int\sqrt{1+h'_{00}}dt'=\int dt'+\frac{1}{2}\int h'_{00}dt'\\ &=\Delta t'+\frac{1}{2}\int h_{00}dt-\int\frac{d\xi_t}{dt}dt\\ &=\Delta t'+\xi_t(P,t_0)-\xi_t(P,t'_0)+\frac{1}{2}\int h_{00}dt\\ &=\Delta t+\frac{1}{2}\int h_{00}dt=\Delta\tau,\end{aligned} \tag{21}$$

where we have used Eq. (20). This shows that we should really consider how proper time rather than the coordinate time is affected by light-cone fluctuations. However, the calculation of the proper time in a general gauge is a rather difficult task, because the source (and detector) may not be at rest with respect to the chosen coordinate system, and thus in general the emission and the subsequent reception may not happen at the same point in space. To find the proper time, we have to integrate along the geodesic between two events, the emission and the subsequent reception. In general, there is a Doppler shift due to fluctuations in the positions of the source and the mirror. However, the analysis can be greatly simplified if we adopt the transverse-tracefree (TT) gauge, which is specified by the conditions

$$h^j_j=\partial_j h^{ij}=h^{0\nu}=0. \tag{22}$$

To see this, let us examine the geodesic equations for a test particle

$$\frac{d^2x^\mu}{d^2\lambda}=-\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\rho}{d\lambda}, \tag{23}$$

which, when written in terms of derivatives with respect to coordinate time t, becomes

$$\frac{d^2x^\mu}{d^2t}+\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{dt}\frac{dx^\sigma}{dt}-\Gamma^t_{\rho\sigma}\frac{dx^\mu}{dt}\frac{dx^\rho}{dt}\frac{dx^\sigma}{dt}=0. \tag{24}$$

For a non-relativistic test particle, $dx^i/dt \ll 1$, so, to the leading order,

$$\frac{d^2x^i}{d^2t}\approx\Gamma^i_{tt}. \tag{25}$$

But, in the TT gauge, $\Gamma^i_{tt}=0$. Therefore, from the above equation, we can see that if the test particle is at rest at $t=0$, then it will subsequently always remain at rest [13]. So, if we are considering the emission and reflection of a light signal between two points (particles) in the TT gauge, then the proper time $\delta\tau$ between emission and reception (after reflection) of the signal is related with the coordinate time by

$$\delta\tau=\int\sqrt{g_{tt}}dt=\int\sqrt{(1+h_{00})}dt=\int dt=\delta t. \tag{26}$$

Here we have appealed to the fact that $h_{00}=0$ in the TT gauge. Therefore, the coordinate time for the round trip in the TT gauge is the proper time, and Δt calculated from Eq. (6) in the TT gauge is actually a gauge invariant quantity. In this gauge, the mean squared fluctuation in the geodesic interval function reduces to

$$\begin{aligned}\langle\sigma_1^2\rangle&=\frac{1}{4}(\Delta r)^2\int_{r_0}^{r_1}dr\int_{r_0}^{r_1}dr'n^in^jn^kn^m\langle h_{ij}(x)h_{km}(x')\rangle_R\\ &=\frac{1}{8}(\Delta r)^2\int_{r_0}^{r_1}dr\int_{r_0}^{r_1}dr'n^in^jn^kn^m\\ &\quad\times\langle h_{ij}(x)h_{km}(x')+h_{ij}(x')h_{km}(x)\rangle_R.\end{aligned} \tag{27}$$

Here $n^i=dx^i/dr$ is the unit three-vector defining the spatial direction of the geodesic.

B. Graviton two-point function in transverse trace-free gauge

If we work in the TT gauge, the gravitational perturbations have only spatial components h_{ij} and they may be quantized using a plane wave expansion as

$$h_{ij}=\sum_{\mathbf{k},\lambda}[a_{\mathbf{k},\lambda}e_{ij}(\mathbf{k},\lambda)f_{\mathbf{k}}+\text{H.c.}]. \tag{28}$$

Here H.c. denotes the Hermitian conjugate, λ labels the polarization states, and

$$f_{\mathbf{k}}=(2\omega(2\pi)^3)^{-1/2}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{29}$$

is the mode function, where

$$\omega=|\mathbf{k}|,\quad |\mathbf{k}|=(k_x^2+k_y^2+k_z^2)^{1/2}, \tag{30}$$

and the $e_{\mu\nu}(\mathbf{k},\lambda)$ are polarization tensors. (Units in which $32\pi G=1$, where G is Newton's constant and in which $\hbar=c=1$, will be used in this paper.)

Now we shall first calculate the Minkowski spacetime Hadamard function for gravitons in the transverse tracefree gauge. Let us define

$$G^{(1)}_{ijkl}(x,x') = \langle 0 | h_{ij}(x) h_{kl}(x') + h_{ij}(x') h_{kl}(x) | 0 \rangle. \tag{31}$$

Then we have

$$G^{(1)}_{ijkl}(x,x') = \frac{2Re}{(2\pi)^3} \int d^3\mathbf{k} \sum_\lambda e_{ij}(\mathbf{k},\lambda) e_{kl}(\mathbf{k},\lambda) \frac{1}{2\omega} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')}. \tag{32}$$

Equation (A7) in the Appendix for the summation of polarization tensors in the transverse tracefree gauge gives

$$\sum_\lambda e_{ij}(\mathbf{k},\lambda) e_{kl}(\mathbf{k},\lambda) = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl} + \hat{k}_i\hat{k}_j\hat{k}_k\hat{k}_l + \hat{k}_i\hat{k}_j\delta_{kl} + \hat{k}_k\hat{k}_l\delta_{ij} - \hat{k}_i\hat{k}_l\delta_{jk} - \hat{k}_i\hat{k}_k\delta_{jl} - \hat{k}_j\hat{k}_l\delta_{ik} - \hat{k}_j\hat{k}_k\delta_{il}, \tag{33}$$

where

$$\hat{k}_i = \frac{k_i}{k}. \tag{34}$$

We find that $G^{(1)}_{ijkl}(x,x')$ can be expressed as [11]

$$\begin{aligned} G^{(1)}_{ijkl}(x,x') &= 2Re(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl} + D_{ij}) \times \frac{1}{(2\pi)^3} \int d^3\mathbf{k} \frac{1}{2\omega} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')} \\ &= 2Re(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl} + D_{ij}) \times \langle 0 | \phi(x)\phi(x') | 0 \rangle, \end{aligned} \tag{35}$$

where we have defined a formal operator

$$D_{ij} = \left(\frac{\partial_i\partial'_j}{\nabla^2}\delta_{kl} + \frac{\partial_k\partial'_l}{\nabla^2}\delta_{ij} - \frac{\partial_i\partial'_k}{\nabla^2}\delta_{jl} - \frac{\partial_i\partial'_l}{\nabla^2}\delta_{jk} - \frac{\partial_j\partial'_l}{\nabla^2}\delta_{ik} - \frac{\partial_j\partial'_k}{\nabla^2}\delta_{il} + \frac{\partial_i\partial'_j\partial_k\partial'_l}{\nabla^4} \right), \tag{36}$$

and $\langle 0 | \phi(x)\phi(x') | 0 \rangle$ is the usual scalar field two-point function. Here the formal operator ∇^{-2} should be understood in the sense of a Green's function, but when we do our calculations in momentum space its effect is to bring in a factor of k^{-2}.

The combination of these results with Eq. (27) gives

$$\langle \sigma_1^2 \rangle = \frac{1}{4}(\Delta r)^2 \int_{r_0}^{r_1} dr \int_{r_0}^{r_1} dr' \left(1 - \frac{2(\nabla\cdot\mathbf{n})(\nabla'\cdot\mathbf{n})}{\nabla^2} + \frac{(\nabla\cdot\mathbf{n})^2(\nabla'\cdot\mathbf{n})^2}{\nabla^4} \right) \langle \phi(x)\phi(x') \rangle_R. \tag{37}$$

Introduce two functions $F_{ij}(x,x')$ and $H_{ijkl}(x,x')$ by

$$\begin{aligned} F_{ij}(x,x') &= Re\frac{\partial_i\partial'_j}{\nabla^2} \langle 0 | \phi(x)\phi(x') | 0 \rangle \\ &= Re\frac{\partial_i\partial'_j}{\nabla^2} \frac{1}{(2\pi)^3} \int d^3\mathbf{k} \frac{1}{2\omega} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')} \\ &= \frac{Re}{(2\pi)^3} \int d^3\mathbf{k} \frac{k_i k_j}{2\omega^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')}, \end{aligned} \tag{38}$$

and

$$\begin{aligned} H_{ijkl}(x,x') &= Re\frac{\partial_i\partial'_j\partial_k\partial'_l}{\nabla^4} \langle 0 | \phi(x)\phi(x') | 0 \rangle \\ &= Re\frac{\partial_i\partial'_j\partial_k\partial'_l}{\nabla^4} \frac{1}{(2\pi)^3} \int d^3\mathbf{k} \frac{1}{2\omega} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')} \\ &= \frac{Re}{(2\pi)^3} \int d^3\mathbf{k} \frac{k_i k_j k_k k_l}{2\omega^5} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')}. \end{aligned} \tag{39}$$

$G^{(1)}_{ijkl}$ can be expressed as

$$G^{(1)}_{ijkl}=2F_{ij}\delta_{kl}+2F_{kl}\delta_{ij}-2F_{ik}\delta_{jl}-2F_{il}\delta_{jk}-2F_{jl}\delta_{ik}-2F_{jk}\delta_{il}+2H_{ijkl}+2D^{(1)}(x,x')(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\delta_{ij}\delta_{kl}), \qquad (40)$$

where

$$D^{(1)}(x,x')=-\frac{1}{8\pi^2\sigma_0^2} \qquad (41)$$

is the usual Hadamard function for massless scalar fields with $2\sigma_0^2=(t-t')^2-(\mathbf{x}-\mathbf{x}')^2$, and $F_{ij}(x,x')$ and $H_{ijkl}(x,x')$, which will be calculated in the Appendix, are given by

$$F_{ij}(x,x')=-\frac{1}{(2\pi)^2}\partial_i\partial_j'\left[\frac{1}{2}\ln(R^2-\Delta t^2)+\frac{\Delta t}{4R}\ln\left(\frac{R+\Delta t}{R-\Delta t}\right)^2\right], \qquad (42)$$

and

$$H_{ijkl}(x,x')=\frac{1}{96\pi^2}\partial_i\partial_j'\partial_k\partial_l'\left[(R^2+3\Delta t^2)\ln(R^2-\Delta t^2)^2+\left(3R\Delta t+\frac{\Delta t^3}{R}\right)\ln\left(\frac{R+\Delta t}{R-\Delta t}\right)^2\right]. \qquad (43)$$

Here $R=|\mathbf{x}-\mathbf{x}'|$.

III. LIGHT-CONE FLUCTUATIONS IN FLAT SPACETIME WITH NONTRIVIAL TOPOLOGIES OR BOUNDARIES

In this section, we study light-cone fluctuations in two cases: flat spacetime with a compactified spatial section, and with a single plane boundary.

A. Flat spacetime with a compactified spatial section

Let us now assume that the spacetime is flat but compactified in the z direction with a periodicity length L (''circumference of the universe''). This means the spatial points z and $z+L$ are identified. The effect of the space closure is to restrict the field modes to a discrete set

$$f_{\mathbf{k}}=(2\omega(2\pi)^2L)^{-1/2}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \qquad (44)$$

with

$$k_z=\frac{2\pi n}{L}, \quad n=0,\pm1,\pm2,\pm3,\ldots \qquad (45)$$

We now analyze the light-cone fluctuations, assuming that the gravitons are in the new vacuum state $|0_L\rangle$ associated with the discrete modes of Eq. (44).

First consider a light ray along z direction, i.e. along the direction of compactification (Fig. 2), propagating from point $(0,0,a)$ to point $(0,0,b)$ in space, then, we have from Eq. (27)

$$\langle\sigma_1^2\rangle=\frac{1}{8}(b-a)^2\int_a^b dz\int_a^b dz'\,\times\langle 0_L|h_{zz}(x)h_{zz}(x')+h_{zz}(x')h_{zz}(x)|0_L\rangle_R=\frac{1}{8}(b-a)^2\int_a^b dz\int_a^b dz'\,G^{(1)R}_{zzzz}(t,0,0,z,t',0,0,z'). \qquad (46)$$

Here we have defined

$$G^{(1)R}_{zzzz}(x,x')=\langle 0_L|h_{zz}(x)h_{zz}(x')+h_{zz}(x')h_{zz}(x)|0_L\rangle_R=\langle 0_L|h_{zz}(x)h_{zz}(x')+h_{zz}(x')h_{zz}(x)|0_L\rangle-\langle 0|h_{zz}(x)h_{zz}(x')+h_{zz}(x')h_{zz}(x)|0\rangle, \qquad (47)$$

and the integral is to be carried out along the geodesic.

If we adopt the notation

$$(t,0,0,z,t',0,0,z')\equiv(t,z,t',z'), \qquad (48)$$

the renormalized two-point function can be found by using the method of images to be

$$G^{(1)R}_{zzzz}(t,z,t',z')=\sum_{n=-\infty}^{+\infty}{}'\;G^{(1)}_{zzzz}(t,z,t',z'+nL)=2\sum_{n=-\infty}^{+\infty}{}'\;[D^{(1)}(t,z,t',z'+nL)-2F_{zz}(t,z,t',z'+nL)+H_{zzzz}(t,z,t',z'+nL)], \qquad (49)$$

where the prime on the summation indicates that the $n=0$ term is omitted. Substituting $R_t=0$ into Eq. (A24) in the Appendix and replacing Δx by Δz, we have

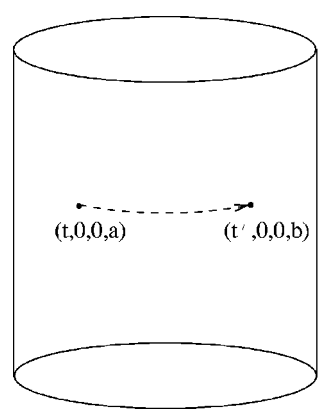

FIG. 2. A light ray (dashed line) propagates in the direction of compactification in a cylindrical ''universe'' from point $(t,0,0,a)$ to point $(t',0,0,b)$. Here only two spatial dimensions are plotted.

$$G^{(1)R}_{zzzz}(t,z,t',z') = -\frac{2}{\pi^2}\sum_{n=-\infty}^{+\infty}{}' \left[\frac{\Delta t^2}{(\Delta z - nL)^4} + \frac{\Delta t^3}{4(\Delta z - nL)^5}\ln\left(\frac{\Delta z - nL - \Delta t}{\Delta z - nL + \Delta t}\right)^2 - \frac{2}{3(\Delta z - nL)^2} - \frac{\Delta t}{4(\Delta z - nL)^3}\ln\left(\frac{\Delta z - nL - \Delta t}{\Delta z - nL + \Delta t}\right)^2\right]. \tag{50}$$

For the null geodesic

$$\Delta t = \Delta z, \tag{51}$$

we get, after an evaluation of the integral,

$$\begin{aligned}\int_a^b dz \int_a^b dz'\, G^{(1)R}_{zzzz}(t,z,t',z')|_{\Delta t=\Delta z} &= \frac{1}{12\pi^2}\sum_{n=-\infty}^{+\infty}{}' \left[\frac{8\epsilon^2(n^2-2\epsilon^2)}{(n^2-\epsilon^2)^2} + \frac{(n+2\epsilon)^3}{2(n+\epsilon)^3}\ln\left(1+\frac{2\epsilon}{n}\right)^2 + \frac{(n-2\epsilon)^3}{2(n-\epsilon)^2}\ln\left(1-\frac{2\epsilon}{n}\right)^2\right] \\ &= \frac{1}{12\pi^2}\sum_{n=1}^{+\infty}\left[\frac{16\epsilon^2(n^2-2\epsilon^2)}{(n^2-\epsilon^2)^2} + \frac{(n+2\epsilon)^3}{(n+\epsilon)^3}\ln\left(1+\frac{2\epsilon}{n}\right)^2 + \frac{(n-2\epsilon)^3}{(n-\epsilon)^3}\ln\left(1-\frac{2\epsilon}{n}\right)^2\right] \\ &\equiv \frac{1}{12\pi^2}\sum_{n=1}^{+\infty} f(n,\epsilon),\end{aligned} \tag{52}$$

where we have defined

$$\epsilon \equiv \frac{(b-a)}{L} = \frac{r}{L}, \tag{53}$$

and

$$f(n,\epsilon) \equiv \frac{16\epsilon^2(n^2-2\epsilon^2)}{(n^2-\epsilon^2)^2} + \frac{(n+2\epsilon)^3}{(n+\epsilon)^3}\ln\left(1+\frac{2\epsilon}{n}\right)^2 + \frac{(n-2\epsilon)^3}{(n-\epsilon)^3}\ln\left(1-\frac{2\epsilon}{n}\right)^2. \tag{54}$$

It appears that there is a singularity in the summand $f(n,\epsilon)$ whenever $n=\epsilon$, i.e., whenever the distance r is an integer multiple of L. However this singularity is illusionary, as it should be from a physical point of view since there is nothing special when $n=\epsilon$. This can be seen if we expand the summand at the point $\epsilon = n$ to get

$$f(n,\epsilon) \approx \frac{19}{3} + \frac{27}{4}\ln(3) + \frac{27\ln(3)+68}{8n}(\epsilon - n) + O((\epsilon-n)^2). \tag{55}$$

So, $f(n,\epsilon)$ is finite as ϵ approaches n. Note also that $2\epsilon = n$ is also not a singularity. The summation converges, as the asymptotic form of $f(n,\epsilon)$ as $n\to\infty$ is

$$f(n,\epsilon) \sim \frac{32\epsilon^2}{n^2} + O(n^{-4}). \tag{56}$$

However, a generic closed form result for the summation is hard to find. So we now discuss two special cases. The first is the one in which the distance traversed by the light ray is much less than the periodicity length, $b-a \ll L$. Then we get

$$\int_a^b dz \int_a^b dz'\, G^{(1)R}_{zzzz}(x,x') \approx \sum_{n=1}^{+\infty} \frac{8\epsilon^2}{3\pi^2}\frac{1}{n^2} = \frac{4\epsilon^2}{9}. \tag{57}$$

Substitution of this result into Eq. (46) yields

$$\langle \sigma_1^2 \rangle \approx \frac{r^4}{18L^2}. \tag{58}$$

Therefore the mean deviation from the classical propagation time is

$$\Delta t = \frac{\sqrt{\langle \sigma_1^2\rangle}}{r} \approx \frac{1}{3\sqrt{2}}\frac{r}{L}. \tag{59}$$

Since we are working in Natural Units, this result reveals that the mean deviation in travel time is less than the Planck time and grows linearly with increasing r when r is small compared to the periodicity length L of the universe.

If $\epsilon \gg 1$, i.e., $r \gg L$, the light loops around the ‘‘universe,’’ and summation Eq. (52) can be approximated by the following integral

$$\int_a^b dz \int_a^b dz'\, G^{(1)R}_{zzzz}(t,z,t',z') \approx \frac{\epsilon}{12\pi^2}\int_{1/\epsilon}^{\infty} dx \left[\frac{(x+2)^3}{(x+1)^3}\ln\left(1+\frac{2}{x}\right)^2 + \frac{(x-2)^3}{(x-1)^3}\ln\left(1-\frac{2}{x}\right)^2 + \frac{16(x^2-2)}{(x^2-1)^2}\right]. \tag{60}$$

Evaluating the integral with the aid of the computer algebra package Maple, series expanding the result and keeping the leading terms only, we arrive at

$$\int_a^b dz \int_a^b dz' G^{(1)R}_{zzzz}(t,z,t',z') \approx \epsilon - \frac{8\ln(2\epsilon)}{3\pi^2}. \qquad (61)$$

Therefore the mean deviation from the classical propagation time is

$$\Delta t = \frac{\sqrt{\langle \sigma_1^2\rangle}}{r} \approx \frac{1}{2\sqrt{2}}\sqrt{\frac{r}{L}}, \qquad (62)$$

where r is assumed to be much greater than L. So the light-cone fluctuations can, in principle, get as large as one would like if the light ray travels around and around. This is interesting in the sense that it suggests that a fluctuation which is much greater than the Planck scale could be achieved.

Now we turn to the case where the light ray moves along the direction perpendicular to that of compactification, for instance, along x direction. If the light ray travels from point $(a,0,0)$ to point $(b,0,0)$, as illustrated in Fig. 3, then

$$\begin{aligned}\langle \sigma_1^2\rangle &= \frac{1}{8}(b-a)^2 \int_a^b dx \int_a^b dx' \langle 0_L | h_{xx}(x)h_{xx}(x') + h_{xx}(x')h_{xx}(x)|0_L\rangle_R \\ &= \frac{1}{8}(b-a)^2 \int_a^b dx \int_a^b dx' G^{(1)R}_{xxxx}(t,x,0,0,t',x',0,0), \\ &= \frac{1}{8}(b-a)^2 \int_a^b dx \int_a^b dx' \sum_{n=-\infty}^{+\infty}{}' \; G^{(1)}_{xxxx}(t,x,0,0,t',x',0,nL). \end{aligned} \qquad (63)$$

Let us now define

$$\rho = x - x', \quad b - a = r, \qquad (64)$$

then if we use Eq. (A24) in the Appendix and bear in mind the fact that for the light ray $\Delta t = \Delta x$, we have

$$G^{(1)R}_{xxxx}(t,x,0,0,t',x',0,0) \equiv g_1(\rho) + g_2(\rho), \qquad (65)$$

where

$$g_1 = 2\sum_{n=-\infty}^{+\infty}{}' \; -\frac{1}{8\pi^2}\frac{\rho^2(nL)^4}{[\rho^2+(nL)^2]^4} - \frac{1}{3\pi^2}\frac{\rho^6}{[\rho^2+(nL)^2]^4} + \frac{47}{12\pi^2}\frac{\rho^4(nL)^2}{[\rho^2+(nL)^2]^4}, \qquad (66)$$

and

$$g_2 = -2\sum_{n=-\infty}^{+\infty}{}' \; \frac{1}{2\pi^2}\ln\left(\frac{\sqrt{\rho^2+(nL)^2}+\rho}{\sqrt{\rho^2+(nL)^2}-\rho}\right)^2 \times\left[-\frac{1}{16}\frac{\rho(nL)^6}{[\rho^2+(nL)^2]^{(9/2)}} - \frac{3}{4}\frac{\rho^3(nL)^4}{[\rho^2+(nL)^2]^{(9/2)}} + \frac{3}{2}\frac{\rho^5(nL)^2}{[\rho^2+(nL)^2]^{(9/2)}}\right]. \qquad (67)$$

We can clearly see that $G^{(1)R}_{xxxx}$ is an even function of ρ, so,

$$\int_a^b dx \int_a^b dx' G^{(1)R}_{xxxx}(t,x,0,0,t',x',0,0) = 2\int_0^r d\rho (r-\rho)(g_1+g_2). \qquad (68)$$

Performing the integration (integrate by parts for those terms involving logarithmic function), we arrive at

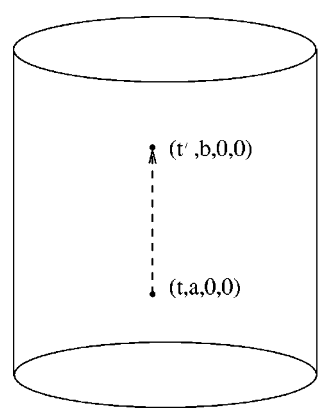

FIG. 3. A light ray (dashed line) propagates perpendicular to the direction of compactification in a cylindrical ''universe'' from point $(t,a,0,0)$ to point $(t',b,0,0)$. Here only two spatial dimensions are plotted.

$$2\int_0^r d\rho(r-\rho)(g_1+g_2)$$
$$=\frac{2}{\pi^2}\sum_{n=-\infty}^{+\infty}{}' \left[-\frac{\epsilon^4}{2(\epsilon^2+n^2)^2}-\frac{\epsilon^2 n^2}{4(\epsilon^2+n^2)^2}+\frac{8\epsilon^5+8n^2\epsilon^3+3n^4\epsilon}{24(n^2+\epsilon^2)^{5/2}}\ln\left(\frac{\sqrt{n^2+\epsilon^2}+\epsilon}{\sqrt{n^2+\epsilon^2}-\epsilon}\right)\right], \tag{69}$$

where $\epsilon=r/L$ as before. The above series can be shown to be convergent. Yet a result in closed form is not easy to find. Let us first examine the case in which $r\ll L$, where

$$\int_a^b dx\int_a^b dx' G^{(1)R}_{xxxx}(x,x')\approx\sum_{n=1}^{+\infty}\frac{64\epsilon^6}{45\pi^2}\frac{1}{n^6}=\frac{64\pi^4\epsilon^6}{45^2\times 21}. \tag{70}$$

Here we have used

$$\sum_{n=1}^{+\infty}\frac{1}{n^6}=\frac{\pi^6}{45\times 21}. \tag{71}$$

Thus the mean deviation from the classical propagation time is

$$\Delta t=\frac{\sqrt{\langle\sigma_1^2\rangle}}{r}\approx\sqrt{\frac{2}{21}\frac{2\pi^2}{45}\left(\frac{r}{L}\right)^3}. \tag{72}$$

This result holds in the small ϵ regime. The time deviation is much smaller than that for light rays propagating along the compactification direction [compare with Eq. (59)]. This reveals that light-cone fluctuations due to topology change are more likely to be felt in the direction of compactification than in the transverse direction, if we perform local experiments in which r, the distance between the source and the detector, is very small as compared to L, the periodicity length.

We now turn our attention to the case in which $r\gg L$, i.e., $\epsilon\gg 1$. Here it is easy to see that the summation in Eq. (69) can be approximated by the following integral

$$\int_a^b dx\int_a^b dx' G^{(1)R}_{xxxx}(t,x,0,0,t',x',0,0)$$
$$\approx\frac{4\epsilon}{\pi^2}\int_{1/\epsilon}^{\infty}dx\left[-\frac{1}{2(1+x^2)^2}-\frac{x^2}{4(1+x^2)^2}+\frac{8+8x^2+3x^4}{24(x^2+1)^{5/2}}\ln\left(\frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}-1}\right)\right]. \tag{73}$$

If we perform the integral and series expand the result, we have, to the order of $O(\epsilon)$,

$$\int_a^b dx\int_a^b dx' G^{(1)R}_{xxxx}(t,x,0,0,t',x',0,0)\approx c_1^2\epsilon-c_2^2\ln(\epsilon), \tag{74}$$

where c_1 and c_2 are constants given, respectively, by

$$c_1^2=\int_0^\infty dx\frac{\ln(x+\sqrt{x^2+1})}{x\sqrt{x^2+1}}\approx 2.468 \tag{75}$$

and

$$c_2^2=\frac{8}{3\pi^2}. \tag{76}$$

Therefore we have for the mean squared geodesic interval fluctuation

$$\langle\sigma_1^2\rangle\approx\frac{1}{8}\left(\frac{c_1 r}{\pi}\right)^2\epsilon, \tag{77}$$

and the mean deviation from the classical propagation time is

$$\Delta t=\frac{\sqrt{\langle\sigma_1^2\rangle}}{r}\approx\frac{c_1}{\pi}\frac{1}{2\sqrt{2}}\sqrt{\frac{r}{L}}. \tag{78}$$

This result applies in the regime where $r\gg L$. Here we have the same functional dependence on r as in the case where light rays loop around the compactified dimension many times. The only difference lies in the proportionality constants. In fact, here the mean time deviation is also smaller than that for light rays traveling in the direction of compactification, since the numerical constant $c_1/\pi\approx 0.5$.

B. Single plane boundary

Let us assume that there is a single plane boundary located at $z=0$ in space such that metric perturbations satisfy the following Neumann boundary condition (the reason that we use the Neumann boundary condition instead of the Dirichlet boundary condition here is to get a positive $\langle\sigma_1^2\rangle$):

$$\partial_z h_{jk}|_{z=0}=0. \tag{79}$$

In the presence of the boundary, the field mode no longer has the form of Eq. (44) but becomes

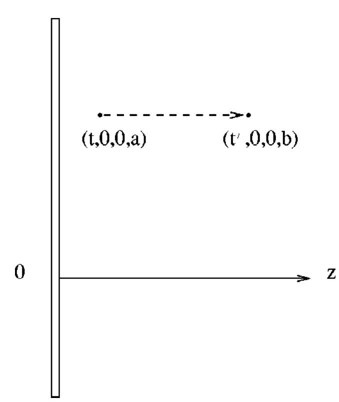

FIG. 4. A light ray (dashed line) propagates in the direction perpendicular to the plane boundary, starting a distance away from the boundary.

$$f_{\mathbf{k}}=(\omega(2\pi)^2\pi)^{-1/2}e^{i(\mathbf{k_t}\cdot\mathbf{x_t}-\omega t)}\cos(k_z z), \tag{80}$$

where $\mathbf{k_t}$ and $\mathbf{x_t}$ denote the components of $\mathbf{k}$ and $\mathbf{x}$, respectively, in directions parallel to the boundary. Now if we assume that the gravitons are in the vacuum state $|0'\rangle$ associated with the modes of Eq. (80), we have, for a light ray propagating perpendicular to the boundary from point $(0,0,a)$ to $(0,0,b)$ (see Fig. 4),

$$\begin{aligned}\langle\sigma_1^2\rangle &= \frac{1}{8}(b-a)^2\int_a^b dz\int_a^b dz'\langle 0'|h_{zz}(x)h_{zz}(x') \\ &\quad + h_{zz}(x')h_{zz}(x)|0'\rangle_R \\ &= \int_a^b dz\int_a^b dz'\, G^{(1)R}_{zzzz}(t,0,0,z,t',0,0,z'). \end{aligned}\tag{81}$$

Here the renormalized graviton two point function $G^{(1)R}_{zzzz}(x,x')$ can be found by the method of images as usual and the only difference is an overall sign change as we go from the Dirichlet boundary condition to the Neumann boundary condition. The reason for this is that, to satisfy the Neumann boundary condition, we need to add the image term instead of subtracting it as in the case of the Dirichlet boundary condition. So, $G^{(1)R}_{zzzz}(x,x')$ may be obtained by picking out the $n=0$ term in Eq. (50) and setting $\Delta z=z+z'$ to get

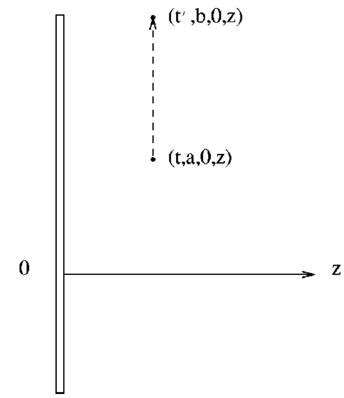

FIG. 5. A light ray (dashed line) propagates in the direction parallel to the plane boundary, starting z distance away from it.

$$\begin{aligned}G^{(1)R}_{zzzz}(t,0,0,z,t',0,0,z') &= -\frac{2(t-t')^2}{\pi^2(z+z')^4}-\frac{(t-t')^3}{2\pi^2(z+z')^5}\ln\left(\frac{z+z'-(t-t')}{z+z+(t+t')}\right)^2+\frac{4}{3\pi^2(z+z')^2} \\ &\quad +\frac{(t-t')}{2\pi^2(z+z')^3}\ln\left(\frac{z+z'-(t-t')}{z+z'+(t+t')}\right)^2 .\end{aligned}\tag{82}$$

Substituting this result into Eq. (81) and performing the integration, we finally get

$$\langle\sigma_1^2\rangle=\frac{(b-a)^3[b^2-a^2+(a^2+4ab+b^2)\ln(b/a)]}{24\pi^2(b+a)^3}. \tag{83}$$

Note that this result is always greater than zero. However, had we chosen the Dirichlet boundary condition, we would have that $\langle\sigma_1^2\rangle<0$. Recall that the formalism which we are using applies only if $\langle\sigma_1^2\rangle>0$.

When the light ray starts very close to the boundary such that $a\ll r$, we have

$$\langle\sigma_1^2\rangle\approx\frac{r^4}{24\pi^2}[1+\ln(r/a)]. \tag{84}$$

The mean deviation in travel time is

$$\Delta t=\frac{\sqrt{\langle\sigma_1^2\rangle}}{r}=\sqrt{\frac{1+\ln(r/a)}{24\pi^2}}, \tag{85}$$

which diverges as a approaches 0. This is not surprising since the energy density of a quantized field blows up on the boundary. However, it has been shown recently [12] that, if one treats the boundaries as quantum objects with a nonzero position uncertainty, the singularity in energy density is removed. The result, Eq. (85), applies whenever $r\gg a$. The other limit is when $r\ll a$, where the mean squared fluctuation in the geodesic interval function is approximated as

$$\langle\sigma_1^2\rangle\approx\frac{r^4}{24\pi^2a^2}, \tag{86}$$

consequently, the mean deviation in time is given by

$$\Delta t=\frac{\sqrt{\langle\sigma_1^2\rangle}}{r}=\frac{1}{2\sqrt{6}\pi}\frac{r}{a}. \tag{87}$$

We now consider a null geodesic which is z distance away from and parallel to the plane boundary (see Fig. 5). The relevant renormalized Hadamard function is given by Eq. (A24) with Δz being replaced by $z+z'$.

Now suppose the geodesic starts at point $(t,a,0,z)$ and ends at point $(t',b,0,z)$, then the mean squared fluctuation in the geodesic interval function is

$$\langle\sigma_1^2\rangle=\frac{1}{8}(b-a)^2\int_a^b dx\int_a^b dx'\, G^R_{xxxx}(t,x,0,z,t',x',0,z). \tag{88}$$

Here $G^R_{xxxx}(t,x,0,z,t',x',0,z)$ is also given by Eqs. (65)–(67) but with a replacement of nL by $2z$. Therefore

$$\int_a^b dx \int_a^b dx' G^{(1)R}_{xxxx}(t,x,0,z,t',x',0,z)$$

$$= \frac{2}{\pi^2}\Bigg[-\frac{\epsilon^4}{2(\epsilon^2+4)^2} - \frac{\epsilon^2}{(\epsilon^2+4)^2} + \frac{8\epsilon^5+32\epsilon^3+48\epsilon}{24(4+\epsilon^2)^{5/2}} \ln\left(\frac{\sqrt{4+\epsilon^2}+\epsilon}{\sqrt{4+\epsilon^2}-\epsilon}\right)\Bigg], \tag{89}$$

where $\epsilon = r/z$. Since the above expression is very complicated, we shall discuss two interesting special cases. One is when $r \gg z$, then we have

$$\langle \sigma^2 \rangle \approx \frac{r^2}{6\pi^2} \ln(r/z) \tag{90}$$

and

$$\Delta t = \frac{\sqrt{\langle \sigma_1^2 \rangle}}{r} = \sqrt{\frac{\ln(r/z)}{6\pi^2}}. \tag{91}$$

This also blows up as z approaches 0, however the functional dependence upon z is different from that of Eq. (85). The other limit is when $r \ll z$. For this case, we find

$$\langle \sigma^2 \rangle \approx \frac{r^8}{720 z^6 \pi^2} \tag{92}$$

and

$$\Delta t = \frac{\sqrt{\langle \sigma_1^2 \rangle}}{r} = \frac{1}{12\sqrt{5}\pi}\left(\frac{r}{z}\right)^3. \tag{93}$$

IV. SUMMARY AND DISCUSSION

In this paper, we have obtained general expressions, in the transverse tracefree gauge, for the vacuum graviton two-point function for various boundary conditions. These were used to study the light-cone fluctuations in flat spacetimes with a compactified spatial section and with a plane boundary. The mean squared fluctuations of the geodesic interval function and therefore the mean deviations from the classical propagation time have been obtained.

In the case of a compactified spatial section, when the travel distance is less than the periodicity length, the fluctuation in the propagation time is less than the Planck time. In this limit, the effect is much larger for propagation in the periodicity direction than for propagation in the transverse direction. Thus the local light-cone fluctuations become anisotropic, reflecting the global structure of the spacetime. When the travel distance is large compared to the periodicity length, the fluctuation in travel time increases with the square root of the distance traveled for propagation in either direction, and the only difference lies in the proportionality constants. Here we have a possibility of having fluctuations larger than Planck scale by several orders of magnitude.

In the case of a plane boundary, as light rays start closer and closer to the boundary, the light-cone fluctuations blow up as the square root of the logarithm of the starting distance both when light rays propagate perpendicular and parallel to the boundary. This is not as surprising as it might seem because the imposition of a fixed boundary can lead to singular expectation values of local observables, such as energy densities. However we expect this singularity to disappear if one treats the boundary as a quantum mechanical object with a nonzero position uncertainty [12]. It is also found that if the starting distance from the boundary is fixed, then the fluctuation in travel time grows as the square root of the logarithm of the distance traversed when this distance is large compared to the starting distance.

In summary, we have demonstrated that in the linearized theory of quantum gravity, changes in the topology of flat spacetime produce light-cone fluctuations. These fluctuations are in general larger in the directions in which topology changes occur and are typically of the order of Planck scale, but they can get larger for path lengths large compared to the compactification scale. It is interesting to note that this effect could become significant in theories which postulate extra dimensions compactified on a very small scale.

ACKNOWLEDGMENTS

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APPENDIX

1. Summation of graviton polarization tensors in the TT gauge

Let us introduce a triad of orthonormal vectors $[\mathbf{e}_1(\mathbf{k}),\mathbf{e}_2(\mathbf{k}),\mathbf{e}_3(\mathbf{k})]$ with

$$\mathbf{e}_3(\mathbf{k}) = \frac{\mathbf{k}}{|\mathbf{k}|} = \hat{\mathbf{k}}, \tag{A1}$$

the unit vector in the direction of propagation. The triad satisfies the orthonormality relation

$$\mathbf{e}_a(\mathbf{k}) \cdot \mathbf{e}_b(\mathbf{k}) = \delta_{ab}, \quad a,b = 1,2,3. \tag{A2}$$

This relation can be written, in terms of the components in the coordinate system characterizing the metric, as

$$e^i_a(\mathbf{k}) e^i_b(\mathbf{k}) = \delta_{ab}, \quad a,b = 1,2,3. \tag{A3}$$

Here the Einstein summation convention is employed. We also have

$$e^i_a(\mathbf{k}) e^j_a(\mathbf{k}) = e^i_1 e^j_1 + e^i_2 e^j_2 + \hat{k}^i \hat{k}^j = \delta_{ij}, \quad i,j = x,y,z. \tag{A4}$$

Therefore, the two independent graviton polarization tensors in the TT gauge are given, in terms of the triad, by

$$e^{ij}(\mathbf{k},+)=e_1^i(\mathbf{k})\otimes e_1^j(\mathbf{k})-e_2^i(\mathbf{k})\otimes e_2^j(\mathbf{k}), \tag{A5}$$

$$e^{ij}(\mathbf{k},\times)=e_1^i(\mathbf{k})\otimes e_2^j(\mathbf{k})+e_2^i(\mathbf{k})\otimes e_1^j(\mathbf{k}), \tag{A6}$$

where we have adopted the notation of Ref. [13]. Hence,

$$\begin{aligned}\sum_\lambda e_{ij}(\mathbf{k},\lambda)e_{kl}(\mathbf{k},\lambda)&=e_{ij}(\mathbf{k},+)e_{kl}(\mathbf{k},+)+e_{ij}(\mathbf{k},\times)e_{kl}(\mathbf{k},\times)\\ &=e_1^ie_1^je_1^ke_1^l-e_1^ie_1^je_2^ke_2^l-e_2^ie_2^je_1^ke_1^l+e_2^ie_2^je_2^ke_2^l+e_1^ie_2^je_1^ke_2^l+e_1^ie_2^je_2^ke_1^l+e_2^ie_1^je_1^ke_2^l+e_2^ie_1^je_2^ke_1^l\\ &=(e_1^ie_1^k+e_2^ie_2^k)(e_1^je_1^l+e_2^je_2^l)+(e_1^ie_1^l+e_2^ie_2^l)(e_1^je_1^k+e_2^je_2^k)-(e_1^ie_1^j+e_2^ie_2^j)(e_1^ke_1^l+e_2^le_2^k)\\ &=(\delta^{ik}-\hat{k}^i\hat{k}^k)(\delta^{jl}-\hat{k}^j\hat{k}^l)+(\delta^{il}-\hat{k}^i\hat{k}^l)(\delta^{jk}-\hat{k}^j\hat{k}^k)-(\delta^{ij}-\hat{k}^i\hat{k}^j)(\delta^{kl}-\hat{k}^k\hat{k}^l)\\ &=\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\delta_{ij}\delta_{kl}+\hat{k}_i\hat{k}_j\hat{k}_k\hat{k}_l+\hat{k}_i\hat{k}_j\delta_{kl}+\hat{k}_k\hat{k}_l\delta_{ij}-\hat{k}_i\hat{k}_l\delta_{jk}-\hat{k}_i\hat{k}_k\delta_{jl}-\hat{k}_j\hat{k}_l\delta_{ik}-\hat{k}_j\hat{k}_k\delta_{il}.\end{aligned} \tag{A7}$$

This result can also be obtained as follows. Let us introduce a 4th-rank tensor

$$T^{ijkl}(\mathbf{k})=\sum_\lambda e^{ij}(\mathbf{k},\lambda)e^{kl}(\mathbf{k},\lambda), \tag{A8}$$

which has the following symmetry properties

$$T^{ijkl}=T^{jikl}=T^{ijlk}=T^{klij}. \tag{A9}$$

However, the objects, which are at our disposal to construct T^{ijkl}, are only k^i and δ^{ij}, thus in general, we have

$$\begin{aligned}T^{ijkl}&=A\delta^{ij}\delta^{kl}+B\delta^{ik}\delta^{jl}+B\delta^{il}\delta^{jk}+C(\hat{k}^i\hat{k}^j\delta^{kl}+\hat{k}^k\hat{k}^l\delta^{ij})\\ &\quad+D(\hat{k}^i\hat{k}^k\delta^{jl}+\hat{k}^i\hat{k}^l\delta^{jk}+\hat{k}^j\hat{k}^l\delta^{ik}+\hat{k}^j\hat{k}^k\delta^{il})\\ &\quad+E\hat{k}^i\hat{k}^j\hat{k}^k\hat{k}^l,\end{aligned} \tag{A10}$$

where A,B,C,D,E are constants to be determined. This tensor is subject to the transversality condition

$$k_iT^{ijkl}=k_jT^{ijkl}=k_kT^{ijkl}=k_lT^{ijkl}=0, \tag{A11}$$

and the trace-free condition

$$T^{iikl}=T^{ijkk}=0. \tag{A12}$$

Applying these constraint conditions to T^{ijkl} and solving the resulting equations leads to

$$a=d=-e=-c=-b. \tag{A13}$$

Therefore T^{ijkl} is the same as the right-hand side of Eq. (A7), apart from a multiplicative normalization constant which can be chosen to be unity.

2. Vacuum graviton Hadamard function in the TT gauge

Here we evaluate the function $F_{ij}(x,x')$ and $H_{ijkl}(x,x')$ defined in Eqs. (42) and (43), respectively. Once these functions are given, the graviton two point functions are easy to obtain. Define

$$\begin{aligned}R&=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2},\\ \Delta t&=t-t',\quad k=|\mathbf{k}|=\omega.\end{aligned} \tag{A14}$$

Then,

$$\begin{aligned}F_{ij}(x,x')&=\frac{Re}{(2\pi)^3}\int d^3\mathbf{k}\frac{k_ik_j}{2\omega^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}e^{-i\omega(t-t')}\\ &=\frac{Re}{(2\pi)^3}\partial_i\partial_j'\int_0^\infty\frac{e^{-ik\Delta t}}{2k}dk\\ &\quad\times\int_0^\pi d\theta\sin\theta e^{ikR\cos\theta}\int_0^{2\pi}d\phi\\ &=\frac{1}{(2\pi)^2}\partial_i\partial_j'\frac{1}{R}\int_0^\infty\frac{dk}{k^2}\sin kR\cos k\Delta t.\end{aligned} \tag{A15}$$

Because there is an infrared divergence in the above integral, we will introduce a regulator β in the denominator of the integrand and then let β approach 0 after the integration is performed:

$$\begin{aligned}F_{ij}(x,x')&=\frac{1}{(2\pi)^2}\partial_i\partial_j'\lim_{\beta\to0}\frac{1}{R}\int_0^{+\infty}\frac{dk}{k^2+\beta^2}\sin kR\cos k\Delta t\\ &=\frac{1}{(2\pi)^2}\partial_i\partial_j'\lim_{\beta\to0}f(\beta,R,\Delta t).\end{aligned} \tag{A16}$$

Here we have used a integral in Ref. [14] and defined

$$f(\beta,R,\Delta t)=\frac{1}{4\beta R}\{e^{\beta(\Delta t-R)}\mathrm{Ei}[\beta(R-\Delta t)]+e^{-\beta(\Delta t+R)}\mathrm{Ei}[\beta(R+\Delta t)]-e^{\beta(\Delta t+R)}\mathrm{Ei}[-\beta(R+\Delta t)]-e^{\beta(R-\Delta t)}\mathrm{Ei}[\beta(\Delta t-R)]\}. \quad \text{(A17)}$$

Here Ei(x) is the exponential-integral function. Making use of the fact that, when x is small,

$$\mathrm{Ei}(x)\approx\gamma+\ln|x|+x+\frac{1}{4}x^2+\frac{1}{18}x^3+O(x^4), \quad \text{(A18)}$$

where γ is the Euler constant, and expanding f around $\beta=0$ to the order of β^2, we get

$$f(\beta,R,\Delta t)\approx 1-\gamma-\ln\beta-\frac{1}{2}\ln(R^2-\Delta t^2)-\frac{\Delta t}{4R}\ln\left(\frac{R+\Delta t}{R-\Delta t}\right)^2+O(\beta^2). \quad \text{(A19)}$$

Taking the limit and keeping in mind that the constant terms (with respect to x and x') vanish under differentiation, we finally obtain

$$F_{ij}(x,x')=-\frac{1}{(2\pi)^2}\partial_i\partial_j'\left[\frac{1}{2}\ln(R^2-\Delta t^2)+\frac{\Delta t}{4R}\ln\left(\frac{R+\Delta t}{R-\Delta t}\right)^2\right]. \quad \text{(A20)}$$

Now let us turn our attention to $H_{ijkl}(x,x')$. We have, proceeding with similar steps as we did for $F_{ij}(x,x')$,

$$H_{ijkl}(x,x')=\frac{Re}{(2\pi)^3}\int d^3\mathbf{k}\,\frac{k_ik_jk_kk_l}{2\omega^5}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}e^{-i\omega\Delta t}$$

$$=-\frac{1}{(2\pi)^2}\partial_i\partial_j'\partial_k\partial_l'\lim_{\beta\to 0}\frac{1}{2\beta R}\frac{\partial}{\partial\beta}\times\int_0^{+\infty}\frac{dk}{k^2+\beta^2}\sin kR\cos k\Delta t$$

$$=-\frac{1}{(2\pi)^2}\partial_i\partial_j'\partial_k\partial_l'\lim_{\beta\to 0}\frac{1}{2\beta}\frac{\partial}{\partial\beta}f(\beta,R,\Delta t). \quad \text{(A21)}$$

Now expand $(1/2\beta)(\partial/\partial\beta)f(\beta,R,\Delta t)$ to order β^2 to find

$$\frac{1}{2\beta}\frac{\partial}{\partial\beta}f(\beta,R,\Delta t)=-\frac{1}{2\beta}-\frac{1}{3}[(\ln\beta+\gamma-1)R^2+3(\ln\beta+\gamma-1)\Delta t^2]-\frac{1}{12R}[(R+\Delta t)^3\ln|R+\Delta t|+(R-\Delta t)^3\ln|R-\Delta t|]. \quad \text{(A22)}$$

Plugging this result into Eq. (A21) and noting that only terms higher than quadratic in R contribute after the differentiation, we obtain

$$H_{ijkl}(x,x')=\frac{1}{96\pi^2}\partial_i\partial_j'\partial_k\partial_l'\times\left[(R^2+3\Delta t^2)\ln(R^2-\Delta t^2)^2+\left(3R\Delta t+\frac{\Delta t^3}{R}\right)\ln\left(\frac{R+\Delta t}{R-\Delta t}\right)^2\right]. \quad \text{(A23)}$$

For convenience, we give the explicit forms for $G^{(1)}_{xxxx}$ and $G^{(1)}_{zzzz}$ here:

$$G^{(1)}_{xxxx}(x,x')=2(D^{(1)}(x,x')-2F_{xx}(x,x')+H_{xxxx}(x,x'))$$
$$=\frac{1}{12\pi^2R^8\sigma^2}\{(\Delta x^2-\Delta t^2)(16\Delta x^6-24\Delta x^4\Delta t^2)-3\Delta t^2R_t^6+(9\Delta t^4+69\Delta x^2\Delta t^2+16\Delta x^4)R_t^4+(-72\Delta x^2\Delta t^4+32\Delta x^4\Delta t^2+32\Delta x^6)R_t^2\}-\frac{\Delta t}{16\pi^2R^9}\ln\left(\frac{R+\Delta t}{R-\Delta t}\right)^2[-R_t^6-(3\Delta t^2+9\Delta x^2)R_t^4+24\Delta t^2\Delta x^2R_t^2-8\Delta x^4\Delta t^2+8\Delta x^6], \quad \text{(A24)}$$

where

$$R_t^2=\Delta y^2+\Delta z^2 \quad \text{(A25)}$$

$$\Delta x=x-x' \quad \text{(A26)}$$

$$R^2=R_t^2+\Delta x^2=\Delta x^2+\Delta y^2+\Delta z^2 \quad \text{(A27)}$$

$$\sigma^2=R^2-\Delta t^2=\Delta x^2+\Delta y^2+\Delta z^2-\Delta t^2. \quad \text{(A28)}$$

To get $G^{(1)}_{zzzz}(x,x')$, all we need to do is to replace R_t^2 in Eq. (A24) by $R_t^2=\Delta y^2+\Delta x^2$.

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