

# Effective action and Hawking radiation for dilaton coupled scalars in two dimensions

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The effective one-loop action for general dilaton theories with an arbitrary dilaton-dependent measure and nonminimal coupling to scalar matter is computed. As an application we determine the Hawking flux to infinity from black holes in  $D$  dimensions. We resolve the recently resurrected problem of an apparent negative flux for nonminimally coupled scalars: For any  $D \geq 4$  black hole the complete flux turns out to be precisely the one of minimal coupling. This result is obtained from a Christensen-Fulling type argument involving the (non)conservation of energy-momentum. It is compared with approaches using the effective action.

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## I. INTRODUCTION

The study of gravity models in two dimensions (2D) provides some important answers to the difficult questions posed by the quantization of gravity. Indeed, the restriction to one time and one space coordinate of Einstein gravity in four and higher dimensions, to a  $D=2$  dilaton theory (spherical reduction, SRG) [1] represents a class of such models with immediate physical relevance.

The last decade has seen substantial progress in this field. Especially the use of a light cone gauge for the Cartan-variables [2] which amounts to choosing an Eddington-Finkelstein gauge for the 2D metric has led to a much better and simpler understanding of the classical theory [3,4] as well as at the quantum level [5]. Most of these results have turned out to be less obvious or even not attainable in the traditional approach where a conformal gauge was used for the 2D metric [6,7].

In order to be able to attribute to 2D models of gravity a sufficient credibility as far as the application of their results to the genuine (higher dimensional) case is concerned, an obvious precondition is that what has been found already at  $D \geq 4$  should be fully reproduced in  $D=2$ , wherever such an overlap occurs and can be tested. Hawking radiation, one of the most interesting features of black hole (BH) physics, is a consequence of precisely this kind. It is known from calculations in  $D=4$  that the thermal radiation from a BH to infinity is related to the Hawking temperature at the horizon according to the laws of blackbody radiation [8]. A central role is played by the scale anomaly of the energy momentum (EM) tensor.

By spherical reduction from  $D \geq 4$  the scalar field acquires nonminimal coupling to the dilaton field. That this

may cause complications in a fully two-dimensional calculation of the Hawking flux has been realized only relatively recently in the pioneering paper by Mukhanov and collaborators [9]. As observed for the first time by these authors, a naive adoption of the 4D approach due to the dependence of the anomaly of the dilaton field leads to a negative flux at infinity. Therefore, these authors added to their integrated effective (Polyakov) action a nonlocal Weyl-invariant term of the Coleman-Weinberg type which depends on a renormalization scale. Then the sign of the resulting total flux was seen to become positive.

This question was taken up in a number of papers with mutually contradicting results for the anomaly [10–14]. Using different expressions for the effective action it was claimed [11,13] that instabilities and “antievaporation” phenomena can occur. Even the conformal anomaly itself became the subject of a discussion [10–12,14,15]. In our opinion this part of the problem has been settled with the enlightening paper by Dowker [16] who confirmed the previous results [9,10,14] for SRG and our result [12] for general dilaton models. A recent summary of the different approaches leading to negative Hawking flux and a general framework for a solution of this problem has been proposed in [17]; another attempt to solve this problem can be found in Ref. [18].

We believe that our present work for the first time provides a complete but perhaps surprising answer to the question of 2D Hawking radiation from nonminimally coupled scalar fields. Our approach is based upon a consistent use of  $\zeta$ -function regularization, not only for the part of the effective action determined by minimally coupled scalars, but also for the part controlled by the dilaton field.

In Sec. II we recall the action of SRG in  $D \geq 4$  dimensions. We summarize our conventions for the solution of this (BH) background part. We also derive in a simple manner the nonconservation relation for the EM tensor, valid for arbitrary dilaton theories in any dimension for nonminimally coupled matter. The integrated effective action is determined in Sec. III. Our result generalizes the Polyakov action [19] to

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arbitrary nonminimal dilaton coupling of matter fields and to arbitrary dilaton dependent measure. Section IV contains the application of the results of Sec. III to the Hawking flux at infinity. We first integrate the nonconservation equation for the EM tensor. Here the input is the 2D anomaly together with the result for the ‘‘dilaton anomaly,’’ not requiring the knowledge of the functionally integrated action. Subsequently we discuss how one may arrive at this flux from the EM tensor, obtained directly from that action.

## II. DILATON THEORY FOR SPHERICAL REDUCTION

### A. Spherically reduced action

SRG in  $D \geq 4$  is based upon the choice of the  $D$ -dimensional metric

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu - \frac{1}{\lambda^2} e^{-[4/(D-2)]\phi} (d\Omega)^2, \quad (1)$$

where  $d\Omega$  is the standard line element on  $S^{D-2}$ ,  $\lambda$  is a scale parameter. The dilaton field  $\phi$  and  $g_{\mu\nu}$  depend on the two first coordinates  $x^\mu$  only. In terms of (1) the  $D$ -dimensional Einstein-Hilbert Lagrangian reduces to

$$\mathcal{L}_{\text{SRG}} = e^{-2\phi} \sqrt{-g} \left( R + \frac{4(D-3)}{D-2} (\nabla\phi)^2 - \frac{\lambda^2}{4} (D-2)(D-3) e^{[4/(D-2)]\phi} \right) \quad (2)$$

which is the particular case  $a = (D-3)/(D-2)$  and  $B = -(\lambda^2/4)(D-2)(D-3)$  of a more general dilatonic Lagrangian treated in Ref. [4] for general values of the parameter  $0 \leq a \leq 1$

$$\mathcal{L}_g = \sqrt{-g} e^{-2\phi} [R + 4a(\nabla\phi)^2 + B e^{4(1-a)\phi}]. \quad (3)$$

The family of models of this type comprises all theories with one horizon, Minkowski asymptotics and (for  $0 < a < 1$ ) with the same (null and non-null incomplete) singularity as the Schwarzschild BH. The dilaton BH [6] is contained as the limit  $a=1$  or  $D \rightarrow \infty$  with  $\lambda^2 \rightarrow \lambda^2/D^2$ . It has null-complete geodesics at the singularity [4].

The most convenient way to obtain the general solution for Eqs. (2) or (3) has been described in Ref. [4]. For our present purposes we need the background solution in conformal gauge. With the proper choice of the coordinates [20,21] which yields the Minkowski metric in the asymptotic region it takes the form

$$(ds)^2 = K(u)(d\tau^2 - dz^2), \quad dU = K(u)dz, \quad (4)$$

$$K(u) = 1 - \left( \frac{u_h}{u} \right)^{D-3}, \quad (5)$$

$$\phi(u) = -\frac{D-2}{2} \log \left( \frac{2u}{D-2} \right), \quad (6)$$

where  $u_h$  is the value of  $u$  at the horizon [defined by the equation  $K(u_h) = 0$ ], the asymptotic region corresponds to  $u = \infty$ . The explicit expression for  $u_h$  is not needed for our calculations,  $u_h$  is proportional to the absolutely conserved quantity  $C$ , which, in turn, is proportional to the Arnoevitt-Deser-Misner (ADM) mass of the BH [21]. For  $D=4$  one obtains the usual Schwarzschild solution. In the solution (4)–(6) we slightly change the notation of Ref. [21] [ $U \rightarrow u, L(U) = -K(u)$ ]. The line element (4) in conformal light cone coordinates reads

$$(ds)^2 = e^{2\rho} dx^+ dx^-, \quad \rho = \frac{1}{2} \log[K(u)], \quad (7)$$

where  $x^\pm = \tau \pm z$ . The range for  $z$  is  $-\infty \leq z \leq +\infty$ . Thus derivatives of light cone coordinates, acting on functions of  $u$  become

$$\partial_+ = -\partial_- = \frac{1}{2} \partial_z = -\frac{1}{2} K(u) \partial_u. \quad (8)$$

For completeness we also quote the Hawking temperature, computed from  $K(u)$  as the surface gravity at the horizon:

$$T_H = \frac{1}{4\pi} K'(u)|_{u=u_h} = \frac{D-3}{4\pi u_h}. \quad (9)$$

Reducing the action for massless matter  $f$ , coupled minimally in  $D=4$  according to Eq. (1) leads to nonminimal coupling in the corresponding 2D Lagrangian

$$\mathcal{L}_{(m)} = \frac{1}{2} e^{-2\phi} \sqrt{-g} g^{\mu\nu} (\partial_\mu f) (\partial_\nu f). \quad (10)$$

The formalism in the following will be developed for a general dilaton factor  $\exp[-2\varphi(\phi)]$ .

Spherical reduction also affects the definition of the covariant measure. This is seen most directly from the path integral whose diffeomorphism invariant definition at general  $D$  requires a factor  $\sqrt[4]{-g_{(D)}}$  where  $g_{(D)}$  is the determinant of the original  $D$ -dimensional metric [22]. By Eq. (1) this yields a factor  $e^{-\phi}$  so that the scalar field  $\tilde{f}$  redefined as  $\tilde{f} = f e^{-\phi}$  possesses a trivial measure. Of course, such a factor is nothing else but the inverse power of the radius required for a proper inclusion of  $s$ -wave excitations. Also here we consider the more general case

$$\tilde{f} = f e^{-\psi(\phi)} \quad (11)$$

and take the standard path integral measure for  $\tilde{f}$ . Namely, we require  $\int (d\tilde{f})^4 \sqrt{-g} \exp(i\int \sqrt{-g} \tilde{f}^2)$  be a field independent (infinite) constant.

### B. Nonconservation of the energy momentum tensor for dilaton coupled fields

For nonminimal coupling of the scalars to the dilaton field the conservation law for the EM tensor must be modified. Classically the matter field action is invariant under the diffeomorphism transformations

$$\begin{aligned}
\delta g_{\mu\nu} &= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \\
\delta\Phi &= \xi^\nu \partial_\nu \Phi, \\
\delta f &= \xi^\nu \partial_\nu f,
\end{aligned} \tag{12}$$

where  $\Phi$  denotes either the dilaton field  $\phi$  or any local function thereof. By applying the transformation (12) to the action  $S_{(m)}$  we obtain on the mass-shell for the scalar field

$$\nabla^\mu T_{\mu\nu} = -(\partial_\nu \Phi) \frac{1}{\sqrt{-g}} \frac{\delta S_{(m)}}{\delta \Phi}, \tag{13}$$

where the EM tensor is defined as usual

$$\frac{\sqrt{-g}}{2} T_{\mu\nu} = \frac{\delta S_{(m)}}{\delta g^{\mu\nu}}. \tag{14}$$

The symmetry (12) is retained also at the quantum level when the scalar field is integrated out. Thus Eq. (13) holds as well for the expectation values, i.e., for the corresponding quantities computed from the one-loop effective action  $W$ . Recently the appearance of a nonconservation equation (13) in  $D=2$  has been noted after reduction from the  $D=4$  case [17,18]. As seen from the simple derivation above such a relation holds in fact for any generic dilaton theory in any dimension. It could also be interpreted as an ‘‘extended’’ conservation law, involving  $\delta W/\delta\Phi$  as part of an extended EM tensor.

### III. EFFECTIVE ACTION FOR DILATON THEORIES

Expressing the classical action related to Lagrangian (10) in terms of the field  $\tilde{f}$  according to Eq. (11) yields the classical action

$$S = -\frac{1}{2} \int \sqrt{-g} d^2x \tilde{f} A \tilde{f}, \tag{15}$$

containing the differential operator

$$\begin{aligned}
A &= -e^{-2\varphi+2\psi} g^{\mu\nu} [\nabla_\mu \nabla_\nu + 2(\psi_{,\mu} - \varphi_{,\mu}) \nabla_\nu \\
&\quad + (\nabla_\mu \nabla_\nu \psi) - 2\varphi_{,\mu} \psi_{,\nu} + \psi_{,\mu} \psi_{,\nu}].
\end{aligned} \tag{16}$$

The one loop effective action is obtained by the path integral for  $\tilde{f}$  as

$$W = \frac{1}{2} \text{Tr} \ln A. \tag{17}$$

$W$  depends on the metric, on  $\varphi$  and  $\psi$  which in the following all will be regarded as independent (background) fields. In Eq. (17)  $W$  represents the Euclidean action. The path integral leading to that equation should be done with  $\sqrt{-g} \rightarrow i\sqrt{g}$  in Eq. (15) to obtain the  $\zeta$ -function regularization method with elliptic differential operator  $A$  after continuation to the Euclidean domain. This is implied in the following, although we retain Minkowski space notation.

In that regularization [23,24]  $W$  can be expressed in terms of the zeta function of the operator  $A$ :

$$W = -\frac{1}{2} \zeta'_A(0), \quad \zeta_A(s) = \text{Tr}(A^{-s}). \tag{18}$$

The prime denotes differentiation with respect to  $s$ .

Evaluation of the  $\zeta'_A(0)$  in general is quite a tedious task. For the case of a generic operator  $A$  no analytic formulas are available. Fortunately, as will be shown below, in the particular case of Eq. (16) variations of the  $\zeta'_A(0)$  with respect to the dilaton field and to the scale transformation of the metric can be reduced to known heat kernel coefficients for certain second order operators.

First, we repeat our derivation [12] of the trace of the EM tensor. The variation of the zeta function with respect to a certain parameter or field is related to the one of the operator  $A$  as [25,26]

$$\delta \zeta_A(s) = -s \text{Tr}[(\delta A) A^{-1-s}]. \tag{19}$$

An infinitesimal conformal transformation  $\delta g_{\mu\nu} = \delta k(x) g_{\mu\nu}$  produces the trace of the (effective) EM tensor

$$\begin{aligned}
\delta W &= \frac{1}{2} \int d^2x \sqrt{-g} \delta g^{\mu\nu} T_{\mu\nu} \\
&= -\frac{1}{2} \int d^2x \sqrt{-g} \delta k(x) T_\mu^\mu(x).
\end{aligned} \tag{20}$$

Due to the multiplicative transformation property  $\delta A = -\delta k A$  of Eq. (16) (valid in  $D=2$  only) powers of  $A$  in Eq. (19) recombine to  $A^{-s}$ . With the definition of a generalized  $\zeta$ -function [27]

$$\zeta(s|\delta k, A) = \text{Tr}(\delta k A^{-s}) \tag{21}$$

the variation in Eq. (20) can be identified with

$$\delta W = -\frac{1}{2} \zeta(0|\delta k, A). \tag{22}$$

Combining Eqs. (22) and (20) one obtains

$$\zeta(0|\delta k, A) = \int d^2x \sqrt{-g} \delta k(x) T_\mu^\mu(x). \tag{23}$$

By a Mellin transformation one can show that  $\zeta(0|\delta k, A) = a_1(\delta k, A)$  [27], where  $a_1$  is defined as a coefficient in a small  $t$  asymptotic expansion of the heat kernel:

$$\text{Tr}[F \exp(-At)] = \sum_n a_n(F, A) t^{n-1}. \tag{24}$$

To evaluate  $a_1$  according to Ref. [27] we rewrite  $A$  as ( $\hat{\nabla}_\mu$  refers to the metric  $\hat{g}_{\mu\nu}$ )

$$A = -(\hat{g}^{\mu\nu} D_\mu D_\nu + E), \quad E = \hat{g}^{\mu\nu} (-\varphi_{,\mu} \varphi_{,\nu} + \hat{\nabla}_\mu \hat{\nabla}_\nu \varphi), \tag{25}$$

where  $\hat{g}^{\mu\nu} = e^{-2\varphi+2\psi} g^{\mu\nu}$ ,  $D_\mu = \hat{\nabla}_\mu + \omega_\mu$ ,  $\omega_\mu = \psi_{,\mu} - \varphi_{,\mu}$ . Then for  $a_1$  follows [27]

$$a_1(\delta k, A) = \frac{1}{24\pi} \text{tr} \int d^2x \sqrt{-\hat{g}} \delta k (\hat{R} + 6E). \quad (26)$$

$\text{tr}$  denotes ordinary trace over all matrix indices (if any). In the present case this is a trivial operation. However, below we will need the heat kernel coefficient  $a_1$  for a matrix operator where the more general formula (26) is essential. Returning to the initial metric and comparing with Eq. (20) the most general form of the ‘‘conformal anomaly’’ for non-minimal coupling in  $D=2$  is found to be [12]

$$T_\mu^\mu = \frac{1}{24\pi} [R - 6(\nabla\varphi)^2 + 4\Box\varphi + 2\Box\psi]. \quad (27)$$

The variation of the effective action with respect to  $\varphi$  or  $\psi$  does not exhibit the same multiplicative property as the conformal variation, because after substituting in Eq. (19) the variation of  $A$  does not recombine to powers of  $A$ . Therefore, the heat kernel technique is not applicable to the evaluation of Eq. (19) as it stands. However, crucial simplifications occur after transition to flat space by means of a conformal transformation. In conformal gauge  $g_{\mu\nu} = e^{2\rho} \eta_{\mu\nu}$  with flat metric  $\eta_{\mu\nu}$  the identity

$$\frac{\delta W(\rho)}{\delta\varphi} = \int_0^\rho d\sigma \frac{\delta^2 W(\sigma)}{\delta\sigma \delta\varphi} + \frac{\delta W(0)}{\delta\varphi} \quad (28)$$

is obvious, with an analogous one for the variation with respect to  $\psi$ . The first term on the right hand side of Eq. (28) can be expressed in terms of the conformal anomaly:

$$\frac{\delta W(\rho)}{\delta\varphi} = - \int_0^\rho d\sigma \sqrt{-g} \frac{\delta[T_\mu^\mu(\sigma)]}{\delta\varphi} + \frac{\delta W(0)}{\delta\varphi}. \quad (29)$$

To evaluate the second term, which at  $\rho=0$  represents the flat space contribution, we rewrite  $W(0)$  as

$$W(0) = \frac{1}{4} \log \int (d\vec{f}) \exp\left(- \int d^2x \sqrt{-\eta} \vec{f} \mathbf{1}_2(A) \vec{f}\right), \quad (30)$$

where we have doubled bosonic degrees of freedom by introducing the two-component field  $\vec{f}$ . In flat space the integral in the exponential in Eq. (30) can be rewritten as

$$\int d^2x \sqrt{-\eta} \vec{f} \mathbf{1}_2(A) \vec{f} = \int d^2x \sqrt{-\eta} \vec{f} D D^\dagger \vec{f}. \quad (31)$$

Here new differential operators in spinor space  $D = i\gamma^\mu e^{\psi} \partial_\mu e^{-\varphi}$  and  $D^\dagger = D(\psi \leftrightarrow -\varphi)$  have been introduced. Indeed, the right hand side of Eq. (31) is equal to

$$\int d^2x \sqrt{-\eta} [\vec{f}(A + 2\gamma^5 \epsilon^{\mu\nu} e^{2(\psi-\varphi)} \varphi_{,\mu} \psi_{,\nu}) \vec{f} + \epsilon^{\mu\nu} e^{2(\psi-\varphi)} \varphi_{,\mu} \partial_\nu (\vec{f} \gamma^5 \vec{f})] \quad (32)$$

which proves Eq. (31) after integration by parts. Therefore, in flat space

$$2W(0) = \frac{1}{2} \log \det(D D^\dagger) \quad (33)$$

holds. For the  $\zeta$  function of the operator  $D D^\dagger$  we use its representation in terms of an inverse Mellin transform of the heat kernel

$$\zeta_{D D^\dagger}(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{Tr} \exp(-t D D^\dagger). \quad (34)$$

This yields the variation of  $\zeta$  with respect to  $\varphi$  and  $\psi$ :

$$\begin{aligned} \delta \zeta_{D D^\dagger}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{Tr} \sum_n \frac{(-t)^n}{n!} [2\delta\psi (D D^\dagger)^n \\ &\quad - 2\delta\varphi (D^\dagger D)^n] \\ &= \frac{2}{\Gamma(s)} \int_0^\infty dt t^s \text{Tr} [-2\delta\psi D D^\dagger \exp(-t D D^\dagger) \\ &\quad + 2\delta\varphi D^\dagger D \exp(-t D^\dagger D)] \\ &= \frac{2\Gamma(1+s)}{\Gamma(s)} \text{Tr} [-2\delta\psi D D^\dagger (D D^\dagger)^{-s-1} \\ &\quad + 2\delta\varphi D^\dagger D (D^\dagger D)^{-s-1}] \\ &= -2s \text{Tr} [(D D^\dagger)^{-s} \delta\psi - (D^\dagger D)^{-s} \delta\varphi]. \quad (35) \end{aligned}$$

Thus the introduction of  $D D^\dagger$  has provided a means to achieve multiplicative factors for the two variations—at least in flat space, but this is sufficient for our purpose. By differentiating Eq. (35) with respect to  $s$  one arrives at

$$\begin{aligned} \delta \zeta'_{D D^\dagger}(0) &= -2[\zeta(0|\delta\psi, D D^\dagger) - \zeta(0|\delta\varphi, D^\dagger D)] \\ &= -2[a_1(\delta\psi, D D^\dagger) - a_1(\delta\varphi, D^\dagger D)]. \quad (36) \end{aligned}$$

To evaluate  $a_1$  in the first term on the right hand side of Eq. (36) we again use the method of Ref. [27]. Introducing yet another type of differential operator in spinor space, we represent the operator  $D D^\dagger$  as

$$\begin{aligned} D D^\dagger &= -(\hat{g}^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu + E), \\ \mathcal{D}_\nu &= \partial_\nu + \psi_{,\nu} - \varphi_{,\nu} - \gamma^5 \epsilon^\mu{}_\nu \varphi_{,\mu}, \quad \hat{g}^{\mu\nu} = e^{2(\psi-\varphi)} \eta^{\mu\nu}, \\ E &= \hat{g}^{\mu\nu} (\hat{\nabla}_\mu \hat{\nabla}_\nu \varphi), \quad (37) \end{aligned}$$

and again use the result (26). The covariant derivatives  $\hat{\nabla}_\mu$  refer to the present metric  $\hat{g}_{\mu\nu}$ . In a similar manner the second heat kernel coefficient  $a_1$  for the operator  $D^\dagger D$  is obtained by the replacement  $\varphi \rightarrow -\psi$ ,  $\psi \rightarrow -\varphi$ . From Eq. (36) with  $\Delta = \eta^{\mu\nu} \partial_\mu \partial_\nu$

$$\begin{aligned} \delta \zeta'_{D D^\dagger}(0) = & -\frac{1}{3\pi} \int d^2x \sqrt{-\eta} [\delta\psi(2\Delta\varphi + \Delta\psi) \\ & + \delta\varphi(2\Delta\psi + \Delta\varphi)] \end{aligned} \quad (38)$$

follows. The curved space version of Eq. (38) can be obtained by means of the identity  $\sqrt{-\eta}\Delta = \sqrt{-g}\square = \sqrt{-g}g^{\mu\nu}\nabla_\mu\nabla_\nu$ . We have retained the determinant also for the flat metric in order to cover the case of light cone coordinates (7) where  $\eta = \det \eta \neq -1$ .

Now all variations of the effective action  $W(\rho, \phi, \psi)$  with respect to all background fields can be summarized:

$$\frac{\delta W}{\delta \varphi} = -\frac{1}{12\pi} \sqrt{-\eta} (6\partial^\mu(\rho\partial_\mu\varphi) + 2\Delta\rho - 2\Delta\psi - \Delta\varphi), \quad (39)$$

$$\frac{\delta W}{\delta \psi} = -\frac{1}{12\pi} \sqrt{-\eta} (\Delta\rho - 2\Delta\varphi - \Delta\psi), \quad (40)$$

$$\frac{\delta W}{\delta \rho} = -\frac{1}{12\pi} \sqrt{-\eta} (-\Delta\rho - 3(\partial_\mu\varphi)^2 + 2\Delta\varphi + \Delta\psi). \quad (41)$$

The solutions to Eqs. (39)–(41) can be found by inspection:

$$\begin{aligned} W = & -\frac{1}{24\pi} \int d^2x \sqrt{-\eta} (-\rho\Delta\rho + 2\psi\Delta\rho - \psi\Delta\psi \\ & - 6\rho(\partial_\mu\varphi)^2 + 4\varphi\Delta\rho - 4\varphi\Delta\psi - \varphi\Delta\varphi). \end{aligned} \quad (42)$$

By the replacements  $\sqrt{-\eta}\Delta = \sqrt{-g}\square$  when acting on  $\varphi$  or  $\psi$ , and  $\sqrt{-\eta}\Delta\rho = -\frac{1}{2}\sqrt{-g}R$  and some partial integrations the integrated effective action can be brought into covariant form:

$$\begin{aligned} W = & -\frac{1}{24\pi} \int d^2x \sqrt{-g} \left[ -\frac{1}{4}R\square^{-1}R + 3(\nabla\varphi)^2\square^{-1}R \right. \\ & \left. - R(\psi + 2\varphi) + (\nabla\psi)^2 + (\nabla\varphi)^2 + 4(\nabla^\mu\psi)(\nabla_\mu\varphi) \right] \\ & + W(\mu, \mu'). \end{aligned} \quad (43)$$

The first term in Eq. (43) represents the Polyakov action [19] for minimal coupling ( $\varphi = \psi = 0$ ) of the scalar fields.  $\varphi(\phi)$  and  $\psi(\phi)$  encode a general dilaton coupling of the scalars and of the dilaton-dependent measure, respectively. Thus Eq. (43) generalizes the Polyakov action for the case of nonminimal coupling to the dilaton field. The appearance of a new nonlocal term should be emphasized. A functional integral applied to a bounded region in space time always contains ambiguities with respect to eventual surface variables. In that case Eq. (43) may acquire further (here undetermined) contributions. The terms  $W(\mu, \mu')$  depending on the renormalization points  $\mu, \mu'$  will be discussed below.

For SGR from  $D$  dimensions the case  $\varphi = \psi = \phi$  is of special interest:

$$\begin{aligned} W_{\text{SRG}} = & \frac{1}{96\pi} \int d^2x \sqrt{-g} [R\square^{-1}R - 12(\nabla\phi)^2\square^{-1}R \\ & + 12\phi R - 24(\nabla\phi)^2] + W(\mu, \mu'). \end{aligned} \quad (44)$$

The second, nonlocal term was not present in the analogous formula for the full effective action in Ref. [9]. The first three terms, however, appear in the ‘‘uncorrected’’ effective action there. In the first Ref. [13] all four terms, but the last two with different factors, can be found.

Usually, the full effective action including conformally invariant part is available as a power series in a small parameter [28]. No such parameter exists for the BH background. Therefore, the closed form of the action (44) is essential. Two previous famous examples where such a closed form could be obtained were the Polyakov and WZNW actions. In those cases the effective actions were completely defined by the corresponding anomalies. For general dilaton theories in  $D=2$  we have here a similar situation, because the ‘‘dilaton anomaly’’ Eqs. (39), (40) ultimately can be interpreted as carrying the information of part of the conformal anomaly belonging to some theory in  $D$  dimensions. But, as will be seen in the next section, Eq. (44) cannot be the whole story.

#### IV. HAWKING RADIATION

As pointed out above, the direct derivation of the Hawking flux to  $\mathcal{I}^+$  from Eq. (44) has to rely on a complete functional integration of the action. This is avoided in the Christensen-Fulling approach [8] where only an ordinary integration is required.

In conformal light cone coordinates Eqs. (7), (8) we separate the conformal anomaly (27) for  $\varphi = \psi = \phi$  as

$$T_{+-} = T_{+-}^{\text{min}} + T_{+-}^{(1)}, \quad (45)$$

$$T_{+-}^{\text{min}} = -\frac{1}{12\pi} \partial_+ \partial_- \rho, \quad (46)$$

$$T_{+-}^{(1)} = \frac{1}{4\pi} [\partial_+ \partial_- \phi - (\partial_+ \phi)(\partial_- \phi)]. \quad (47)$$

From Eqs. (39) and (40) we obtain

$$\frac{1}{\sqrt{-g}} \frac{\delta W}{\delta \phi} = -\frac{1}{4\pi} [2\partial^\mu(\rho\partial_\mu\phi) + \Delta\rho - 2\Delta\phi]. \quad (48)$$

Equation (13) for the minus component of the index  $\nu$  becomes

$$\begin{aligned} \partial_+ T_{--} = & -\partial_- T_{+-} + 2(\partial_- \rho) T_{+-} \\ & - \frac{(\partial_- \phi)}{2\pi} [\partial_+ \partial_- \rho + \partial_+(\rho\partial_- \phi) + \partial_-(\rho\partial_+ \phi) \\ & - 2(\partial_+ \partial_- \phi)]. \end{aligned} \quad (49)$$

From Eqs. (7) and (8) the external fields only depend on  $z(u)$ , therefore Eq. (49) may be integrated straightforwardly. Choosing the limits  $z_h = -\infty$  and  $z = \infty$  for  $T_{--}$  we take into

account the condition of a finite flux (in Kruskal coordinates) at the horizon [8] which means vanishing  $T_{--}$  at  $z = -\infty$ . The integrated first term on the right hand side of Eq. (49) only contributes at the limits of the integral. It vanishes there [see Eqs. (6)–(8)]. The integral from  $T_{+-}^{\min}$  in the second term on the right hand side of Eq. (49) produces the flux for minimal coupling [see Eq. (9)]:

$$T_{--}^{\min} = \frac{(D-3)^2}{192 \pi u_h^2} = \frac{\pi}{12} T_H^2. \quad (50)$$

If expressed in terms of the Hawking temperature it does not depend on the dimension. Of course, the corresponding flux of the unreduced theory acquires an additional factor proportional to  $T_H^{D-2}$  from proper counting of further degrees of freedom on  $S^{D-2}$ . Inserting the nonminimal contribution Eq. (47) from the anomaly  $T_{+-}^{(1)}$  into the same term of Eq. (49) after integration yields

$$\begin{aligned} T_{--}^{(1)} &= -2 \int_{-\infty}^{\infty} dz' (\partial_{z'} \rho) T_{+-}^{(1)} \\ &= -\frac{1}{8\pi} \int_{-\infty}^{\infty} dz' (\partial_{z'} \rho) [(\partial_{z'} \phi)^2 - \partial_{z'}^2 \phi] \\ &= -\frac{9(D-2)}{2(D-1)} T_{--}^{\min}. \end{aligned} \quad (51)$$

In terms of the parameter  $a$  in the models of Eq. (3) this result has been obtained already in Ref. [12]. Together with  $T_{--}^{\min}$  this would yield the unphysical result of a negative flux. However, the nonconservation also implies the additional terms in the second line of Eq. (49). The last  $\rho$ -independent one only contributes a total derivative to the  $z$ -integral which from the explicit expression of  $\phi$  in Eq. (5) with Eq. (8) again vanishes at the limits of integration. The remaining terms by partial integrals may be written as

$$\begin{aligned} T_{--}^{(2)} &= -T_{--}^{(1)} + \frac{1}{8\pi} \int_{-\infty}^{\infty} dz' \partial_{z'} [2(\partial_{z'} \phi)(\partial_{z'} \rho) \\ &\quad + \rho(\partial_{z'} \phi)^2], \end{aligned} \quad (52)$$

where the second term on the right hand side of Eq. (52) vanishes. We thus observe complete cancellation of the dilaton dependent terms in the flux.

It should be noted that the nonconservation equation (13) is nothing else than the  $D$ -dimensional conservation condition for the energy-momentum tensor. Hence we are allowed to apply the Christensen-Fulling procedure [8] without changes.

We finally compare this result to the one of a direct computation of  $T_{--}|_{z \rightarrow \infty}$  from the functionally integrated effective action. The contribution from the last term in square brackets of Eq. (44) to the functional derivative with respect to  $g_{\mu\nu}$  leads to a term proportional to  $g_{\mu\nu}$  itself and to one proportional to  $(\partial_\mu \phi)(\partial_\nu \phi)$ . For  $T_{--}$  in conformal gauge  $g_{--} = 0$  and  $(\partial_- \phi)^2 \propto K^2/u^2$  vanishes at infinity and at the horizon.

In the framework of the zeta function regularization this is taken into account by the terms

$$W(\mu, \mu') = \zeta_A(0) \log \mu + \frac{1}{2} \zeta_{DD^*}(0) \log \mu' \quad (53)$$

in  $W_{\text{SRG}}$ , Eq. (44). Here we need two normalization parameters,  $\mu$  and  $\mu'$ , since we use zeta functions of two different operators. From  $\zeta_A(0) = a_1(1, A)$  and the explicit formulas (25), (26), and (37) it can be verified that the only term in Eq. (53) which is not a total derivative reads

$$\log \mu \int d^2x \sqrt{-g} \left( -\frac{1}{4\pi} (\nabla \phi)^2 \right). \quad (54)$$

However, the contribution of Eq. (54) to  $T_{--}$  vanishes at infinity as well as at the horizon.

The first three terms in Eq. (44) coincide with the integrated effective action of Ref. [9] before there a compensating expression has been added. Their contribution to  $T_{--}$  has been discussed recently in Ref. [17]. Direct insertion of the background solution (4)–(7) corresponds to the choice of the so-called Boulware vacuum [8]. It implies vanishing flux at infinity for all those terms, including the one for minimal coupling. At the same time the conformal flux at the horizon is finite which entails a divergent flux in global (Kruskal) coordinates. On the other hand, if—as in our Christensen-Fulling approach—the Unruh vacuum is chosen (vanishing conformal flux at the horizon), the total flux from Eq. (44) becomes negative, because the negative dilaton dependent contribution is larger than the (“correct”) positive one from the Polyakov action.

In this context we should recall that the derivation of the radiation flux to infinity is known to be quite a delicate matter [29]. Even without dilatons (as in the full  $D=4$  theory) the choice of the asymptotics for the inverse d’Alembertian has a decisive influence on the result. The direct integration of the EM conservation also is completely insensitive to back scattering effects which appear when the equations of motion for the scalar field are solved in a BH background. This back scattering is influenced strongly by the inclusion of dilaton fields [9].

Of course, by certain explicit assumptions which treat the different terms in Eq. (44) in a different manner, our result (50) could be obtained also from that equation; for example, different choices could be made for the asymptotic behavior of the inverse d’Alembertian in the Polyakov term (leading to  $T_{--}^{\min}$ ) and in the first dilaton term (leading to a vanishing “Boulware” flux at infinity). This certainly does not seem satisfactory; it just underlines our opinion that the effective action approach has a fundamental weakness: it encodes a UV effect from quantum corrections, i.e., in coordinate space is certainly only correct locally. This is in agreement with the rules for functional differentiation which in an expression such as Eq. (44) require sufficiently strong vanishing of the fields at the infinite boundaries of the integration in order to be able to perform partial integrations without surface contributions. But the region where the flux is needed here is precisely at that boundary. There the functional derivative

with respect to the metric, the flux, should give a nonvanishing result. Also the metric itself does not vanish there but becomes Minkowskian. The main advantage of the approach used in our paper is that the input from the one loop quantum effects entered locally. The subsequent integral from the horizon to infinity is a trivial, well defined ordinary one.

We conclude this section by noting that in the presence of general nonminimal coupling to the dilaton in the 2D context the very definitions of various vacua must probably be changed. Due to the dilaton ordinary plane waves are no longer solutions of the field equations. Hence the arguments based successfully in 4D [30] on selection of positive frequency modes fail. Again we do not need to worry about such issues in our approach.

## V. CONCLUSIONS

With Eq. (43) of our present paper we are able to present—to best of our knowledge—the first complete derivation of the local part of the effective one-loop action in  $D=2$  for a general dilaton theory. The nonminimal coupling to scalar fields encoded by  $\varphi(\phi)$  and the dilaton measure  $\psi(\phi)$  may be specialized to any given dilaton model. This expression, as well as the one of Eq. (44) with  $\varphi=\psi=\phi$  for spherically reduced gravity from  $D$  dimensions beside the Polyakov term contains another nonlocal contribution. Our derivation consistently uses  $\zeta$ -function regularization for *all* terms. We are able to tie in the functional derivative for the dilaton field with a kind of integrability condition involving the 2d scale anomaly together with a contribution which refers to a flat background. Integrating the nonconservation of the energy

momentum tensor we find that the Hawking radiation at infinity is identical to the one for minimally coupled scalars.

The cancellation of dilaton dependent terms in the flux does not seem too surprising in view of the fact that this is true also in the  $D=4$  calculation. Thus the input for a complete  $D=2$  computation should therefore be the same one as from the  $D=4$  anomaly. In  $D=2$  the effect of that anomaly is separated into the information encoded in the  $D=2$  anomaly plus another contribution which is expressible as a functional derivative of the effective action with respect to the dilaton field. Indeed, the relation of the latter quantity to the “transversal” part of the  $D=4$  anomaly has been pointed out already in Refs. [9] and [17].

However, taking the functionally integrated effective action as starting point, we encounter the usual problems [9,17] related to the derivation of a global result—flux at infinity—from a quantity in which only local quantum corrections are encoded. Our method to integrate completely the effective action, however, seems to allow interesting applications in other fields, upon which we hope to be able to report soon.

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- [1] P. Thoni, B. Isaak, and P. Hajicek, *Phys. Rev. D* **30**, 1168 (1984); P. Hajicek, *ibid.* **30**, 1178 (1984); S. R. Lau, *Class. Quantum Grav.* **13**, 1541 (1996).
- [2] W. Kummer and D. J. Schwarz, *Phys. Rev. D* **45**, 3628 (1992).
- [3] H. Grosse, W. Kummer, P. Prešnajder, and D. J. Schwarz, *J. Math. Phys.* **33**, 3892 (1992); T. Klösch and T. Strobl, *Class. Quantum Grav.* **13**, 965 (1996); **14**, 1689 (1997); W. Kummer, and P. Widerin, *Phys. Rev. D* **52**, 6965 (1995); W. Kummer and G. Tieber, *ibid.* **59**, 044001 (1999).
- [4] M. O. Kaktanaev, W. Kummer, and H. Liebl, *Phys. Rev. D* **53**, 5609 (1996); *Nucl. Phys.* **B486**, 353 (1997).
- [5] W. Kummer and D. J. Schwarz, *Nucl. Phys.* **B382**, 171 (1992); F. Haider and W. Kummer, *Int. J. Mod. Phys. A* **9**, 207 (1994); P. Schaller and T. Strobl, *Class. Quantum Grav.* **11**, 331 (1994); W. Kummer, H. Liebl, and D. V. Vassilevich, *Nucl. Phys.* **B493**, 491 (1997); **B513**, 723 (1998); **B544**, 403 (1999).
- [6] G. Mandal, A. Sengupta, and S. R. Wadia, *Mod. Phys. Lett. A* **6**, 1685 (1991); S. Elitzur, A. Forge, and E. Rabinovici, *Nucl. Phys.* **B359**, 581 (1991); E. Witten, *Phys. Rev. D* **44**, 314 (1991); C. G. Callan, S. B. Giddings, J. A. Harvey, and A. Strominger, *ibid.* **45**, 1005 (1992); J. Russo, L. Susskind, and L. Thorlacius, *Phys. Lett. B* **292**, 13 (1992).
- [7] S. P. deAlwis, *Phys. Lett. B* **289**, 278 (1992); T. Banks, A. Dabholkar, M. Douglas, and M. O’Laughlin, *Phys. Rev. D* **45**, 3607 (1992); S. D. Odintsov and I. L. Shapiro, *Phys. Lett. B* **263**, 183 (1991).
- [8] S. M. Christensen and S. A. Fulling, *Phys. Rev. D* **15**, 2088 (1977).
- [9] V. Mukhanov, A. Wipf, and A. Zelnikov, *Phys. Lett. B* **332**, 283 (1994).
- [10] T. Chiba and M. Siino, *Mod. Phys. Lett. A* **12**, 709 (1997).
- [11] R. Bousso and S. Hawking, *Phys. Rev. D* **56**, 7788 (1997); S. Nojiri and S. Odintsov, *ibid.* **57**, 2363 (1998); *Mod. Phys. Lett. A* **12**, 2083 (1997); *Phys. Rev. D* **57**, 4847 (1998); M. Buric, V. Radovanovic, and A. Mikovic, *ibid.* **59**, 084002 (1999).
- [12] W. Kummer, H. Liebl, and D. V. Vassilevich, *Mod. Phys. Lett. A* **12**, 2683 (1997).
- [13] R. Bousso and S. Hawking, *Phys. Rev. D* **57**, 2436 (1998); R. Bousso, *ibid.* **58**, 083511 (1998); S. Nojiri and S. D. Odintsov, *Mod. Phys. Lett. A* **13**, 2695 (1998); *Phys. Rev. D* **59**, 044003 (1999).
- [14] S. Ichinose, *Phys. Rev. D* **57**, 6224 (1998).
- [15] W. Kummer, H. Liebl, and D. V. Vassilevich, *Phys. Rev. D* **58**, 108501 (1998).
- [16] J. S. Dowker, *Class. Quantum Grav.* **15**, 1881 (1998).
- [17] R. Balbinot and A. Fabbri, *Phys. Rev. D* **59**, 044031 (1999).
- [18] F. C. Lombardo, F. D. Mazzitelli, and J. G. Russo, *Phys. Rev. D* **59**, 064007 (1999).

- [19] A. M. Polyakov, Phys. Lett. **103B**, 207 (1981).
- [20] M. O. Katanaev, W. Kummer, and H. Liebl, Nucl. Phys. **B486**, 353 (1997).
- [21] H. Liebl, D. V. Vassilevich, and S. Alexandrov, Class. Quantum Grav. **14**, 889 (1997).
- [22] K. Fujikawa, U. Lindström, N. K. Nielsen, M. Rocek, and P. van Nieuwenhuizen, Phys. Rev. D **37**, 391 (1988); D. J. Toms, *ibid.* **35**, 3796 (1987).
- [23] J. S. Dowker and R. Critchley, Phys. Rev. D **13**, 3224 (1976).
- [24] S. W. Hawking, Commun. Math. Phys. **55**, 133 (1977).
- [25] M. F. Atiyah, V. K. Patodi, and I. M. Singer, Math. Proc. Camb. Philos. Soc. **79**, 71 (1976).
- [26] G. Esposito, A. Y. Kamenshchik, and G. Pollifrone, *Euclidean Quantum Gravity on Manifolds with Boundary* (Kluwer, Dordrecht, 1997).
- [27] P. B. Gilkey, J. Diff. Geom. **10**, 601 (1975); *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem* (CRC Press, Boca Raton, 1994).
- [28] G. A. Vilkovisky, in *Quantum Theory of Gravity*, edited by S. M. Christensen (Hilger, Bristol, 1984); A. O. Barvinsky and G. A. Vilkovisky, Nucl. Phys. **B282**, 163 (1987); **B333**, 471 (1990); Yu. Gusev and A. Zelnikov, Class. Quantum Grav. **15**, L13 (1998).
- [29] I. D. Novikov and V. P. Frolov, *Physics of Black Holes* (Kluwer, Dordrecht, 1989).
- [30] W. Unruh, Phys. Rev. D **14**, 870 (1976).