Quantum corrected geodesics

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(Received 20 April 1999; published 24 September 1999)

We compute the graviton-induced corrections to the trajectory of a classical test particle. We show that the motion of the test particle is governed by an effective action given by the expectation value (with respect to the graviton state) of the classical action. We analyze the quantum corrected equations of motion for the test particle in two particular backgrounds: a Robertson Walker spacetime and a $(2+1)$-dimensional spacetime with rotational symmetry. In both cases we show that the quantum corrected trajectory is not a geodesic of the background metric. [S0556-2821(99)09118-3]

PACS number(s): 04.60.−m, 11.15.Kc

I. INTRODUCTION

A full quantum theory of gravity is still out of reach. However, in situations where the spacetime curvature is well below Planck's curvature, it is possible to compute some quantum gravity effects. Indeed, metric fluctuations can be quantized using standard methods. The nonrenormalizability of the resulting quantum field theory is not an impediment for making meaningful quantum corrections. The key point is to consider general relativity as an effective field theory [1].

Although the leading long distance quantum corrections are expected to be too small in realistic situations, the analysis of general relativity as an effective field theory is of conceptual interest. Moreover, tiny but measurable quantum gravity effects could show up when measuring the decoherence of wavepackets of a nonrelativistic particle subjected to the gravitational potential [2]. On the other hand, recent speculations raise the length scale relevant for quantum gravity effects from Planck length to a TeV scale [3]. In this situation, the effects of metric fluctuations could be easier to observe.

In the context of effective field theories, it is in principle possible to compute an effective action and effective field equations for the mean value of the spacetime metric. The effective field equations (known as semiclassical Einstein equations or back reaction equations) include the back reaction of quantum matter fields and of the metric fluctuations on the spacetime metric. These equations should be the starting point to investigate interesting physical problems like, for example, the dynamical evolution of a black hole geometry taking into account the evaporation process.

The back reaction equations have been investigated by several authors in the last twenty years or so [4]. However, due to the complexity of the problem (and also to the nonrenormalizability of the theory) most works considered scalar or spinor quantum matter fields, but the graviton contribution was simply omitted.

It is in general stressed that the graviton effects should be similar to those of a couple of massless, minimally coupled scalar fields. While this is true at the level of the back reaction equations, there is an important physical difference that has been pointed only recently [5]. When metric fluctuations are taken into account, the background geometry (i.e. the metric that solves the back reaction equations), turns out to be nonphysical. The reason is the following: any classical or quantum device used to measure the spacetime geometry will also feel the graviton fluctuations. As the coupling between the device and the metric is nonlinear, the device will not measure the background geometry, which therefore is not the relevant physical quantity to compute. In particular, in Ref. [5] we have shown that, working in the Newtonian approximation, the trajectory of a classical test particle is not a geodesic of the background metric. Instead its motion is determined by a quantum corrected equation that takes into account its coupling to the gravitons. Moreover, while the back reaction equations and their solutions depend on the gauge fixing of the gravitons, this dependence cancels out in the quantum corrected equation of motion for the test particle.

The aim of this paper is to analyze the effect of the gravitons on the motion of a test particle beyond the Newtonian approximation. In order to avoid technical complications, we will assume we know a solution to the back reaction equations, and will focus only on the departure of the test particle's equation of motion from the geodesic equation of the background metric. Moreover, we will consider models where it is easy to fix completely the gauge of the gravitons and quantize the theory by taking into account the remaining degrees of freedom.

The paper is organized as follows. In Sec. II we prove that the effective action that governs the motion of the test particle is the mean value of the classical action. In Sec. III we consider Robertson Walker universes. We first briefly describe how to quantize the metric fluctuations in terms of massless scalar fields. Then we compute the quantum correc-

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tions to the geodesic equation and solve the quantum corrected equations of motion perturbatively. In particular, we find the graviton corrections to the cosmological redshift. In Sec. IV we consider three-dimensional gravity coupled to a Maxwell field. Following Ref. [6], we first show that this model is exactly soluble: one can fully fix the gauge and show that the degrees of freedom reside in the Maxwell field. Then we compute the quantum corrected equation of motion for the test particle. We show that, even in regions where the background metric is locally flat, the trajectory of the test particle is not a straight line. Section V contains our final remarks.

II. EFFECTIVE ACTION FOR A TEST PARTICLE

In this section we will show that, when quantum metric fluctuations are taken into account, the effective action for the test particle is the mean value of its classical action. This result is summarized in Eq. (5) below (the reader may want to accept this as a reasonable assumption and skip this section).

Consider pure gravity described by Einstein-Hilbert action[1] $S_G=(2/\kappa^2)\int d^4x\sqrt{-g}R$, where $\kappa^2=32\pi G$, and imagine that in addition we have some type of matter content described by an action S_M. The effect of quantum metric fluctuations can be analyzed with the background field method, expanding the whole action S_G+S_M around a background metric as $g_{\mu\nu}\rightarrow g_{\mu\nu}+\kappa h_{\mu\nu}$, and integrating over the graviton field $h_{\mu\nu}$ to get an effective action for the background metric. In order to fix the gauge one chooses a gauge-fixing function $\chi^\mu[g,h]$, a gauge-fixing action $S_{gf}[g,h]=-(1/2)\int d^4x\sqrt{-g}\chi^\mu g_{\mu\nu}\chi^\nu$, and the corresponding ghost action S_{gh}.

Imagine that in addition we have a classical test particle that moves in the above background metric and we wish to study the effects of metric fluctuations on it. We couple gravity to the particle by means of the standard action $S_m[x]=-m\int\sqrt{-g_{\mu\nu}dx^\mu dx^\nu}$, where x^μ denotes the path of this test particle. The complete effective action $S_{\rm eff}$ for the background metric $g_{\mu\nu}$ and for the test particle m is obtained by integrating the whole action $S\equiv S_G+S_M+S_{gf}+S_m+S_{gh}$ over the graviton and ghost fields. To evaluate it in the one loop approximation we first expand S up to second order in gravitons. The second order term reads

$$S^{(2)}=\int d^4y\sqrt{-g}h_{\mu\nu}F^{\mu\nu\rho\sigma}h_{\rho\sigma}-\int d^4y\sqrt{-g}h_{\mu\nu}m^{\mu\nu\rho\sigma}h_{\rho\sigma} \tag{1}$$

where $\hat{F}\equiv F^{\mu\nu\rho\sigma}$ is a second order differential operator that depends on the background metric, and $m^{\mu\nu\rho\sigma}$ is a tensor depending on the position and velocity of the test particle,

$$m^{\mu\nu\rho\sigma}(y)=\frac{m\kappa^2}{8}\int d\tau\delta^4(y-x(\tau))\dot{x}^\mu\dot{x}^\nu\dot{x}^\rho\dot{x}^\sigma. \tag{2}$$

There is also a second order term in ghost fields, that for gauge-fixing functions linear in the metric fluctuations decouple from the gravitons, and couple only to the background metric.

The result of the path integral is the classical action $S_{\rm clas}=S_G+S_M+S_m$ plus the sum of two functional determinants,

$$S_{\rm eff}=S_{\rm clas}+\frac{i}{2}{\rm Tr}\ln(\hat{F}-\hat{m})-i{\rm Tr}\ln\hat{G} \tag{3}$$

where $\hat{G}$ is also a second order differential operator that arises from integrating over ghosts. Once the effective action is evaluated, one can derive the equations of motion for the background metric $g_{\mu\nu}$, the so-called semiclassical Einstein equations, i.e. $\delta S_{\rm eff}/\delta g_{\mu\nu}=0$. To solve these equations one can discard all contributions coming from the test particle, as they are vanishingly small. As they stand, these equations (obtained from the standard *in-out* effective action) are neither real nor causal. In order to get real and causal equations of motion for the background metric, the *in-in* effective action must be evaluated [7]. Alternatively, one can take twice the real and causal part of the propagators in the *in-out* field equations. In both ways one gets semiclassical Einstein equations suitable for initial value problems.

From the effective action given above one can also derive the quantum corrected equation of motion for the test particle, i.e. $\delta S_{\rm eff}/\delta x^\rho=0$, which will be our main concern in what follows. The same comments about reality and causality apply to this equation of motion. In this paper we will work with the usual *in-out* effective action and use the adequate propagators in the quantum corrected equations.

In general it is extremely complicated, if not impossible, to work out the functional traces in Eq. (3), so several approximation methods have been developed to deal with them. However, in this paper we will only focus on the quantum effects of the coupling between the test particle and gravitons. We can make use of the fact that the test particle has a small mass, so we can expand Eq. (3) in powers of m and just keep the leading contribution. In this way we find that the whole effective action reads

$$S_{\rm eff}[g_{\mu\nu},x]=S_{\rm clas}+\frac{i}{2}{\rm Tr}\ln\hat{F}-i{\rm Tr}\ln\hat{G}-\int d^4x\sqrt{-g}\langle h_{\mu\nu}m^{\mu\nu\rho\sigma}h_{\rho\sigma}\rangle. \tag{4}$$

The expectation value is taken with respect to the graviton state. The effective action for the test particle will be the sum of the classical term $S_m[x]$ and this last term, so that we conclude that in fact that effective action is the expectation value of the classical one

$$S_{\rm eff}[x]=\langle S_m[x]\rangle. \tag{5}$$

[1]Our metric has signature $(-+++)$ and the curvature tensor is defined as $R^\mu{}_{\nu\alpha\beta}=\partial_\alpha\Gamma^\mu_{\nu\beta}-\ldots$, $R_{\alpha\beta}=R^\mu{}_{\alpha\mu\beta}$ and $R=g^{\alpha\beta}R_{\alpha\beta}$. We use units $\hbar=c=1$.

It is important to stress that due to the nonlinear nature of the coupling between gravity and test particle, the effective Lagrangian is not the same as the classical lagrangian evaluated in the expectation value for the particle's path.

The calculation described so far preserves the covariance in the background metric $g_{\mu\nu}$. Alternatively, one can fully fix the gauge of the quantum fluctuations of the geometry and quantize the remaining degrees of freedom. As can be easily proved, the argument leading to Eq. (5) remains unchanged, since it relies only on the fact that the test particle mass is small.

III. QUANTUM CORRECTIONS TO GEODESICS FOR FLAT ROBERTSON-WALKER METRICS

A. Noncovariant quantization

In this subsection we briefly review the noncovariant method of quantization for flat Robertson-Walker (RW) universes. The metrics we are dealing with are therefore of the form $ds^2=-dt^2+a^2(t)d\mathbf{x}^2$, where $a(t)$ is the expansion coefficient. The action for the matter content in RW metrics has the form

$$S_{\mathrm{M}}=\int d^4x\sqrt{-g}\left[\frac{1}{2}(\rho+p)u^{\mu}u^{\nu}g_{\mu\nu}+\frac{1}{2}(\rho+3p)\right] \quad (6)$$

where u^{μ}, ρ and p and the fluid's four-velocity, density and pressure, respectively. The associated classical Einstein equations are

$$R_{\mu\nu}=-\frac{1}{2}\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T^{\lambda}_{\lambda}\right) \quad (7)$$

where the classical energy-momentum tensor is $T_{\mu\nu}=(\rho+p)u_{\mu}u_{\nu}-pg_{\mu\nu}$.

There are different ways to quantize the theory. One is based on the background field method, which was described above. Here we follow another quantization procedure that starts from the classical theory of perturbations in RW metrics, developed in [8]. One considers perturbations such that $\delta\rho=\delta p=\delta u^{\mu}=0$, and metric perturbations $h_{\mu\nu}$ that satisfy $u^{\mu}h_{\mu\nu}=0$, and further imposes the gauge conditions $h^{\mu\nu}{}_{;\nu}=0$. Finally one ends up with only two independent components of the metric, h_+ and $h_\times$, which can be expressed in terms of the original components of $h_{\mu\nu}$, and that correspond to the two polarizations of a gravitational wave. The above conditions on the metric imply that $h_{0\mu}=0$ and a transversality condition $\tilde{\nabla}_j h^{ij}=0$, where $\tilde{\nabla}_j$ denotes the covariant derivative with respect to the spatial part of the metric. Both components h_+ and $h_\times$, and also h_i^j, verify the field equation for a minimally coupled massless scalar field in RW metrics

$$\Box\phi=-a^{-3}\frac{\partial}{\partial t}\left(a^3\frac{\partial}{\partial t}\phi\right)+\nabla^2\phi=0. \quad (8)$$

To quantize we use the noncovariant quantization procedure of [9,10]. First one writes the second order term of the action $S_{\mathrm{G}}+S_{\mathrm{M}}$ in terms of the two independent degrees of freedom of the field, h_+ and $h_\times$

$$S^{(2)}_{\mathrm{G+M}}=\frac{1}{2}\int d^4x\sqrt{-g}[\partial_{\mu}h_+(x)\partial^{\mu}h_+(x)+\partial_{\mu}h_\times(x)\partial^{\mu}h_\times(x)] \quad (9)$$

and then imposes equal-time canonical conmutation relations for the two scalar fields $[h_a(\mathbf{x},t),\Pi_b(\mathbf{x}',t)]=i\delta_{ab}\delta(\mathbf{x}-\mathbf{x}')$, where $a,b=+,\times$ and Π_a is the canonical momentum conjugate to h_a. This quantization procedure is equivalent to that for the individual modes h_i^j. Instead of using canonical quantization, one can also do path integrals. One expands the action in terms of the individual modes h_i^j (or in terms of h_+ and $h_\times$) and integrates over them in order to get an effective action for the background metric. For the one loop effective action one needs the second order term of the expansion of the action in terms of metric perturbations, namely $S^{(2)}_{\mathrm{G+M}}=1/2\int d^4y\sqrt{-g}h_j^i\Box h_i^j$, where $\Box$ denotes the scalar D'Alambertian operator. Finally one has to evaluate the functional determinant of this differential operator.

B. Quantum corrected geodesic equation

Having summed up how to quantize metric perturbations in RW universes, let us see how such quantum metric fluctuations affect the motion of a classical test particle. As described in the previous section, the effective action for the test particle is the expectation value of the classical action, namely

$$S_{\mathrm{eff}}[x]=-m\int\sqrt{-g_{\mu\nu}(x)dx^{\mu}dx^{\nu}}-\frac{m\kappa^2}{8}\int d\tau\langle h_{ij}(x)h_{lm}(x)\rangle\dot{x}^i\dot{x}^j\dot{x}^l\dot{x}^m \quad (10)$$

where the dot denotes the derivative with respect to τ. The graviton two-point function can be expressed in terms of the scalar two-point function $\langle\phi(x)\phi(x')\rangle$ as

$$\langle h_{ij}(x)h_{lm}(x')\rangle=-\frac{1}{3}a^2(t)a^2(t')\left(\delta_{ij}\delta_{lm}-\frac{3}{2}\delta_{il}\delta_{jm}-\frac{3}{2}\delta_{im}\delta_{jl}\right)\langle\phi(x)\phi(x')\rangle. \quad (11)$$

We recall that in these expressions the metric $g_{\mu\nu}$ is the solution to the semiclassical Einstein equations that follow from quantizing gravity in a RW universe. In the following we will assume that these equations have been solved and that the quantum corrected expansion factor $a(t)$ has been found.

The geodesic equation for the test particle follows from $\delta S_{\mathrm{eff}}[x]/\delta x^{\rho}=0$. For the temporal component we get

$$\frac{d^2t}{d\tau^2}+a(t)a'(t)\left(\frac{d\mathbf{x}}{d\tau}\right)^2-\frac{\kappa^2}{8}\dot{x}^i\dot{x}^j\dot{x}^l\dot{x}^m\frac{\partial}{\partial t}G_{ijlm}[x(t)]=0 \quad (12)$$

where $a'(t) \equiv da/dt$ and $G_{ijlm}[x(t)]$ is the coincident limit of the graviton two-point function, evaluated along the trayectory of the particle. For the n-th spatial component $(n=1,2,3)$ we obtain

$$\frac{d}{d\tau}\left(a^2(t)\frac{dx^n}{d\tau}-\frac{\kappa^2}{2}G_{ijkl}[x(t)]\delta^{in}\dot{x}^j\dot{x}^k\dot{x}^l\right)=0. \tag{13}$$

Now let us solve Eqs. (12,13) for $d\mathbf{x}/d\tau$ and $dt/d\tau$. From Eq. (13) we see that the expression in parentheses is conserved. These conserved three quantities reflect the spatial translational invariance of RW metric, which is preserved upon the quantization procedure. Therefore

$$a^2(t)\frac{dx^n}{d\tau}-\frac{\kappa^2}{2}G_{ijkl}[x(t)]\delta^{in}\dot{x}^j\dot{x}^k\dot{x}^l=\alpha^n \tag{14}$$

where α^n is a dimensionless constant three-vector that depends on the initial velocity of the particle. Plugging this identity into Eq. (12) we find[2]

$$\frac{dt}{d\tau}=\sqrt{1+a^{-2}(t)\sum_{n=1}^{3}\left(\alpha^n+\frac{\kappa^2}{2}G_{ijkl}[x(t)]\delta^{in}\dot{x}^j\dot{x}^k\dot{x}^l\right)^2}. \tag{15}$$

Now we solve Eqs. (14,15) perturbatively in terms of the coupling between the test particle and gravitons. Let us assume that the initial velocity of the test particle is in the $x=x^1$ direction; i.e. $\alpha^n=\alpha\delta^{n1}$. The zeroth order approximation corresponds to neglecting the coupling between the particle and gravitons, which results in

$$\frac{dx}{d\tau}=\frac{\alpha}{a^2(t)}, \tag{16}$$

$$\frac{dt}{d\tau}=\sqrt{1+\alpha^2a^{-2}(t)}. \tag{17}$$

Note that the limiting case of a light ray (null limit) $dx/dt=1/a$ is obtained when $\alpha^2a^{-2}\gg 1$.

When the coupling is taken into account, we see that the particle still moves in the same x direction, and we get

$$\frac{dx}{d\tau}=\frac{\alpha}{a^2(t)}\left(1+\frac{\alpha^2\kappa^2}{3a^2(t)}\langle\phi^2(t)\rangle\right), \tag{18}$$

$$\frac{dt}{d\tau}=\sqrt{1+\alpha^2a^{-2}(t)}\left(1+\frac{\alpha^4\kappa^2}{3a^4(t)}\frac{\langle\phi^2(t)\rangle}{1+\alpha^2a^{-2}(t)}\right) \tag{19}$$

where we expressed the graviton two-point function in terms of the scalar two-point function as $G_{xxxx}(t)=(2/3)a^4(t)\langle\phi^2(t)\rangle$. The speed of the particle results

$$\frac{dx}{dt}=\frac{\alpha a^{-2}(t)}{\sqrt{1+\alpha^2a^{-2}(t)}}\left(1+\frac{32}{3}\pi G\langle\phi^2(t)\rangle\frac{\alpha^2a^{-2}(t)}{1+\alpha^2a^{-2}(t)}\right). \tag{20}$$

This is the main result of this section. It expresses the quantum corrections to the velocity of a test particle that moves in a flat Robertson-Walker quantum background. In the null limit

$$dx/dt\approx a^{-1}(t)[1+(32/3)\pi G\langle\phi^2(t)\rangle] \tag{21}$$

describes the graviton correction to the cosmological redshift. This expression is valid as long as the quantum correction remains small. Since, as we will shortly see, the two-point function in the coincident limit is proportional to the scalar curvature, the quantum correction is proportional to $R/R_{\rm Planck}$ times α^2a^{-2}. For $R/R_{\rm Planck}\ll 1$ it is possible to satisfy both the null limit condition and the smallness of the quantum correction at the same time, i.e. $\alpha^2a^{-2}\gg 1$ and $\alpha^2a^{-2}(R/R_{\rm Planck})\ll 1$.

To estimate the effect of this quantum correction on the classical trajectory of the test particle, we first have to evaluate the two-point function in the coincident limit, $\langle\phi^2(t)\rangle$. As is well known, this coincident limit is divergent, so a renormalization procedure is compelling. In the following we will calculate $\langle\phi^2(t)\rangle$ for particular RW metrics, namely $a(t)=a_0e^{Ht}$ (de Sitter) and $a(t)=a_0t^c$.

For de Sitter spacetime, the two-point function not only has UV problems but also IR ones. However, in the late time limit $t=t'\gg H^{-1}$ it is possible to give an approximate form for the renormalized function. It was shown by several authors [11–14] that the coincident limit grows linearly with the coordinate time, $\langle\phi^2(t)\rangle\approx H^3t/2\pi^2$. Using that $\kappa^2\propto R_{\rm Planck}^{-1}$ and that for de Sitter the curvature is constant $R\propto H^2$, we conclude that the quantum correction is proportional to $(R/R_{\rm Planck})F(t)$, where the function $F(t)=\alpha^2Hta_0^{-2}e^{-2Ht}/(1+\alpha^2a_0^{-2}\exp^{-2Ht})$ decreases exponentially for late times. The velocity of the test particle in the late time limit is therefore given by

$$\frac{dx}{dt}=\frac{\alpha a_0^{-2}e^{-2Ht}}{\sqrt{1+\alpha^2a_0^{-2}e^{-2Ht}}}\left(1+\frac{16G\hbar H^2F(t)}{3\pi c^5}\right) \tag{22}$$

where we have restored units $\hbar$ and c.

As we pointed out before, the scale factor $a(t)$ should be a solution to the semiclassical Einstein equations. A perturbative solution will be of the form $a(t)=a_{\rm clas}(t)+\delta a(t)$, $a_{\rm clas}$ being the classical scale factor and $\delta a\ll a_{\rm clas}$. It is well known that the semiclassical Einstein equations admit de Sitter solutions [15] $a(t)=a_0e^{Ht}$ with $H=H_{\rm clas}[1+\gamma(H_{\rm clas}^2/R_{\rm Planck})]$, $\gamma=O(1)$. Therefore, as long as $H_{\rm clas}^3t/R_{\rm Planck}\ll 1$ the correction to the scale factor is given by $\delta a/a_{\rm clas}\simeq\gamma(H_{\rm clas}^3t/R_{\rm Planck})$. Replacing $a(t)=a_{\rm clas}(t)+\delta a(t)$ in Eq. (22) we obtain, to first order in all quantum corrections

[2]This equation also follows from the very definition of the proper time. Indeed, from $1=(dt/d\tau)^2-a^2(t)(dx/d\tau)^2$ we easily get Eq. (15).

$$\frac{dx}{dt}=\frac{\alpha a_0^{-2}e^{-2H_{\rm clas}t}}{\sqrt{1+\alpha^2 a_0^{-2}e^{-2H_{\rm clas}t}}}\left(1+\frac{16G\hbar H^2 F(t)}{3\pi c^5}-\gamma\frac{2+\alpha^2 a_{\rm clas}^{-2}(t)}{1+\alpha^2 a_{\rm clas}^{-2}(t)}\frac{H_{\rm clas}^3 t}{R_{\rm Planck}}\right) \tag{23}$$

where $F(t)$ is to be evaluated with the classical value for the Hubble parameter. This shows that the quantum correction to the geodesics coming from the graviton coupling [second term in Eq. (23)] and the one coming from the semiclassical Einstein equations (third term) are of the same order of magnitude.

Consider now metrics with $a(t)=a_0 t^c$. Although these are not solutions to the semiclassical Einstein equation, they are useful to illustrate the corrections to the geodesics. In this case there are no infrared divergencies. In the Appendix we give some details as to how to evaluate the renormalized two-point function. The result is $\langle\phi^2(t)\rangle\propto t^{-2}\log(t^2\mu^2)$, where μ is an (arbitrary) renormalization scale. Since for these metrics the curvature is $R\propto t^{-2}$, we obtain that the quantum correction also has the form $(R/R_{\rm Planck})F(t)$, where now $F(t)=\alpha^2 a_0^{-2}t^{-2c}\log(t^2\mu^2)/(1+\alpha^2 a_0^{-2}t^{-2c})$, which also decreases for long times. The velocity of the test particle is

$$\frac{dx}{dt}=\frac{\alpha a_0^{-2}t^{-2c}}{\sqrt{1+\alpha^2 a_0^{-2}t^{-2c}}}\left(1+\frac{2c(2c-1)G\hbar F(t)}{3\pi c^5 t^2}\right). \tag{24}$$

As the two-point function is divergent in the coincidence limit, a counterterm is needed in the effective action Eq. (10). The theory is not renormalizable because the counterterm needed is not of the form of the classical action. Indeed, the counterterm must have the following schematic form $A_0 R\dot{x}^4$, where A_0 is a bare constant, R is the Ricci scalar and $\dot{x}^4$ denotes contractions of the components of the three-velocity of the test particle. After absorbing the pole of the divergence of the two-point function into the bare constant, the finite part of the counterterm reads $AR\dot{x}^4$, where A denotes the dressed constant. As usual, this dressed constant must depend on the scale μ in such a way that the complete effective action is independent of μ. In this paper, for simplicity we have omited the finite part of the counterterm. This can be justified in the following situations. On the one hand, one can assume that the dressed constant vanishes for some particular value of the scale μ, which, of course, is an unnatural assumption. On the other hand, one can treat the theory that describes the dynamics of the test particle as an effective, non-renormalizable theory. As the quantum fluctuations involve massless particles, in the low energy regime the finite, nonanalytic part of the quantum correction will be more important than the finite counterterm (see [1] for a general discussion). In the particular case we are analyzing, this regime corresponds to the limit $\log(\mu t)\gg 1$.

IV. QUANTUM CORRECTIONS TO GEODESICS IN THREE DIMENSIONAL GRAVITY

A. Three dimensional general relativity

In this section we will consider $(2+1)$ gravity coupled to Maxwell fields. Under the assumption of rotational symmetry, this model is exactly soluble. Moreover, it is possible to associate a well defined quantum operator to the spacetime metric. Therefore, it is particularly useful to analyze the effective action for a test particle and the corrections to the geodesics. In this subsection we will follow closely Refs. [6,16].

At the classical level, the theory is governed by the Einstein-Maxwell equations, which read

$$R_{ab}=8\pi G\nabla_a\phi\nabla_b\phi, \tag{25}$$

$$g^{ab}\nabla_a\nabla_b\phi=0 \tag{26}$$

where the electromagnetic field has been written in terms of a scalar field as $F_{ab}=\epsilon_{abc}\nabla^c\phi$. Assuming rotational symmetry, the above equations can be easily solved. The metric can be written as

$$g_{ab}dx^a dx^b=e^{G\Gamma(r,t)}[-dt^2+dr^2]+r^2 d\theta^2. \tag{27}$$

Moreover, the scalar field decouples from the metric

$$g^{ab}\nabla_a\nabla_b\phi=0\rightarrow(-\partial_t^2+\partial_r^2)\phi=0. \tag{28}$$

Therefore, one can solve the $(1+1)$ Klein Gordon equation for ϕ and then determine Γ from the Einstein equation. The result is

$$\Gamma(r,t)=\frac{1}{2}\int_0^r dr' \quad r' \quad [(\partial_t\phi)^2+(\partial_{r'}\phi)^2]. \tag{29}$$

Note that, as $r\rightarrow\infty$, Γ tends to a constant value $\Gamma(\infty,t)=H_0$. The metric becomes locally flat with a deficit angle $2\pi(1-e^{-GH_0/2})$.

To quantize the theory, one can promote ϕ to an operator $\hat{\phi}$ describing a free quantum scalar field in $(1+1)$ dimensions. The spacetime metric is a secondary operator that can be expressed in terms of $\hat{\phi}$ as

$$\hat{g}_{rr}=-\hat{g}_{tt}=e^{G\hat{\Gamma}} \tag{30}$$

where $\hat{\Gamma}$ is the operator defined by Eq. (29) with $\phi\rightarrow\hat{\phi}$.

For simplicity in what follows we will consider the metric operator in the asymptotic region $r\rightarrow\infty$, where the operator $\hat{\Gamma}$ is time independent. For a given coherent state of the scalar field (denoted by $|F\rangle$ and peaked around a classical configuration $F(r,t)$), it is easy to show that

$$\langle F|\hat{\phi}|F\rangle=F(r,t),$$

$$\langle F|\hat{g}_{rr}|F\rangle=\exp\left[\frac{1}{\hbar}\int_0^\infty dw|F(w)|^2(e^{G\hbar w}-1)\right]. \tag{31}$$

For sufficiently low frequencies (i.e. when the Fourier transform of the classical configuration is peaked around a low frequency), the mean value of the metric operator can be approximated by

$$\langle F|\hat{g}_{rr}|F\rangle = g_{rr}\left(1+\hbar\frac{G^2}{2}\int_0^\infty dw w^2 \ \ |F(w)|^2\right). \qquad (32)$$

The first term is the value of the metric we would obtain from the classical field equations for a classical scalar field configuration given by $F(r,t)$. The second term represents a small quantum correction. As in the classical case, for $r\to\infty$ the mean value of the metric describes a locally flat spacetime, but with a quantum corrected deficit angle.

B. Effective action for a test particle

According to our general discussion in Sec. II, the effective action for a test particle moving in the $(2+1)$-dimensional spacetime is given by

$$S_{\rm eff}[x] = \langle S_m[x]\rangle = -m\left\langle \int dt\sqrt{e^{G\hat{\Gamma}}(1-\dot{r}^2)-r^2\dot{\theta}^2}\right\rangle \qquad (33)$$

where the mean value is taken with respect to the coherent state $|F\rangle$. Here a dot denotes derivative with respect to *t*. As in the previous section we will consider only the asymptotic region where the metric operator is time independent.

We write the metric operator as $e^{G\hat{\Gamma}} = \langle e^{G\hat{\Gamma}}\rangle + \hat{\Delta}$. The effective Lagrangian then becomes

$$L_{\rm eff} = -m\bar{L}\left\langle \sqrt{\left[1+\frac{\hat{\Delta}(1-\dot{r}^2)}{\bar{L}^2}\right]}\right\rangle \qquad (34)$$

where $\bar{L}$ is proportional to the classical Lagrangian evaluated in the mean value of the metric

$$\bar{L} = \sqrt{\langle e^{G\hat{\Gamma}}\rangle(1-\dot{r}^2)-r^2\dot{\theta}^2}. \qquad (35)$$

Note that, after a redefinition of the angular variable $\theta\to\sqrt{\langle e^{G\hat{\Gamma}}\rangle}\theta$, $\bar{L}$ becomes proportional to the Lagrangian of the test particle in a locally flat spacetime. The deficit angle is given by $2\pi(1-\sqrt{\langle e^{G\hat{\Gamma}}\rangle})$.

Assuming that the quantum fluctuations around the mean value are small[3] we get

$$L_{\rm eff} = -m\bar{L}\left[1-\frac{1}{8}\frac{(1-\dot{r}^2)^2\Delta^2}{\bar{L}^4}\right] \qquad (36)$$

where $\Delta^2 = \langle\hat{\Delta}^2\rangle = \langle(e^{G\hat{\Gamma}}-\langle e^{G\hat{\Gamma}}\rangle)^2\rangle$. The above equation is the starting point to describe the quantum corrections to the trajectory of the test particle.

Let us first consider a nonrelativistic motion of the particle. In this situation we have

$$\bar{L} \simeq \sqrt{\langle e^{G\hat{\Gamma}}\rangle}\left[1-\frac{\dot{r}^2}{2}-\frac{r^2\dot{\theta}^2}{2\langle e^{G\hat{\Gamma}}\rangle}\right]. \qquad (37)$$

Therefore, the effective Lagrangian can be approximated by

$$L_{\rm eff} \simeq -m\sqrt{\langle e^{G\hat{\Gamma}}\rangle}\left[1-\frac{1}{8}\left(\frac{\Delta g}{g}\right)^2\right]\left[1-\frac{\dot{r}^2}{2}-\frac{r^2\dot{\theta}^2}{2\langle e^{G\hat{\Gamma}}\rangle}\times\left(1+\frac{1}{2}\left(\frac{\Delta g}{g}\right)^2\right)\right] \qquad (38)$$

where $(\Delta g/g)^2 = \Delta^2/\langle e^{G\hat{\Gamma}}\rangle^2$.

We can see from Eq. (38) that in this nonrelativistic limit the effective Lagrangian has, up to an irrelevant constant factor, the same form as $\bar{L}$, but with a different deficit angle. Indeed, after the redefinition of the angular variable $\theta\to\sqrt{\langle e^{G\hat{\Gamma}}\rangle}[1-\frac{1}{4}(\Delta g/g)^2]\theta$, the effective Lagrangian becomes proportional to the flat spacetime Lagrangian. Therefore the trajectories will be straight lines in a locally flat spacetime. However, the global properties of the trajectories will be different from the ones obtained with the mean value of the metric $\langle e^{G\hat{\Gamma}}\rangle$, since the deficit angle for the effective Lagrangian is now given by $2\pi\{1-\sqrt{\langle e^{G\hat{\Gamma}}\rangle}[1-\frac{1}{4}(\Delta g/g)^2]\}$.

In the general case (a relativistic particle), the situation is different. Indeed, one can prove that it is not possible to redefine θ in order to bring $L_{\rm eff}$ [Eq. (36)] to a flat spacetime form. As a consequence, although the mean value of the metric is locally flat, the test particle ''sees'' a much more complex geometry.

The conclusion of this section is that, again, the trajectories of the test particle do not coincide with the geodesics of the mean value of the metric.

V. FINAL REMARKS

Let us summarize the new results contained in this paper. We have computed the quantum corrections to the trajectory of a test particle by taking into account the quantum fluctuations of the spacetime metric. We have analyzed two particular models where it is easy to fix completely the gauge of the quantum fluctuations and quantize the remaining degrees of freedom.

For a Robertson-Walker spacetime, the fluctuations of the metric can be described by two massless, minimally coupled scalar fields. The quantum corrected trajectory has the same symmetries as the classical trajectory. However, it contains a quantum correction proportional to the graviton two-point function and to the initial velocity of the test particle. This additional term produces, in particular, a quantum correction to the gravitational redshift.

Let us assume that we solve the back reaction equations perturbatively and find a solution $a(t) = a_c(t)+\delta a(t)$, where $a_c(t)$ is the classical scale factor. Had we neglected the coupling between gravitons and test particle, we would have concluded that the test particle's trajectories coincide with the geodesics of the metric $a(t) = a_c(t)+\delta a(t)$. However,

[3]This is not always the case. See Ref. [16].

this coupling induces an additional correction to the equation of motion that is of the same order of magnitude as the one produced by $\delta a(t)$ (we have shown this in the particular case of a de Sitter solution and, in a previous paper [5], in the Newtonian approximation). As a consequence, it is meaningless to compute $\delta a(t)$ and neglect the graviton effects on the motion of the particle, which is the physical observable.

An interesting feature of our result is that the quantum corrections to the geodesic depend on the velocity of the test particle in such a way that one cannot define an ''effective metric'' for the trajectory, i.e. a metric such that its geodesics coincide with *all* the quantum corrected trajectories. It is worth to note that if one tries to define observationally an ''effective spacetime curvature'' through a geodesic deviation equation, this effective curvature will be dependent on the initial four velocity of the geodesics under consideration. The effective metric and effective scale factor we are talking about here should be looked upon as average quantities, after integration of the graviton fluctuations.

In the case of three-dimensional general relativity, there are no propagating degrees of freedom associated with the geometry. At the classical level one can make the degrees of freedom to reside in the matter field. At the quantum level, the operator associated with the metric can be written in terms of the matter field operator.

In this model, given a quantum state of the matter fields, it is easy to compute the mean value of the metric and of any function of it. In particular, we computed the mean value of the Lagrangian for a test particle. We have shown that, even in the asymptotic region, where both the classical metric and the mean value of the quantum metric operator describe locally flat spacetimes, the test particle ''feels'' the quantum fluctuations and the trajectory is not a straight line.

Now we would like to comment about related works. To our knowledge, the fact that the mean value of the metric is not enough to describe the spacetime geometry when the graviton contribution is taken into account, was first pointed out in Ref. [17]. It was stressed there that one can assign an effective metric to a given observable $\mathcal{O}(g_{\mu\nu})$, through the identity

$$g_{\mu\nu}^{\rm eff}=\mathcal{O}^{-1}\langle\mathcal{O}(g_{\mu\nu})\rangle. \tag{39}$$

The effective metric obviously depends on $\mathcal{O}$. We agree with this point of view. Indeed, from our results it is easy to illustrate this fact. Consider for example the quantum corrected velocity of the test particle given in Eq. (20). Taking into account the classical result for the velocity, one can introduce an ''effective scale factor'' through the identity

$$\frac{\alpha a_{\rm eff}^{-2}(t)}{\sqrt{1+\alpha^2 a_{\rm eff}^{-2}(t)}}=\frac{\alpha a^{-2}(t)}{\sqrt{1+\alpha^2 a^{-2}(t)}}\times\left(1+\frac{32}{3}\pi G\langle\phi^2(t)\rangle\frac{\alpha^2 a^{-2}(t)}{1+\alpha^2 a^{-2}(t)}\right). \tag{40}$$

This gives

$$\frac{a_{\rm eff}}{a}\simeq 1-\frac{\alpha^2+a^2}{\alpha^2+2a^2}\,\frac{32}{3}\,\pi G\langle\phi^2(t)\rangle\frac{\alpha^2 a^{-2}(t)}{1+\alpha^2 a^{-2}(t)}.$$

The ''effective scale factor'' depends on the initial velocity of the particle.

In Ref. [18] the authors analyzed the graviton induced fluctuations of horizons in Robertson Walker and Schwarszchild spacetimes. The analysis was based on the study of the effects of gravitons on (nearly) null geodesics. They pointed out that due to the interaction with the fluctuations of the metric, there are two effects on the trajectories of photons: the mean geodesic will deviate from the classical geodesic, and there will be stochastic fluctuations around the mean value. They studied the stochastic fluctuations and neglected the deviation of the mean value. In this sense, our work is complementary to Ref. [18], since we computed the mean value corrections. In our framework, the stochastic fluctuations could be analyzed by using the closed time path formalism to compute the effective action for the test particle. It can be shown that the imaginary part of this closed time path effective action introduces a noise term in the equation of motion (similar ideas have been applied to the semiclassical Einstein equations, see for example [19]).

In this paper we fixed completely the gauge of the metric fluctuations before quantization. Alternatively, one could use the covariant method described in Sec. II. We showed in a previous work [5] that the solution to the back reaction equation and the quantum corrections to the geodesics are both dependent on the gauge fixing procedure. In the Newtonian approximation, this dependence cancels when computing the trajectory of the test particle. Whether this is true or not beyond the Newtonian approximation is an open question that will be addressed in a forthcoming paper.

ACKNOWLEDGMENTS

We acknowledge the support from Universidad de Buenos Aires, Fundación Antorchas and CONICET (Argentina). We would like to thank M. Bañados and M. Tiglio for pointing out Refs. [6,16] to us.

APPENDIX A

In this appendix we calculate the renormalized two-point function $\langle\phi(x)\phi(x')\rangle$ in the coincide limit $x'\to x$ for a massless minimally coupled scalar field in flat Roberton-Walker metrics with $a(t)=a_0 t^c$. Throughout this appendix we work in conformal time, $\eta=[a_0(1-c)]^{-1}t^{1-c}$. The metric reads $ds^2=C(\eta)(-d\eta^2+d\mathbf{x}^2)$ where $C(\eta)=a^2(t)=a_0^{2/(1-c)}(1-c)^{2c/(1-c)}\eta^{2c/(1-c)}$.

The two-point function we wish to evaluate is basically the Hadamard function $D^{(1)}=\langle\{\phi(x),\phi(x')\}\rangle$. By means of the point-splitting technique, we separate the points x,x' only in their temporal component $\Delta\eta\equiv\eta-\eta'=\epsilon\to 0$. The Hadamard function then takes the form [20]

$$D^{(1)}(x,x')=-\frac{C^{-1/2}(\eta)C^{-1/2}(\eta')}{2\pi^2\Delta\eta^2}-\frac{R}{24\pi^2}\left[\frac{1}{2}\log\left|\frac{\epsilon^2}{C\eta^2}\right|+\gamma+\frac{1}{2}\psi\left(\frac{3}{2}+\nu\right)+\frac{1}{2}\psi\left(\frac{3}{2}-\nu\right)\right]+\frac{R}{48\pi^2}+\mathcal{O}(\epsilon^2) \tag{A1}$$

where $\nu=|1-3c|/(2|1-c|)$, γ is Euler's constant and ψ is Euler's function. The first term on the right is the expression for $D^{(1)}$ in the conformally coupled case, which can also be expanded in powers of ϵ

$$-\frac{C^{-1/2}(\eta)C^{-1/2}(\eta')}{2\pi^2\Delta\eta^2}=-\frac{1}{8\pi^2\epsilon^2\Sigma}+\frac{1}{24\pi^2}\left[R_{\alpha\beta}\frac{t^\alpha t^\beta}{\Sigma}-\frac{1}{6}R\right]+\mathcal{O}(\epsilon^2) \tag{A2}$$

where t^μ is a unit vector that parametrizes the direction of splitting and $\Sigma=t_\mu t^\mu$.

To renormalize we substract the second order adiabatic expansion for the Hadamard function, namely

$$D^{(1)}_{\rm ad}(\eta,\eta')=-\frac{1}{8\pi^2\epsilon^2\Sigma}-\frac{R}{24\pi^2}\left[\frac{1}{2}\log(\mu^2\epsilon^2)+\gamma\right]+\frac{1}{24\pi^2}R_{\alpha\beta}\frac{t^\alpha t^\beta}{\Sigma}+\mathcal{O}(\epsilon^2) \tag{A3}$$

where μ is an arbitrary scale with dimensions of energy. Finally

$$D^{(1)}_{\rm ren}(x,x)=\lim_{\epsilon\to 0}(D^{(1)}(\eta,\eta')-D^{(1)}_{\rm ad}(\eta,\eta'))=\frac{R}{48\pi^2}\log(C\mu^2\eta^2) \tag{A4}$$

all constants having been absorbed into a redefinition of μ. We can now go back to coordinate time, and on using that for these metrics the scalar curvature is $R=6c(2c-1)t^{-2}$, we get the final result

$$\langle\phi^2(t)\rangle=\frac{6c(2c-1)}{96\pi^2}t^{-2}\log(t^2\mu^2).$$

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