Classical self-force

Fritz Rohrlich*
Department of Physics, Syracuse University, Syracuse, New York 13244-1130
(Received 8 March 1999; published 24 September 1999)

The self-force for the classical dynamics of finite size particles is obtained. It is to replace the one of von Laue type obtained for point particles. Such particles are beyond the validity domain of classical mechanics, and their self-force leads to pathological solutions. Both electromagnetic and gravitational self-interactions are considered. The approximation made neglects nonlinear terms in the derivatives of the acceleration. A by-product is the fact that the new self-force destroys the time-reversal invariance of the equations of motion. [S0556-2821(99)05418-1]

PACS number(s): 04.25.−g, 03.50.De

I. INTRODUCTION

The self-force has a long history which has recently been reviewed [1]. It is best known in the Lorentz-Abraham-Dirac equation [2] where this self-force (also known as the von Laue four-vector) is responsible for unphysical solutions such as runaway and noncausal ones [3]. The symptomatic infinite self-energy of a point charge need hardly be mentioned. As pointed out a long time ago, the von Laue self-force is due to the point particle approximation (see [3], p. 186). This approximation lies outside the domain of validity of classical physics, which is characterized roughly by the Compton wavelength.

When gravitation was taken into account, either in the presence of a charge [4,5] or for neutral particles [6,7,8], the offending terms remained even though additional terms were found to be present and even though some of the approximations [7,8] did not assume point particles. The following study is directed at those terms and answers the question as to by what terms these are to be replaced when the internal interaction of classical finite size particles is taken into account. The other self-force terms in the equations of motion such as the ''tail terms'' are not affected by these considerations.

The next section, Sec. II, is devoted to the self-force due to an electric charge, Sec. III to the self-force due to the particle's mass, and Sec. IV presents the general conclusions dealing also with the breaking of time-reversal invariance. All equations will be given in Gaussian units with $c=1$ and the metric tensor of positive signature.

II. CHARGED PARTICLE

A. Charge in a nongravitational force field

In the Lorentz-Abraham-Dirac equation the self-force of a point charge is given by the von Laue four-vector

$$F^{\mu}{}_{S}(e,pt)=\frac{2}{3}e^{2}(\ddot{v}^{\mu}-\dot{v}^{\alpha}\dot{v}_{\alpha}v^{\mu}). \qquad (1)$$

This vector is sometimes incorrectly referred to as the four-vector of ''radiation reaction.'' Only the last term is the four-vector of the rate of energy-momentum loss due to radiation; the first term, sometimes called the Schott term, has nothing to do with radiation. Its meaning will become clear later on (see end of Sec. II B).

When inserted into the equation of motion of a point charge, $F^{\mu}{}_{S}(e,pt)$ leads to various pathological solutions. The divergence of the electrostatic self-energy of a point charge is well known. Then there are the self-accelerating (runaway) solutions which are due to the term involving the second derivative of velocity (third derivative of position). The latter also violates the mandate that dynamical equations be of second order of position. The self-acceleration can be avoided by requiring an extra condition: that asymptotically the velocity be finite. The resulting integro-differential equation ([3], Sec. 6-6), however, gives noncausal solutions because the acceleration now depends on the *future* behavior of the force. Additional pathological behavior includes the head-on collision of particles. All can be traced to the assumption that the particle is a point. But a classical theory cannot be applied to characteristic distances smaller than a Compton wavelength (not to speak of a particle radius shrinking to zero) because those distances fall into the domain of quantum mechanics. The validity domain of classical physics is outside of λ_C.

In order to construct the self-force for a particle of finite size (assume a sphere of radius $a>\lambda_C$), one can proceed as follows. The equation of motion is

$$m_{B}\dot{v}^{\mu}=F^{\mu}+F^{\mu}{}_{S}, \qquad (2)$$

where m_B is the bare mass and F^{μ} is an external force, for example the Lorentz force. Since the other two terms in the equation are orthogonal to the velocity, F_S^{μ} must be too. Therefore, it must be of the form

$$F^{\mu}{}_{S}=P^{\mu}{}_{\nu}X^{\nu}, \qquad (3)$$

where the projection into the subspace orthogonal to v^{μ} is

$$P^{\mu}{}_{\nu}=\delta^{\mu}{}_{\nu}+v^{\mu}v_{\nu}. \qquad (4)$$

In the instantaneous rest frame of the particle, $F^{\mu}{}_{S}(0)=X^{\mu}(0)$. It is therefore sufficient to compute the self-force in the instantaneous rest frame.

*Email address: rohrlich@syr.edu

Consider a particle at rest, and let it be a rigid sphere of radius a. The self-force is the result of the interaction of each small element of charge of the sphere with every other element. In the rest frame, only the electric field $\mathbf{E}=-\nabla\phi-\dot{\mathbf{A}}$ is involved. The potentials are to be computed in the standard way from the charge density $e\rho$ and the current density $e\rho v$ by using the integrals involving the *retarded* times. Here ρ is the charge density distribution with $\int\rho\, d^3x=1$. At this stage, this becomes a graduate student exercise; the calculation can be found in Jackson's textbook [9] (Sec. 17.3 in the 2nd edition and Sec. 16.3 in the 3rd edition). The result, obtained by *neglecting all nonlinear powers of the acceleration and its derivatives*, can be written as (note Jackson's $d\mathbf{p}/dt=-\mathbf{F}_S$)

$$\mathbf{F}_S(e,a)=-\frac{2}{3}e^2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}I_n\left(\frac{\partial}{\partial t}\right)^n\dot{\mathbf{v}}\quad(\mathbf{v}=0),\tag{5}$$

where

$$I_n=\int\int d^3\mathbf{x}\,d^3\mathbf{x}'\,\rho(\mathbf{x})|\mathbf{x}-\mathbf{x}'|^{n-1}\rho(\mathbf{x}').\tag{6}$$

In the point particle limit, I_0 diverges corresponding to the infinite self-energy of a point particle, $I_1=1$, and $I_n=0\,(n>1)$. One recovers the well-known damping term $(2/3)e^2\ddot{\mathbf{v}}$ of the Lorentz equation.

For a general inertial reference frame, Eqs. (3) and (4) tell us that the self-force for an arbitrary charge distribution is

$$F^{\mu}{}_S(e,a)=-\frac{2}{3}e^2[\eta^{\mu\nu}+v^{\mu}(\tau)v^{\nu}(\tau)]\times\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}I_n\left(\frac{\partial}{\partial\tau}\right)^n\dot{v}_{\nu}(\tau).\tag{5$'$}$$

Note that Jackson does not take us beyond Eq. (5). When ρ is a volume distribution, the integral (6) is too complicated a function of n to permit the summation in Eq. (5′) to be carried out. However, when the charge is distributed uniformly over the *surface* of the sphere,

$$I_n=(2a)^{n-1}\frac{2}{n+1}.\tag{6$'$}$$

and Eq. (5) can be summed to

$$\frac{2}{3}\frac{e^2}{a}\frac{1}{\tau_\alpha}\mathbf{v}(t-\tau_\alpha)\quad(\mathbf{v}=0),\tag{7}$$

where $\tau_\alpha=2a$ is the time it takes light to cross the diameter of the sphere. Note that this is fully relativistic, but written in the instantaneous rest frame of the particle.

The result (7) was previously obtained by Yaghjian in the last appendix of his book [10]. He used the well-known expression for $\mathbf{E}$ that is obtained after substitution of the Liénard-Wiechert potentials [see, for example, [9], (14.14)]. An expansion remaining fully relativistic, but neglecting all higher powers of the acceleration and its derivatives then leads to Eq. (7). His calculation is completely equivalent to the one used above using Jackson.

The last step, not carried out by Yaghjian, is to insert Eq. (7) into Eqs. (3) and (4) or I_n directly into Eq. (5′). This results in the final expression for the self-force of a surface charged sphere;

$$F^{\mu}{}_S(e,a)=\frac{2}{3}\frac{e^2}{a}\frac{1}{\tau_\alpha}[v^{\mu}(\tau-\tau_\alpha)+v^{\mu}(\tau)v^{\alpha}(\tau)v_{\alpha}(\tau-\tau_\alpha)].\tag{8}$$

This remarkable equation was first conjectured by Caldirola [11], but he was unable to prove it. The equation of motion has now become a differential-difference equation. The self-force also spoils time reversal invariance [12] which is still enjoyed by the point particle Lorentz-Abraham-Dirac equation despite expression (1) (see [3], Sec. 9-2b for a proof of time-reversal invariance).

When Eq. (8) is expanded for small τ_α, one obtains the point particle results (1) as well as an inertial term due to self-interaction:

$$F^{\mu}{}_S(e,a)=-m_{ed}\dot{v}^{\mu}+F^{\mu}{}_S(e,pt)+O(\tau_\alpha),\tag{9a}$$

$$m_{ed}=4/3m_{es},\quad m_{es}=e^2/(2a),\tag{9b}$$

where m_{ed} and m_{es} are the electrodynamic and the electrostatic self-energies which diverge in the point limit. The inertial term can be combined with the bare mass inertial term in Eq. (2) to yield the observed rest mass $m_o=m_B+m_{ed}$ (mass renormalization).

Finally, the nonrelativistic limit of Eq. (8) yields

$$\mathbf{F}_S(e,a)_{\rm NR}=\frac{2}{3}\frac{e^2}{a}\frac{1}{\tau_\alpha}[\mathbf{v}(t-\tau_\alpha)-\mathbf{v}(t)].\tag{8$'$}$$

which can also be obtained from Eq. (7) by a Galilean boost. This is an old result first derived by Sommerfeld in 1904 [13] and later by Page [14]. Its point limit gives the inertial term correction and the Lorentz term for the self-force:

$$\mathbf{F}_S(e,pt)_{\rm NR}=-m_{ed}\dot{\mathbf{v}}+\frac{2}{3}e^2\ddot{\mathbf{v}}.\tag{10}$$

It is to be noted that the equations of motion for a finite size particle such as Eqs. (8) and (8′) have no pathological solutions.

The above deduction of Eqs. (9) and (10) from Eq. (8) provides an explanation of the mysterious $\ddot{v}$ term, the Schott term. It is seen to arise from the nonlocality in time due to the occurrence of both τ and $\tau-\tau_\alpha$, or, more generally, from the fact that the equation of motion is an infinite order differential equation.

B. Charge in a gravitational field

This problem was first successfully treated by Dewitt and Brehme [4] and later corrected by Hobbs [5]. The only interest in their result for the present purpose is the von Laue self-force (1) which is part of their result. They are still using the point particle approximation.

From the above discussion it follows that this offending term (1) can now be replaced by the self-force (8) for a finite size particle. The other terms, the tail term and the Ricci term found by Hobbs, do not show point particle pathologies and remain unaltered.

III. NEUTRAL PARTICLE IN A GRAVITATIONAL FIELD

This problem was studied by Havas and Goldberg [6] in 1962. In the point particle (single pole) approximation which they used, the self-force of an electrically neutral mass m in a gravitational field was found to be the Lorentz-invariant expression

$$F^{\mu}{}_{S}(G,pt) = -\frac{11}{3}Gm^2[\ddot{v}^{\mu}(\tau) - v^{\mu}(\tau)\dot{v}^{\alpha}(\tau)\dot{v}_{\alpha}(\tau)]. \tag{11}$$

This force is in many ways analogous to the von Laue self-force $F^{\mu}{}_{S}(e,pt)$ of a point particle (1).

Very recent calculations of the gravitational self-force by Mino, Sasaki, and Tanaka [7] and by Quinn and Wald [8] were done in a much more sophisticated approximation. The former describe a finite size particle by a spherically symmetric black hole. Their result of [8] nevertheless reproduces Eq. (11) [[8], Eq. (50)], but also has additional terms that depend on the curvature and includes a Dewitt-Brehme-type tail term. The authors of [8] use an axiomatic approach which reproduces the results of [7].

The present calculation is based on a much more simple-minded approximation, namely, the linear gravity approximation. Therefore, neither curvature-dependent terms nor tail terms are obtained. However, as a result, the self-force (11) is replaced by the self-force due to the internal (retarded) gravitational interaction of a macroscopic particle with radius $a \gg \lambda_C$. Such a self-force does not lead to unphysical solutions.

The procedure will be as follows. First, the self-force will be derived for a particle which is instantaneously at rest. This will be done by use of the linearized gravity approximation (Wald [15], Sec. 4.4, or Rindler [16], Sec. 8.12). In this approximation, the particle is assumed to be rigid in the sense that all points of the small but finite size particle (radius a) are simultaneously at rest when its center is at rest. Then, a boost will be applied so that an expression is obtained valid for a moving particle.

In the linear gravity approximation it is assumed that the gravitational field is weak, so that one can expand the metric tensor and keep only the first term, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. There exists a well-known choice of gauge in which the gravitational field equations take on an especially simple form: one defines $\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ with $h = \eta^{\mu\nu}h_{\mu\nu}$ and chooses a gauge in which $\gamma_{\mu\nu}$ is divergence free, $\gamma_{\mu\nu,}{}^{\nu} = 0$. The field equations then become

$$\Box\gamma_{\mu\nu} = -8\pi G T_{\mu\nu}, \tag{12}$$

with $T_{\mu\nu} = m\rho v_{\mu}v_{\nu}$ and $\int\rho\, d^3x = 1$ as before. These field equations can be solved in terms of the retarded fields $\gamma_{\mu\nu}$ exactly as in the electromagnetic case. However, since we first assume that the particle is in its instantaneous rest frame, only T_{00} will contribute.

When the self-field is completely neglected, the trajectory of a neutral particle in a gravitational field is a geodesic, $\ddot{x}^{\mu} = -\Gamma^{\mu}{}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}$. In our approximation, we neglect all nonlinear terms involving the velocity and its derivatives. As is well known, this equation then reduces in the rest system of the particle to an equation reminiscent of the Lorentz force equation (see, e.g., Wald [15] or Rindler [16]),

$$\ddot{\mathbf{x}} = -\frac{1}{4}\nabla\gamma - \frac{\partial}{\partial t}\boldsymbol{\gamma}, \quad \gamma = \gamma^{0}{}_{0}, \quad \boldsymbol{\gamma} = (\gamma^{k}_{0}).$$

With the new fields $\gamma/4 = -\phi$ and $\boldsymbol{\gamma}/4 = -\mathbf{A}$, this equation can be written in the familiar form (inserting gratuitous factors of m)

$$m\ddot{\mathbf{x}} = -m\mathbf{E}, \quad \mathbf{E} = -\nabla\phi - 4\dot{\mathbf{A}}. \tag{13}$$

In Rindler [16] the last term was omitted since it played no role in the calculation done there. Note that the term $-\mathbf{v}\times(\nabla\times 4\mathbf{A})$ is absent in Eqs. (13) because we are working in the instantaneous rest frame. The gratuitous factor m reminds us of the weak equivalence principle, which states the equality of inertial and gravitational mass, $m_i = m_g = m$.

The problem has thus been reduced to the previous problem of electrodynamics except for the factor of 4 on $\mathbf{A}$ and an overall minus sign relative to electrodynamics. The latter is a consequence of the fact that masses attract one another, while charges of the same sign repel one another.

It is now possible to exploit this similarity with electrodynamics and to proceed by using the calculation in Jackson's textbook that was used in Sec. II above, the only difference being a factor of 4 in front of $\mathbf{J}$, when it occurs as the source of $\mathbf{A}$. The result, analogous to Eq. (5), is the self-force in the instantaneous rest frame:

$$\mathbf{F}_S(G,a) = \frac{11}{3}Gm^2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}I_n\left(\frac{\partial}{\partial t}\right)^n\dot{\mathbf{v}} \quad (\mathbf{v}=0). \tag{14}$$

The first term in this sum is an inertial term, $\delta m\dot{\mathbf{v}}$, where

$$\delta m = \frac{11}{3}Gm^2 I_0. \tag{15}$$

Therefore, it could be renormalized away by combining it with the left hand side of the equation of motion (13). The result would be that the inertial mass now becomes $m_i = m - \delta m < m$, where m is the gravitational mass. *The weak equivalence principle would therefore be violated.* (One notes that δm would make the renormalized mass *smaller*, which is contrary to the electromagnetic case, a difference attributable to the attraction between masses, while charges of equal sign repel one another.)

However, such a violation of the equivalence principle is due to the approximations made in our calculation; it is not a true phenomenon. We have ignored the fact that a mass distribution such as is used to represent our particle is not stable without the addition of stresses that would prevent its implosion. Such an omission would not have been possible in an exact treatment due to the nonlinear nature of general relativity [17]. To correct for this, the first term in the sum of Eq. (14) must be omitted,

$$\mathbf{F}_S(G,a)=\frac{11}{3}Gm^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}I_n\left(\frac{\partial}{\partial t}\right)^n\dot{\mathbf{v}}\quad(\mathbf{v}=0). \tag{16}$$

Next, one can make a Lorentz boost to a reference frame in which the particle moves with velocity v^μ. But since the force must be orthogonal to the velocity, the factor (4) must be supplied. The result is

$$F^\mu{}_S(G,a)=\frac{11}{3}Gm^2[\eta^{\mu\nu}+v^\mu(\tau)v^\nu(\tau)]\times\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}I_n\left(\frac{\partial}{\partial\tau}\right)^n\dot{v}_\nu(\tau). \tag{17}$$

It differs from the electromagnetic case (5′) only by the numerical coefficient and the missing inertial term (first term) in the summation. The similarity with electromagnetism of Eq. (13) makes this not entirely surprising.

The self-force (17) gives a Lorentz-invariant equation of motion for a finite size particle in a weak gravitational field,

$$m\dot{v}^\mu=F^\mu{}_S(G,a), \tag{18}$$

which is just the geodesic equation in our approximation. It is interesting to note that in the point particle limit $a\to 0$, the self-force (17) reproduces exactly the result (11) obtained by Havas and Goldberg [6].

The physical meaning of the self-force can be seen from the special case of a mass distribution in the form of a thin mass shell analogous to an electric surface charge distribution. From Eq. (6′) it then follows that the sum in Eq. (17) can be carried out and one obtains

$$F^\mu{}_S(G,a)=-\frac{11}{3}\frac{Gm^2}{a}\frac{1}{\tau_\alpha}[v^\mu(\tau-\tau_\alpha)+v^\mu(\tau)v^\alpha(\tau)v_\alpha(\tau-\tau_\alpha)]-\delta m\dot{v}^\mu, \tag{19}$$

where δm is the self-energy term that follows from Eqs. (15) and (6′),

$$\delta m=\frac{11}{3}Gm^2/a. \tag{20}$$

The nonrelativistic limit of Eq. (19) is

$$\mathbf{F}_S(G,a)_{\rm NR}=-\frac{11}{3}\frac{Gm^2}{a}\frac{1}{\tau_\alpha}[\mathbf{v}(t-\tau_\alpha)-\mathbf{v}(t)]-\delta m\,\dot{\mathbf{v}}. \tag{19′}$$

Both equations show that, as in the electromagnetic case, the self-force introduces a finite shift in time.

IV. SUMMARY AND DISCUSSION

A. Self-force

The above derivations proved that when nonlinear terms in the derivatives of the velocity are neglected, an extended particle in relativistic motion has a self-force due to the internal interaction of its charge or mass distribution, which is given by

$$F^\mu{}_S(a)=-C[\eta^{\mu\nu}+v^\mu(\tau)v^\nu(\tau)]\times\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}I_n\left(\frac{\partial}{\partial\tau}\right)^n\dot{v}_\nu(\tau)-I_0\dot{v}_\nu\right\}. \tag{21}$$

Here a is the radius of the (assumed) spherical particle, ρ its charge or mass distribution, with $\int\rho\, d^3x=1$, and I_n is given by the rest frame integral (6). The constant C is

$$C=\frac{2}{3}e^2 \qquad \text{for a charge distribution,} \tag{22a}$$

$$C=-\frac{11}{3}Gm^2 \qquad \text{for a mass distribution.} \tag{22b}$$

with e and m being the total charge and (gravitational) mass, respectively. The last term in the curly brackets eliminates the inertial (first) term of the sum. In the electromagnetic case, the same term must be added to the left side of the equation of motion thus providing for a mass renormalization. In the gravitational case, no mass renormalization is involved.

$F^\mu{}_S(a)$ of Eq. (21) turns the equation of motion into a differential equation of infinite order. However, in the special case of a sphere of uniformly distributed surface charge or a thin mass shell, respectively, the sum in Eq. (21) can be written in a closed form. The equation of motion then becomes a differential-difference equation (a special form of a differential equation of infinite order).

The self-force (21) is to replace the von Laue type self-force (1) or (11), respectively, in the equations of motion. The great virtue of doing so is of course that they will then have no pathological solutions. It also keeps the *classical* theory within its validity domain (the Compton wavelength λ_C) by describing particles of size $a>\lambda_C$ and by therefore excluding point particles.

B. Arrow of time

It is a well-known fact that all the fundamental dynamical equations of physics are time-reversal invariant: Newton's, Hamilton's, Einstein's gravitational equations, and Schrödinger's. Also the Lorentz-Abraham-Dirac equation (2) with $F^\mu{}_S$ given by Eq. (1) is time-reversal invariant ([3], Sec. 9-2). It may therefore be surprising that the self-force for a finite size particle, Eq. (21), spoils this invariance. The reason is simple: the self-interaction of a finite size particle involves the *retarded* field of one element acting on another element of charge (or mass) some distance away. This introduces an asymmetry in time. In the point limit, the retarded and advanced actions can no longer be distinguished because the interaction distance between the charge (mass) elements shrinks to zero. Therefore, in that limit the equations of motion are time-reversal invariant.

The fact that the equations of motion of *extended* particles are *not* time-reversal invariant when the self-force is taken into account [12] removes an old puzzle: when one tries to explain the unidirectionality in time (the arrow of time) of physical processes (the entropy law, the emission of radiation, the Hubble expansion, etc.), one has to start with the *time-symmetric* fundamental dynamical equations. This qualitative difference between the fundamental dynamics and the actual physical processes which has been so puzzling in the past is now eliminated by inclusion of the self-force for finite size particles.

ACKNOWLEDGMENTS

I want to thank Arthur Komar for a valuable discussion and for drawing his paper [17] to my attention.

[1] F. Rohrlich, Am. J. Phys. **65**, 1051 (1997).
[2] P. A. M. Dirac, Proc. R. Soc. London **A167**, 148 (1938).
[3] F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Redwood City, CA, 1990).
[4] B. S. Dewitt and R. W. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960).
[5] J. M. Hobbs, Ann. Phys. (N.Y.) **47**, 141 (1968).
[6] P. Havas and J. N. Goldberg, Phys. Rev. **128**, 398 (1962).
[7] Y. Mino, M. Sasaki, and T. Tanaka, Phys. Rev. D **55**, 3457 (1997).
[8] T. C. Quinn and R. M. Wald, Phys. Rev. D **56**, 3381 (1997).
[9] J. D. Jackson, Classical Electrodynamics, 3rd ed. (Wiley, New York, 1999).
[10] A. D. Yaghjian, *Relativistic Dynamics of a Charged Sphere* (Springer-Verlag, Berlin, 1992).
[11] P. Caldirola, Nuovo Cimento Suppl. **3**, 297 (1956).
[12] F. Rohrlich, Found. Phys. **28**, 1045 (1998).
[13] A. Sommerfeld, Verh.-K. Ned. Akad. Wet., Afd. Natuurkd., Eerste Reeks **13**, 346 (1904).
[14] L. Page, Phys. Rev. **11**, 377 (1918).
[15] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[16] W. Rindler, *Essential Relativity* (Springer-Verlag, New York, 1977).
[17] A. Komar, Int. J. Theor. Phys. **4**, 45 (1971).