Quantum entropy of a nonextreme stationary axisymmetric black hole due to a minimally coupled quantum scalar field

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By using the 't Hooft ''brick wall'' model and the Pauli-Villars regularization scheme we calculate the statistical-mechanical entropy arising from minimally coupled scalar fields which rotate with azimuthal angular velocity $\Omega_0=\Omega_H$ (Ω_H is the angular velocity of the black hole horizon) in general four-dimensional nonextreme stationary axisymmetric black hole space-time. We also show, for the Kerr-Newman and the Einstein-Maxwell dilaton-axion black holes, that the statistical-mechanical entropy obtained from our derivation and the quantum thermodynamical entropy by the conical singularity method are equivalent. [S0556-2821(99)02918-5]

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I. INTRODUCTION

Since Bekenstein and Hawking found that black hole entropy is proportional to the event horizon area by comparing black hole physics with thermodynamics and from the discovery of black hole evaporation [1–3], much effort has been devoted to the study of the statistical origin of black hole entropy. Especially, the idea to relate the entropy of the black hole to quantum excitations of the black hole has attracted much attention [4–24]. The thermodynamical entropy of the black hole is related to the covariant Euclidean free energy $F^E[g,\beta]=\beta^{-1}W[g,\beta]$ [25], where β is the inverse temperature. The function $W[g,\beta]$ is given on Euclidean manifolds with the period β in Euclidean time τ. We can calculate the free energy F^E by the conical singularity method. This procedure was consistently carried out for studies of static black holes and the rotating charged Kerr black hole [5,19,26,27,23]. On the other hand, canonical statistical-mechanical entropy can be derived from the free energy F^C of a system [25], where F^C can be defined in terms of the one-particle spectrum. One of the ways to calculate F^C is the ''brick wall'' model (BWM) proposed by 't Hooft [4]. He argued that black hole entropy is identified with the statistical-mechanical entropy arising from a thermal bath of quantum fields propagating outside the horizon. In this model, in order to eliminate the divergence which appears due to the infinite growth of the density of states close to the horizon, 't Hooft introduces a ''brick wall'' cutoff: a fixed boundary Σ_h near the event horizon within the quantum field does not propagate and the Dirichlet boundary condition was imposed on the boundary, i.e., the wave function $\phi=0$ for $r=r(\Sigma_h)$. Later, Demers, Lafrance, and Myers [28] pointed out that the Dirichlet condition can be removed if we use the Pauli-Villars regulated theory. The BWM has been successfully used in studies of the statistical-mechanical entropy for many black holes [4,10–14,17,22].

Recently, Frolov and Fursaev [25] reviewed studies of the relation between the thermodynamic entropy and the statistical-mechanical entropy of black holes. They showed that for general static black holes the covariant Euclidean free energy F^E and the statistical-mechanical free energy F^C are equivalent when one uses the ultraviolet regularization method [25].

As in the static case, the quantum entropy for stationary axisymmetric black holes has also been studied by many authors recently. Mann and Solodukhin [23] investigated the covariant Euclidean formulation for the Kerr-Newman black hole. They showed that an Euclidean manifold which was obtained by Wick rotation of the Kerr-Newman geometry with a Killing horizon has a conical singularity similar to the one which appears in static black holes. The one-loop quantum correction to the entropy of the charged Kerr black hole was calculated by applying the method of conical singularities. They found an interesting result, that the logarithmic term of the quantum entropy for the Kerr-Newman black hole can be written as a constant plus a term proportional to the charge and so, for the Schwarzschild and the Kerr black holes, the logarithmic parts in the entropy are exactly equal.

Cognola [24], through the Euclidean path integral and using a heat kernel and ζ-function regularization scheme, studied the one-loop contribution to the entropy for a scalar field in the Kerr black hole. In the calculation he took an approximation of the metric, which, after a conformal transformation, takes a Rindler-like form. He pointed out in Ref. [24] that the result is valid also for the Kerr-Newman black hole. Nevertheless, the result is in contrast with the corresponding one obtained in Ref. [23].

In Refs. [11–13], by using the 't Hooft BWM Lee and Kim and Ho, Kim and Park discussed the statistical-mechanical entropy of some stationary black holes, such as the Kerr black hole, Kerr-Newman black hole, and Kaluza-Klein black hole. The results showed that the entropies can be expressed as kA_H/ε^2, where A_H is the area of the event horizon, and $\varepsilon=\int_{r_H}^{r_H+h}dr\sqrt{g_{rr}}$ is the proper distance from the horizon to the r_H+h and the h the cutoff in the radial coordinate near the horizon. However, the logarithmically divergent term of the statistical-mechanical entropy for the four-dimensional stationary axisymmetric black hole space-time was not investigated.

Although much attention has been paid to the study of the quantum entropy of stationary axisymmetric black holes, the relation between the statistical-mechanical entropy and the thermodynamical entropy for rotating black holes has not been investigated yet [25]. The aim of this paper is to obtain an expression of the general statistical-mechanical entropy for the general four-dimensional nonextreme stationary axisymmetric black hole by using the BWM and the Pauli-Villars regularization scheme, and then make a comparison of the statistical-mechanical entropy obtained by using BWM and the thermodynamical entropy by the conical singularity method for some well-known stationary axisymmetric black holes.

The paper is organized as follows: In Sec. II, the general stationary axisymmetric black hole is introduced and some properties of the black hole that are necessary to understand the thermodynamics of quantum fields are studied. In Sec. III, making use of 't Hooft's BWM [21] we deduce a formula of the statistical-mechanical entropy for the general nonextreme stationary axisymmetric black hole. In the last section, the statistical-mechanical entropies for the Kerr-Newman and the Einstein-Maxwell dilaton-axion (EMDA) black holes are studied by using the formula. Then the results are compared with the entropy obtained by the conical singularity method. Finally, we end with some conclusions.

II. SPACE-TIME OF THE GENERAL NONEXTREME STATIONARY AXISYMMETRIC BLACK HOLE

In Boyer-Lindquist coordinates the most general line element for a stationary axisymmetric black hole in four-dimensional space-time can be expressed as

$$ds^2=g_{tt}dt^2+g_{rr}dr^2+g_{t\varphi}dt\,d\varphi+g_{\theta\theta}d\theta^2+g_{\varphi\varphi}d\varphi^2, \quad (1)$$

where g_{tt}, g_{rr}, $g_{t\varphi}$, $g_{\theta\theta}$, and $g_{\varphi\varphi}$ are functions of the coordinates r and θ only. Because the space-time (1) is a stationary and axisymmetric one, a stationary Killing vector field $\xi^\mu=(1,0,0,0)$ and an axial Killing field $\Psi^\mu=(0,0,0,1)$ exist [29]. By taking a linear combination of ξ^μ and Ψ^μ we obtain a new Killing field

$$l^\mu=\xi^\mu+\Omega_H\Psi^\mu, \quad (2)$$

which is normal to the horizon of the black hole. In Eq. (2) the constant Ω_H is called the angular velocity of the event horizon. An interesting feature of the black hole worthy of note is that the norm of the Killing field l^μ is zero on the horizon since the horizon is a null surface and the vector l^μ is normal to the horizon. That is to say, the black hole horizon is a surface where the Killing field l^μ is null. Substituting l^μ into the formula of the surface gravity [30], $\kappa^2=-\frac{1}{2}l^{\mu;\nu}l_{\mu;\nu}$, we obtain

$$\kappa=\frac{-1}{2}\lim_{r\to r_H}\left[\sqrt{\frac{-1}{g_{rr}(g_{tt}-g_{t\varphi}^2/g_{\varphi\varphi})}}\frac{d}{dr}\left(g_{tt}-\frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)\right]=\frac{2\pi}{\beta_H}, \quad (3)$$

where r_H represents the outermost event horizon, $1/\beta_H$ is the Hawking temperature, and here and hereafter the metric signature is taken as $(-,+,+,+)$. We know that the event horizon is a null hypersurface determined by

$$g^{\mu\nu}\frac{\partial H}{\partial x^\mu}\frac{\partial H}{\partial x^\nu}=0. \quad (4)$$

For the stationary axisymmetric black hole (1) the function H is a function of r and θ only. Substituting the metric (1) into Eq. (4) and discussing carefully we find

$$\frac{1}{g^{tt}(r_H)}=\left(g_{tt}-\frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)_{r_H}=0, \quad (5)$$

solutions of which determine the location of the event horizons. From Eq. (5) we know that for a nonextreme stationary axisymmetric black hole $1/g^{tt}$ can be expressed as

$$\left(g_{tt}-\frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)\equiv G_1(r,\theta)(r-r_H), \quad (6)$$

where $G_1(r,\theta)$ is a regular function in the region $\infty>r\geqslant r_H$ and its value is nonzero on the outermost event horizon r_H. On the other hand, since $\kappa=$const and $1/g^{tt}=0$ on the event horizon $r=r_H$, we find from Eq. (3) that g^{rr} must take the following form:

$$g^{rr}\equiv G_2(r,\theta)(r-r_H), \quad (7)$$

where $G_2(r,\theta)$ is a well-defined function in the region $\infty>r\geqslant r_H$ and is nonzero on the horizon r_H too. Making use of Eqs. (6) and (7), we obtain

$$g_{rr}\left(g_{tt}-\frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)=\frac{G_1(r,\theta)}{G_2(r,\theta)}\equiv-f(r,\theta), \quad (8)$$

where the $f(r,\theta)$ is a constant or a regular function on the outermost event horizon and outside the horizon.

III. STATISTICAL-MECHANICAL ENTROPY OF THE GENERAL NONEXTREME STATIONARY AXISYMMETRIC BLACK HOLE

We now try to find an expression of the statistical-mechanical entropy due to the minimally coupled quantum scalar fields in a general four-dimensional stationary axisymmetric black hole. We first seek the total number of modes with energy less than E by using the Klein-Gordon equation, and then calculate a free energy. The statistical-mechanical entropy of the black hole is obtained by variation of the free energy with respect to the inverse temperature and setting $\beta=\beta_H$.

Using a WKB approximation with

$$\phi=\exp[-iEt+im\varphi+iW(r,\theta)],\tag{9}$$

and substituting the metric (1) into the Klein-Gordon equation of the scalar field ϕ with mass μ and nonminimal $\xi R\phi^2$ (R is the scalar curvature of the spacetime) coupling

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)-(\mu^2+\xi R)\phi=0,\tag{10}$$

we find [31]

$$p_r^2=\frac{1}{g^{rr}}[-g^{tt}E^2+2g^{t\varphi}Em-g^{\varphi\varphi}m^2-g^{\theta\theta}p_\theta^2-(\mu^2+\xi R)],\tag{11}$$

where $p_r\equiv\partial_r W(r,\theta)$ and $p_\theta\equiv\partial_\theta W(r,\theta)$. If the scalar curvature R takes a nonzero value at the horizon, then this region can be approximated by some sort of constant curvature space. However, the exact results for such a black hole showed that the mass parameter in the solution enters only in the combination $(\mu^2-R/6)$ [21,32]. Therefore, inserting the covariant metric into Eq. (11) we arrive at

$$p_r^2=-\frac{g_{rr}g_{\varphi\varphi}}{g_{tt}g_{\varphi\varphi}-g_{t\varphi}^2}\left[(E-\Omega m)^2+\left(g_{tt}-\frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)\left(\frac{m^2}{g_{\varphi\varphi}}+\frac{p_\theta^2}{g_{\theta\theta}}+M^2(r,\theta)\right)\right],\tag{12}$$

where $\Omega\equiv-g_{t\varphi}/g_{\varphi\varphi}$ and $M^2(r,\theta)\equiv\mu^2-(\frac{1}{6}-\xi)R$. In this paper our discussion is restricted to studying minimally coupled ($\xi=0$) scalar fields. We know from Eq. (12) that $W(r,\theta)$ can be expressed as

$$W(r,\theta)=\pm\int^r\sqrt{\frac{-g_{rr}g_{\varphi\varphi}}{g_{tt}g_{\varphi\varphi}-g_{t\varphi}^2}}K(r,\theta)dr+c(\theta),\tag{13}$$

where

$$K(r,\theta)=\sqrt{(E-\Omega m)^2+\left(g_{tt}-\frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)\left(\frac{m^2}{g_{\varphi\varphi}}+\frac{p_\theta^2}{g_{\theta\theta}}+M^2(r,\theta)\right)}.\tag{14}$$

Therefore, in phase space the number of modes with E, m, and p_θ is shown by [33]

$$n(E,m,p_\theta)=\frac{1}{\pi}\int d\theta\int_{r_H+h}^{r_E}\sqrt{\frac{-g_{rr}g_{\varphi\varphi}}{g_{tt}g_{\varphi\varphi}-g_{t\varphi}^2}}K(r,\theta)dr.\tag{15}$$

Zhao and Gui [34] pointed out that ''a physical space'' must be dragged by the gravitational field with an azimuth angular velocity Ω_H in the stationary axisymmetric spacetime (1). Apparently, a quantum scalar field in thermal equilibrium at temperature $1/\beta$ in the stationary axisymmetric black hole must be dragged too. Therefore, it is rational to assume that the scalar field is rotating with angular velocity $\Omega_0=\Omega_H$. For such an equilibrium ensemble of states of the scalar field, the free energy is given by

$$\begin{aligned}\beta F&=\int dm\int dp_\theta\int dn(E,m,p_\theta)\ln[1-e^{-\beta(E-\Omega_0 m)}]\\&=\int dm\int dp_\theta\int dn(E+\Omega_0 m,m,p_\theta)\ln(1-e^{-\beta E})\\&=-\beta\int dm\int dp_\theta\int\frac{n(E+\Omega_0 m,m,p_\theta)}{e^{\beta E}-1}dE\\&=-\beta\int\frac{n(E)}{e^{\beta E}-1}dE,\end{aligned}\tag{16}$$

with

$$\begin{aligned}n(E)&=\int dm\int dp_\theta\int n(E+\Omega_0 m,m,p_\theta)\\&=\frac{1}{3\pi}\int d\theta\int_{r_H+h}^{r_E}\frac{dr\sqrt{g_4}}{\left[\left(g_{tt}-\frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)\left(1+\frac{g^2_{\varphi\varphi(\Omega-\Omega_0)^2}}{g_{tt}g_{\varphi\varphi}-g_{t\varphi}^2}\right)\right]^2}\\&\quad\times\left[E^2+\left(g_{tt}-\frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)\left(1+\frac{g^2_{\varphi\varphi(\Omega-\Omega_0)^2}}{g_{tt}g_{\varphi\varphi}-g_{t\varphi}^2}\right)\times M^2(r,\theta)\right]^{3/2},\end{aligned}\tag{17}$$

where the function $n(E)$ presents the total number of modes with energy less than E. The integrations of the m and p_θ in the Eq. (16) are taken only over the value for which the square root in Eq. (14) exists.

Taking the integration of the r in Eq. (17) for the case $\Omega_0=\Omega_H$ we have

$$\begin{aligned}n(E)=&-\frac{1}{2\pi}\int d\theta\left\{\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\left[\frac{2}{3}\left(\frac{E\beta_H}{4\pi}\right)^3C(r,\theta)+M^2(r,\theta)\times\left(\frac{E\beta_H}{4\pi}\right)\right]\ln\left(\frac{E^2}{E_{min}^2}\right)\right\}_{r_H}-\frac{1}{3\pi}\left(\frac{\beta_H}{4\pi}\right)\\&\times\int d\theta\left\{\sqrt{g_{\theta\theta}g_{\varphi\varphi}}M^2(r,\theta)\left(E-\frac{E^3}{E_{min}^2}\right)\right\}_{r_H},\end{aligned}\tag{18}$$

where

$$C(r,\theta)=\frac{\partial^2 g^{rr}}{\partial r^2}+\frac{3}{2}\frac{\partial g^{rr}}{\partial r}\frac{\partial\ln f}{\partial r}-\frac{2\pi}{\beta_H\sqrt{f}}\left(\frac{1}{g_{\theta\theta}}\frac{\partial g_{\theta\theta}}{\partial r}+\frac{1}{g_{\varphi\varphi}}\frac{\partial g_{\varphi\varphi}}{\partial r}\right)-\frac{2g_{\varphi\varphi}}{f}\left[\frac{\partial}{\partial r}\left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)\right]^2,$$

$$E^2_{min}=-M^2(r_H,\theta)\left(g_{tt}-\frac{g^2_{t\varphi}}{g_{\varphi\varphi}}\right)_{\Sigma_h}, \tag{19}$$

$$\overline{K^2}=E^2+\left(g_{tt}-\frac{g^2_{t\varphi}}{g_{\varphi\varphi}}\right)_{\Sigma_h}M^2(r_H,\theta),$$

here and hereafter $f\equiv f(r,\theta)$, which is defined by Eq. (8).

Now let us use the Pauli-Villars regularization scheme [28] by introducing five regulator fields $\{\phi_i, i=1,\ldots,5\}$ of different statistics with masses $\{\mu_i, i=1,\ldots,5\}$ dependent on the UV cutoff [28]. If we rewrite the original scalar field $\phi=\phi_0$ and $\mu=\mu_0$, then these fields satisfy $\Sigma^5_{i=0}\Delta_i=0$ and $\Sigma^5_{i=0}\Delta_i\mu_i^2=0$, where $\Delta_0=\Delta_3=\Delta_4=+1$ for the commuting fields and $\Delta_1=\Delta_2=\Delta_5=-1$ for the anticommuting fields. Since each of the fields makes a contribution to the free energy of Eq. (16), the total free energy can be expressed as

$$\beta\overline{F}=\sum_{i=0}^{5}\Delta_i\beta F_i. \tag{20}$$

Substituting Eq. (16) into Eq. (18) and then taking the integration over E we find

$$\begin{aligned}\overline{F}=&\frac{-1}{48}\frac{\beta_H}{\beta^2}\int d\theta\{\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\}_{r_H}\sum_{i=0}^{5}\Delta_i M_i^2(r_H,\theta)\ln M_i^2(r_H,\theta)-\frac{1}{2880}\frac{\beta_H^3}{\beta^4}\\&\times\int d\theta\left(\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\left\{\frac{\partial^2 g^{rr}}{\partial r^2}+\frac{3}{2}\frac{\partial g^{rr}}{\partial r}\frac{\partial\ln f}{\partial r}-\frac{2\pi}{\beta_H\sqrt{f}}\left(\frac{1}{g_{\theta\theta}}\frac{\partial g_{\theta\theta}}{\partial r}+\frac{1}{g_{\varphi\varphi}}\frac{\partial g_{\varphi\varphi}}{\partial r}\right)-\frac{2g_{\varphi\varphi}}{f}\left[\frac{\partial}{\partial r}\left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)\right]^2\right\}\right)_{r_H}\\&\times\sum_{i=0}^{5}\Delta_i\ln M_i^2(r_H,\theta),\end{aligned} \tag{21}$$

where $M_i^2(r_H,\theta)=\mu_i^2-\frac{1}{6}R$. Then the total statistical-mechanical entropy at the Hawking temperature $1/\beta=1/\beta_H$ is given by

$$\begin{aligned}S=&\left[\beta^2\frac{\partial\overline{F}}{\partial\beta}\right]_{\beta=\beta_H}=\frac{1}{48\pi}\int d\theta d\varphi(\sqrt{g_{\theta\theta}g_{\varphi\varphi}})_{r_H}\sum_{i=0}^{5}\Delta_i M_i^2(r_H,\theta)\ln M_i^2(r_H,\theta)\\&+\left(\frac{1}{32\times 45\pi}\int d\theta d\varphi\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\left\{\frac{\partial^2 g^{rr}}{\partial r^2}+\frac{3}{2}\frac{\partial g^{rr}}{\partial r}\frac{\partial\ln f}{\partial r}-\frac{2\pi}{\beta_H\sqrt{f}}\left(\frac{1}{g_{\theta\theta}}\frac{\partial g_{\theta\theta}}{\partial r}+\frac{1}{g_{\varphi\varphi}}\frac{\partial g_{\varphi\varphi}}{\partial r}\right)\right.\right.\\&\left.\left.-\frac{2g_{\varphi\varphi}}{f}\left[\frac{\partial}{\partial r}\left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)\right]^2\right\}\right)_{r_H}\sum_{i=0}^{5}\Delta_i\ln M_i^2(r_H,\theta).\end{aligned} \tag{22}$$

Using the assumption that the scalar curvature R at the horizon is much smaller than each μ_i and noting that the area of the event horizon is given by $A_\Sigma=\int d\varphi\int d\theta\{\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\}_{r_H}$, we obtain at last the following expression of the statistical-mechanical entropy:

$$\begin{aligned}S=&\frac{A_\Sigma}{48\pi}\sum_{i=0}^{5}\Delta_i\mu_i^2\ln\mu_i^2+\left[-\frac{1}{6\times 48\pi}\int d\theta d\varphi(R\sqrt{g_{\theta\theta}g_{\varphi\varphi}})_{r_H}+\frac{1}{32\times 45\pi}\int d\theta d\varphi\left(\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\left\{\frac{\partial^2 g^{rr}}{\partial r^2}+\frac{3}{2}\frac{\partial g^{rr}}{\partial r}\frac{\partial\ln f}{\partial r}\right.\right.\right.\\&\left.\left.\left.-\frac{2\pi}{\beta_H\sqrt{f}}\left(\frac{1}{g_{\theta\theta}}\frac{\partial g_{\theta\theta}}{\partial r}+\frac{1}{g_{\varphi\varphi}}\frac{\partial g_{\varphi\varphi}}{\partial r}\right)-\frac{2g_{\varphi\varphi}}{f}\left[\frac{\partial}{\partial r}\left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)\right]^2\right\}\right)_{r_H}\right]\sum_{i=0}^{5}\Delta_i\ln\mu_i^2,\end{aligned} \tag{23}$$

which is valid for the general nonextreme stationary axisymmetric black holes which the metric can be expressed as Eq. (1) in the Boyer-Lindquist coordinates and their signature is $(-,+,+,+)$. For black holes with signature $(+,-,-,-)$ a corresponding formula can be obtained by replacing the β_H with $-\beta_H$ in Eq. (23).

IV. DISCUSSION AND SUMMARY

In this section, let us begin a discussion with the study of the statistical-mechanical entropy of the Kerr-Newman black hole and EMDA black hole by using formula (23).

A. Entropy of the Kerr-Newman black hole

In Boyer-Lindquist coordinates, the metric of the Kerr-Newman black hole [35,36] takes the form

$$g_{tt}=-\frac{\Delta-a^2\sin^2\theta}{\rho^2},\quad g_{t\varphi}=-\frac{a\sin^2\theta(r^2+a^2-\Delta)}{\rho^2},$$

$$g_{rr}=\frac{\rho^2}{\Delta},\quad g_{\theta\theta}=\rho^2,\quad g_{\varphi\varphi}=\left(\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\rho^2}\right)\sin^2\theta, \tag{24}$$

with

$$\rho^2=r^2+a^2\cos^2\theta,\quad \Delta=(r-r_+)(r-r_-), \tag{25}$$

where $r_+=r_H=M+\sqrt{M^2-Q^2-a^2}$, $r_-=M-\sqrt{M^2-Q^2-a^2}$, and M and Q represent the mass and charge of the black hole, respectively. Using the metric (24) we get

$$\begin{aligned}&\left\{\frac{\partial^2 g^{rr}}{\partial r^2}+\frac{3}{2}\frac{\partial g^{rr}}{\partial r}\frac{\partial \ln f}{\partial r}-\frac{2\pi}{\beta_H\sqrt{f}}\left(\frac{1}{g_{\theta\theta}}\frac{\partial g_{\theta\theta}}{\partial r}+\frac{1}{g_{\varphi\varphi}}\frac{\partial g_{\varphi\varphi}}{\partial r}\right)-\frac{2g_{\varphi\varphi}}{f}\left[\frac{\partial}{\partial r}\left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)\right]^2\right\}_{r_+}\\
&\quad=\frac{16r_+^2[(r_+r_--a^2)-(r_++r_-)r_+]+4[(a^2-r_+r_-)+3(r_++r_-)r_+]\rho^2}{\rho^6}+\frac{2(a^2-r_+r_-)}{\rho^4}\\
&\qquad+\frac{2a^2(1+\cos^2\theta)\rho^2-8a^2(r_+^2+a^2)\cos^2\theta}{\rho^6},\\
&R=0.\end{aligned} \tag{26}$$

Inserting Eq. (26) into Eq. (23) and then taking the integrations of the θ and φ we find that the statistical-mechanical entropy of the Kerr-Newman black hole is given by

$$S_{KN}=\frac{A_\Sigma}{48\pi}\sum_{i=0}^{5}\Delta_i\mu_i^2\ln\mu_i^2-\frac{1}{90}\left\{1+\frac{3(a^2-r_+r_-)}{4r_+^2}\left[1+\frac{r_+^2+a^2}{ar_+}\arctan\left(\frac{a}{r_+}\right)\right]\right\}\sum_{i=0}^{5}\Delta_i\ln\mu_i^2, \tag{27}$$

where $A_\Sigma=4\pi(r_+^2+a^2)$. Noting $r_+r_--a^2=Q^2$ and the Pauli-Villars regularization scheme caused a factor of $-\frac{1}{2}$ for the second part in Eq. (27), we know that the statistical-mechanical entropy (27) coincides with the Mann-Solodukhin's result [23] which was obtained by using the conical singularity method.

B. Entropy of the stationary axisymmetric EMDA black hole

The stationary axisymmetric EMDA black hole metric [we take the solution $b=0$ in Eq. (14) in Ref. [2]; the reason we use this solution is that the solution $b\neq 0$ cannot be interpreted properly as a black hole] is described by [37]

$$g_{tt}=-\frac{\Delta-a^2\sin^2\theta}{\Sigma},\quad g_{t\varphi}=-\frac{a\sin^2\theta[(r^2+a^2-2dr)-\Delta]}{\Sigma},$$

$$g_{rr}=\frac{\Sigma}{\Delta},\quad g_{\theta\theta}=\Sigma,\quad g_{\varphi\varphi}=\left(\frac{(r^2+a^2-2dr)^2-\Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta, \tag{28}$$

with

$$\Sigma=r^2-2dr+a^2\cos^2\theta,\quad \Delta=r^2-2mr+a^2=(r-r_+)(r-r_-), \tag{29}$$

where $r_+=m+\sqrt{m^2-a^2}$, $r_-=m-\sqrt{m^2-a^2}$. The mass M, the angular momentum J, the electric charge Q, and the magnetic charge P of the black hole are, respectively, given by

$$M=m-d,\quad J=a(m-d),\quad Q=\sqrt{2\omega d(d-m)},\quad P=g. \tag{30}$$

By using the metric (28) we obtain

$$\left\{\frac{\partial^2 g^{rr}}{\partial r^2}+\frac{3}{2}\frac{\partial g^{rr}}{\partial r}\frac{\partial \ln f}{\partial r}-\frac{2\pi}{\beta_H\sqrt{f}}\left(\frac{1}{g_{\theta\theta}}\frac{\partial g_{\theta\theta}}{\partial r}+\frac{1}{g_{\varphi\varphi}}\frac{\partial g_{\varphi\varphi}}{\partial r}\right)-\frac{2g_{\varphi\varphi}}{f}\left[\frac{\partial}{\partial r}\left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)\right]^2\right\}_{r_+}$$

$$=\frac{16r_+^2[(2d-r_+-r_-)r_+]+4d(8r_+-3d)(r_+^2-2dr_++a^2)}{\Sigma^3}+\frac{4[(3r_+-2d)(r_++r_--2d)-d^2]}{\Sigma^2}$$

$$+\frac{2d(r_++r_--2d)(\Sigma-2dr_+)}{\Sigma^3}+\frac{2a^2(1+\cos^2\theta)\Sigma-8a^2(r_+^2+a^2-2dr_+)\cos^2\theta}{\Sigma^3},$$

$$R=\frac{2a^2d^2\sin^2\theta}{\Sigma^3}. \tag{31}$$

Substituting Eq. (31) into Eq. (23) and then taking the integration of the θ and φ we find that the statistical-mechanical entropy of the EMDA black hole is

$$S=\frac{A_\Sigma}{48\pi}\sum_{i=0}^{5}\Delta_i\mu_i^2\ln\mu_i^2-\frac{1}{90}\times\left\{1+\frac{9d^2}{8r_+^2-16dr_+}+\frac{9d\left\{3a^2d+(r_+^2-2dr_+)\left[\frac{4}{3}(r_++r_-)-d\right]\right\}}{16(r_+^2-2dr_+)^2}\times\left[1+\frac{r_+^2+a^2-2dr_+}{a\sqrt{r_+^2-2dr_+}}\arctan\left(\frac{a}{\sqrt{r_+^2-2dr_+}}\right)\right]\right\}\times\sum_{i=0}^{5}\Delta_i\ln\mu_i^2, \tag{32}$$

where $A_\Sigma=4\pi(r_+^2+a^2-2dr_+)$. In order to compare the entropy (32) with the thermodynamical entropy obtained by the covariant Euclidean formulation, we [38] calculated the thermodynamical entropy of the EMDA black hole by using the conical singularity method of Ref. [23]. We also find that the results obtained by the two methods are equivalent.

From Eq. (27) or Eq. (32) we find the same result for the Kerr black hole [setting $Q=0$ in Eq. (27) or $d=0$ in Eq. (32)] as that found by Mann and Solodukhin in Ref. [23]; i.e., the quantum entropy does not depend on the rotation parameter a and coincides with the quantum entropy of the Schwarzschild black hole. We think the reason is that the quantum entropy is mainly caused by quantum scalar fields near the event horizon and in the region the scalar fields are corotating with the black hole.

In summary, by using the BWM and with the Pauli-Villars regularization scheme, we investigate the statistical-mechanical entropy arising from minimally coupled quantum scalar fields rotating with angular velocity Ω_0 in the general four-dimensional nonextreme stationary axisymmetric black hole space-time. An expression of statistical-mechanical entropy is obtained for the case $\Omega_0=\Omega_H$. The Kerr-Newman black hole and the EMDA black hole are studied. It is shown that the statistical-mechanical entropy obtained by using formula (23) and the quantum thermodynamical entropy derived from the covariant Euclidean formulation (by using the conical singularity method) are equivalent for the Kerr-Newman and the EMDA black holes. The result fills in the gaps mentioned in Ref. [25], that the relation between the canonical and covariant Euclidean formulations in the rotating black hole has not been investigated. The study may provide us with a better understanding of the relationship between the different entropies to the black holes.

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