Quantum flux from a moving spherical mirror

Warren G. Anderson
Department of Physics, University of Wisconsin–Milwaukee, P.O. Box 413, Milwaukee, Wisconsin 53201

Werner Israel
Canadian Institute for Advanced Research Cosmology Program, Department of Physics and Astronomy, University of Victoria, Victoria, British Columbia, Canada V8W 3P6

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We calculate the flux from a spherical mirror which is expanding or contracting with nearly uniform acceleration. The flux at an exterior point (which could in principle be a functional of the mirror's past history) is actually found to be a local function, depending on the first and second time derivatives of acceleration at the retarded time. [S0556-2821(99)02718-6]

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I. INTRODUCTION

Some of the most remarkable predictions of quantum field theory arise from zero point fluctuations of quantum states. One of the best known examples of this is the Unruh effect [1–3], in which an accelerating detector measures the zero point fluctuations of the inertial (Minkowski) vacuum state and finds they have a thermal spectrum. Another is the Casimir effect [4], in which the walls (or boundaries) of a box experience a net force due to the difference between the zero point fluctuations of the states inside the box and out.

It seems reasonable, then, to expect accelerating boundaries to produce interesting effects, and indeed they do. Following studies of the Casimir effect between moving mirrors in 1+1 dimensions by Moore [5], DeWitt pointed out that the single moving mirror problem would be interesting and could be solved exactly in (1+1)D [6]. Fulling [1] and DeWitt [7] have shown that a *uniformly* accelerating mirror will indeed alter the quantum state in the vicinity of the mirror. However, a far more interesting result was obtained by Davies and Fulling [8,9] for a mirror experiencing *non-uniform* acceleration in (1+1)D. Such a mirror actually emits fluxes of quantum radiation, as though the mirror were knocking zero point quanta out of the vacuum and off to infinity. More precisely, they found that a mirror with 2-velocity $u^\mu = dx^\mu/d\tau\ (u\cdot u = -1)$ and acceleration a^μ emits a flux

$$\frac{dE}{d\tau} = -\langle T_{\mu\nu}\rangle u^\mu n^\nu = -\frac{\hbar}{12\pi}\frac{d}{d\tau}\ (a\cdot n), \qquad (1.1)$$

in the direction of a unit spatial vector n^ν orthogonal to u^μ. This holds for either choice ("left" or "right") of n^μ. Thus, a mirror whose acceleration is increasing (algebraically) toward the right will emit a stream of negative energy to the right and a numerically equal positive stream to the left.

The implications of this result are intriguing. Davies [10] and Ford [11] first raised the possibility that the negative-energy flux from a moving mirror could be used to cool a hot body and thus violate the second law of thermodynamics in a quantum context, and this paradox was further discussed by Deutsch, Ottewill and Sciama [12]. Limitations on the extent of such violations in flat spacetime ("quantum interest") have been formulated by Ford and Roman [13] and others ([14], [15], etc.).

More recently, Anderson [16] has used this result in the context of the Geroch gedankenexperiment [17]. In this experiment, a box with mirrored walls is filled with radiation and lowered adiabatically toward a black hole. Unruh and Wald [18] have shown that such a box is subject to a buoyancy force and will eventually reach a floating point above the black hole. Anderson has examined this further and shown that the ground state inside the box is the Boulware state, whose energy (which becomes increasingly negative as the box descends) is fed by Davies-Fulling fluxes from the reflecting walls. This accounts for the buoyancy felt by the box.

The volume of literature on moving mirrors is impressive, but it bears noting that all the results mentioned above are obtained in 1+1 dimensions. Indeed, if one includes the result for moving mirrors in curved space-times obtained by Ottewill and Takagi [19] with those reviewed above, the (1+1)D theory of moving mirrors can be considered essentially complete. This is due largely to the conformal properties of quantum field equations in (1+1)D, which allow boundaries, and even space-time itself, to be flattened, thereby enabling one to obtain results for complicated geometries from those for much simpler geometries.

This is not the case in 3+1 dimensions, where only partial results are available. The case of constant acceleration has been solved for both plane [20,21] and spherical [22] mirror geometries. Ford and Vilenkin [23] have extended the plane mirror result to include non-constant acceleration for the case when the acceleration and its derivatives are small. More recently, Hadasz *et al.* [24] have considered arbitrary (radial) motion of a spherical mirror, but have restricted their attention to the "S-wave approximation" where only spherically symmetric modes are considered. Because of this restriction, their result can be related to the 1+1 dimensional results of Davies and Fulling [8,9].

Consideration of quasi-stationary processes (e.g. slow descent of a mirror in a strong gravitational field) requires knowledge of the flux emitted by a mirror whose acceleration is changing slowly, though it may be large. Our objective in this paper is to derive an interesting and relatively simple result of this type in 3+1 dimensions. The central tool is the use of a Green's function perturbation technique. Evaluating the perturbation is much more manageable if the Green's functions for the unperturbed problem are available in closed form. This is actually the case for a uniformly accelerated spherical mirror, as shown by Frolov and Serebriany [22]. The mirror's history is then a three-dimensional pseudo-sphere of radius b, say, and the unperturbed problem is just the Minkowski-signature analogue of finding the four-dimensional electrostatic potential of a point charge in the presence of an earthed conducting 3-sphere of radius b. This is easily solved by the method of images.

Our objective in this paper is to solve the perturbed Frolov-Serebriany problem, i.e. to examine the effect of small spherically symmetric non-uniformities of the acceleration of a spherical mirror.

It would be good to stress at the outset that the solution for a plane mirror cannot be derived from ours by a straightforward limiting process. The single parameter b, whose reciprocal gives the unperturbed acceleration, also gives the minimum radius attained by the mirror as seen from its center. Thus, the plane limit $b\to\infty$ is inseparable from small acceleration. [There are reasons to expect the planar case to be considerably more complicated. Formally, the Wightman Green's function is now an infinite sum of McDonald (Bessel) functions. Geometrically, any light ray reflected non-orthogonally off a uniformly accelerated plane mirror will re-encounter the mirror an infinite number of times; in the spherical case there is just one encounter.]

Also, we concern ourselves only with calculating the outward flux, which we expect to be the most interesting stress-energy component. In fact, it turns out to be somewhat more interesting than one might expect. We find that it has a remarkable property. Although it could, in principle, depend on the entire retarded history of the mirror, to first order in the mirror perturbation it depends only on the behavior of the mirror at the most recent retarded time; i.e., it is local.

This article is organized as follows: in Sec. II we review the Frolov-Serebriany result for a mirror expanding with uniform acceleration. In Sec. III we present our Green's function perturbation scheme, and in Sec. IV we use it to evaluate the corrections to the Frolov-Serebriany Green's function, with some of the more cumbersome details relegated to Appendix A. Section V is concerned with calculating the quantum flux from these perturbations, with details again left to Appendixes B and C. Finally, in Sec. VI we offer some concluding remarks.

II. UNIFORMLY ACCELERATING SPHERICAL MIRROR

In the case where the mirror's acceleration is uniform, the Green's functions for the massless fields can be obtained in simple closed form by the method of images, as noted by Frolov and Serebriany [22]. In this section we shall briefly review these results.

Consider first the static potential due to a point charge q' in Euclidean 4-space at a distance R' from the center of an earthed conducting 3-sphere of radius b. The Dirichlet boundary condition can be reproduced by introducing a co-radial image charge $q''=-(b/R')^2q'$ at radius $R''=b^2/R'$.

In the Lorentzian analogue of this problem, we are concerned with Green's functions for the wave equation $\Box\varphi=0$ in Minkowski space-time, with Dirichlet boundary conditions on the pseudo-sphere (i.e. the time-like hyperboloid of one sheet) $R=b$, where now

$$R^2\equiv\eta_{\mu\nu}x^\mu x^\nu=x^2+y^2+z^2-t^2=\rho^2+z^2-t^2, \tag{2.1}$$

in a self-evident notation.

The pseudo-sphere $R=b$ represents the history of a spherical mirror of radius b (constant as measured in its instantaneous rest frame), whose center is fixed at the spatial origin $x=y=z=0$ and which moves with uniform acceleration $a=b^{-1}$.

The image construction gives for the retarded Green's function $G_{ret}(x,x')$, satisfying

$$\Box G_{ret}(x,x')=-\delta^4(x,x'), \tag{2.2}$$

the expression

$$G_{ret}(x,x')=\frac{1}{2\pi}\ \theta(t-t')\left\{\delta[(xx')^2]-\left(\frac{b}{R'}\right)^2\delta[(xx'')^2]\right\}. \tag{2.3}$$

Here θ is the unit step (Heaviside) function, δ is the Dirac distribution, $(xx')^2$ is the squared Minkowski interval

$$(xx')^2\equiv\eta_{\mu\nu}(x^\mu-x'^\mu)(x^\nu-x'^\nu), \tag{2.4}$$

and the image source is located at

$$x''^\mu=\left(\frac{b}{R'}\right)^2x'^\mu. \tag{2.5}$$

Similarly, the Wightman function

$$W(x,x')=\langle 0|\varphi(x)\varphi(x')|0\rangle \tag{2.6}$$

for a massless scalar field takes the form

$$W(x,x')=\frac{1}{4\pi^2}\left\{\frac{1}{(xx')^2+i(t-t')\epsilon}-\left(\frac{b}{R'}\right)^2\frac{1}{(xx'')^2+i(t-t'')\epsilon}\right\}, \tag{2.7}$$

with $\epsilon\to+0$.

III. NEARLY UNIFORM ACCELERATION: PERTURBING THE BOUNDARY

The corresponding Green's functions for a spherical mirror whose acceleration is slightly non-uniform can be derived from the preceding results by superposing the effect of a small perturbation on the history of the mirror, i.e. the time-like 3-space Σ on which Dirichlet boundary conditions are imposed.

Consider generally the problem of solving

$$\Box\Phi=0,\quad \Phi=0 \quad \text{on} \quad \Sigma. \tag{3.1}$$

Suppose that Σ is a small perturbation of a simpler time-like 3-space Σ_0, obtained by displacing Σ_0 a distance $\delta n(x)$ along its outward normal n, and that we know the solution Φ_0 of the problem

$$\Box\Phi_0=0,\quad \Phi=0 \quad \text{on} \quad \Sigma_0, \tag{3.2}$$

with the same initial boundary conditions.

Then Eq. (3.1) can be reformulated as a problem with boundary conditions specified on the unperturbed boundary Σ_0:

$$\Box\Phi=0,\quad \Phi=-\frac{\partial\Phi_0}{\partial n}\delta n(x) \quad \text{on} \quad \Sigma_0. \tag{3.3}$$

The causal solution for the perturbation $\delta\Phi\equiv\Phi-\Phi_0$ follows from Green's identity:

$$\delta\Phi(x')=\int_{\Sigma_0}d\Sigma\ \frac{\partial G_{ret}(x',x)}{\partial n}\delta n(x)\ \frac{\partial}{\partial n}\Phi_0(x), \tag{3.4}$$

where G_{ret} is the retarded Green's function for the unperturbed boundary value problem (3.2) and x is in Σ_0. Equation (3.4) defines a linear operation applied to Φ_0, which we shall write for brevity as

$$\delta\Phi(x')=L_{\delta n}(x')\ \ \Phi_0(\cdot). \tag{3.5}$$

To obtain the effect of the perturbation on the Wightman function (2.6), we note that it can be written as a mode sum

$$W(x,x')=\int\frac{d^3k}{(2\pi)^3}\ f_{\boldsymbol{k}}(x)\overline{f_{\boldsymbol{k}}(x')}, \tag{3.6}$$

where $\{f_{\boldsymbol{k}}(x)\}$ is a complete set of initially positive frequency solutions of Eq. (3.1) and the overbar denotes complex conjugation. Applying Eq. (3.5) to each mode separately and summing the results yields

$$\delta W(x,x')=L_{\delta n}(x)W_0(\cdot,x')+L_{\delta n}(x')W_0(x,\cdot), \tag{3.7}$$

where W_0 denotes the unperturbed Wightman function given by Eq. (2.7).

IV. WIGHTMAN FUNCTION FOR NEARLY UNIFORM ACCELERATION

Evaluation of the expression (3.7) for δW in the case where the mirror's acceleration departs slightly from uniformity is somewhat lengthy but straightforward. Here, we outline the results of this calculation reserving the technical details for Appendix A.

It will be assumed that the mirror remains spherical as viewed by an observer at its center $x=y=z=0$. Then its history is

$$R=b[1+f(t)],\quad |f(t)|\ll 1 \tag{4.1}$$

in terms of this observer's time t, where $f(t)$ is arbitrary but small. The corresponding advanced and retarded times are written

$$v=t+r,\quad u=t-r,\quad r=\sqrt{x^2+y^2+z^2}. \tag{4.2}$$

The radial energy flux measured by a stationary observer outside the mirror is

$$F=T_{uu}-T_{vv}. \tag{4.3}$$

Each of these terms can be dealt with by a similar procedure. Let us consider T_{uu}; it is evident from Eq. (2.6) that its expectation value at an event p_1 outside the mirror is

$$\langle T_{uu}(p_1)\rangle=[\partial_{u_1}\partial_{u_2}W(p_1,p_2)]_{p_2=p_1} \tag{4.4}$$

for a minimally coupled massless scalar field, and a similar but more complicated expression for conformal coupling [see Eq. (5.4) below]. Note that regularization and symmetrization are not needed to evaluate this component of the stress-energy tensor.

We can now present the results for δW. The spherical symmetry of Eq. (4.1) allows us to take both p_1 and p_2 in the z-t plane $(x=y=0)$. Calculation shows (see Appendix A) that the first term of Eq. (3.7) contributes

$$\frac{8\pi^2\zeta^2}{(R_1^2-b^2)(R_2^2-b^2)}L_{\delta n}(p_1)W(\cdot,p_2)$$
$$=\frac{z_1z_2}{\zeta}\int_{-\infty}^{t_1^*}dt\frac{f(t)}{(t-t_0)^3}-\frac{z_1z_1^*}{(R_1^2-b^2)(t_1^*-t_0)^2} \tag{4.5}$$

where ζ and t_0 are defined by

$$\zeta=z_1t_2-z_2t_1,\quad \zeta t_0=\frac{1}{2}(z_1-z_2)(z_1z_2-b^2)+(z_1t_2^2-z_2t_1^2), \tag{4.6}$$

and p_i^* (with coordinates x_i^*,t_i^*) is the event nearest to p_i $(i=1,2)$ at which the past light cone of p_i intersects the (unperturbed) mirror (see Fig. 1).

The contribution of the second term in Eq. (3.7) is obtained by interchanging p_1 and p_2 in Eq. (4.5). Because ζ is an odd function of p_1, p_2 (t_0 is even), the term involving the integral changes sign. Thus, the sum of the two contributions,

$$-\frac{8\pi^2\zeta^2}{(R_1^2-b^2)(R_2^2-b^2)}\delta W(p_1,p_2)$$
$$=\frac{z_1z_2}{\zeta}\int_{t_1^*}^{t_2^*}dt\frac{f(t)}{(t-t_0)^3}+\sum_{i=1}^{2}\frac{z_iz_i^*}{(R_i^2-b^2)(t_i^*-t_0)^2}f(t_i^*) \tag{4.7}$$

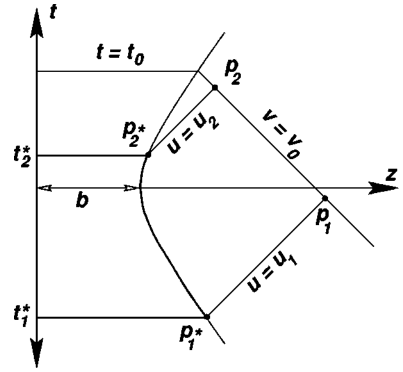

FIG. 1. The z-t plane. The mirror profile is the hyperbola. p_1 and p_2 are the two events at which the Wightman function is to be evaluated and v_0 is their common advanced time. p_1^* and p_2^* are the intersections of their past light cones with the mirror in the z-t plane.

depends only on the mirror's history between the retarded times t_1^* and t_2^*. Particle and anti-particle modes interfere destructively in the case of a spherical mirror to eliminate the effects of the past history and produce (when we take the coincidence limit $p_2 \rightarrow p_1$) a purely local expression for the flux.

To perform the partial derivatives ∂_{u_1} and ∂_{u_2} in the expression (4.4) for T_{uu}, we require mutual independence of the u coordinates of the events p_1 and p_2. But these are the only coordinates which need be independent. To evaluate T_{uu}, it suffices to consider $\delta W(p_1, p_2)$ in a partial pre-coincidence limit $v_1 = v_2$, as in Fig. 1.

These remarks in principle apply, *mutatis mutandis*, also to the evaluation of T_{vv}, but there is a complication. In the pre-coincidence limit $u_2 \rightarrow u_1$, the points p_1^* and p_2^* tend to coincidence with each other and with the point on the mirror having coordinate t_0, so that the integrand in Eq. (4.7) becomes infinite while the interval of integration shrinks to zero. The evaluation of T_{vv} is discussed further in Appendix C. Here, we merely note that T_{vv} makes no radiative contribution (proportional to r^{-2}) to the flux (4.4). This follows at once from the identity

$$\partial_u(r^2 \partial_u(r^2 T_{vv})) = \partial_v(r^2 \partial_v(r^2 T_{uu})), \tag{4.8}$$

which is a consequence of the conservation of $T_{\mu\nu}$ and the vanishing of the trace T^α_α for a conformal scalar field in flat space. In Eq. (4.8), the derivative ∂_v increases the falloff with distance, but this does not hold for ∂_u, which can operate on the retarded displacement $f(u)$ in Eq. (4.1). Thus, T_{vv} falls off more strongly than T_{uu}. The detailed calculation (Appendix C) shows that $T_{vv} \sim r^{-6}$ as $r \rightarrow \infty$.

V. FLUX FROM A NON-UNIFORMLY ACCELERATING MIRROR

The expectation value of the stress-energy tensor is derivable from the partial derivatives of the Wightman function $W(x,x') = W_0 + \delta W$ in the coincidence limit, with the unperturbed part W_0 given by Eq. (2.7) and the perturbation δW by Eq. (4.7). The unperturbed part becomes singular in the limit $x' \rightarrow x$, but is easily regularized by subtracting the value of W_0 in free space without the mirror, i.e. the first term of Eq. (2.7), leaving the second (image) term as the sole contribution to $(W_0)_{reg}$. The perturbation δW is regular in the coincidence limit.

For a massless scalar field, two different stress tensors are commonly considered: (a) the minimal stress-energy tensor, given classically by

$$(T_{\mu\nu})_{min} = \varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2} g_{\mu\nu} (\nabla\varphi)^2 \tag{5.1}$$

and quantum mechanically by

$$\langle T_{\mu\nu}(x)\rangle_{min} = \left[\left(\partial_\mu \partial_{\nu'} - \frac{1}{2} g_{\mu\nu} \partial^\alpha \partial_{\alpha'} \right) W_{reg}(x,x') \right]_{sym;x'=x}, \tag{5.2}$$

in which sym indicates symmetrization in (x,x') and in the partial derivatives; (b) the conformal (trace-free) stress-energy tensor, defined classically by

$$(T_{\mu\nu})_{conf} = \frac{2}{3} \varphi_{,\mu}\varphi_{,\nu} - \frac{1}{3} \varphi\varphi_{,\mu\nu} - \frac{1}{6} g_{\mu\nu} (\nabla\varphi)^2 \tag{5.3}$$

and quantum mechanically by

$$\langle T_{\mu\nu}(x)\rangle_{conf} = \frac{1}{3} \left[\left(2\partial_\mu \partial_{\nu'} - -\partial_{\mu'} \partial_{\nu'} - \frac{1}{2} g_{\mu\nu} \partial^\alpha \partial_{\alpha'} \right) W_{reg}(x,x') \right]_{sym;x'=x}. \tag{5.4}$$

We begin by reviewing the Frolov-Serebriany [22] results for *uniform* acceleration. Differentiating the regularized form of Eq. (2.7), we easily find

$$\langle T^{(0)}_{\mu\nu}(x)\rangle_{min} = -\frac{b^2}{\pi^2 (R^2 - b^2)^4} \left(x^\mu x^\nu - \frac{1}{2} g^{\mu\nu} R^2 \right), \tag{5.5}$$

$$\langle T^{(0)}_{\mu\nu}(x)\rangle_{conf} = 0. \tag{5.6}$$

This last result is quite remarkable, because the conformal stress is not likely to vanish for a spherical mirror at rest (it certainly does not for electromagnetic fields [25,10,26]). It appears that the effects of uniform acceleration exactly cancel the static Casimir stresses.

The effects of non-uniform acceleration are more complicated. We shall simply quote the result for the conformal radial out-flux $\langle T_{uu}\rangle_{conf}$ at a point (r,t) outside the mirror, leaving to Appendix B an outline of the derivation:

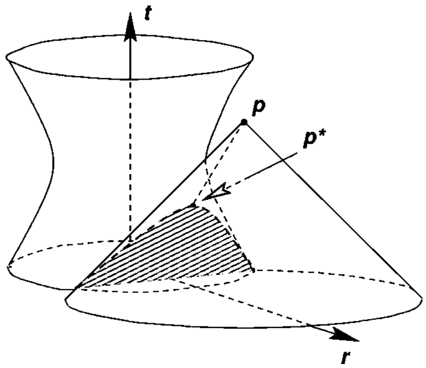

FIG. 2. Intersection of the history of a uniformly accelerating sphere (hyperboloidal cylinder) and the past null cone of an exterior point p. The intersection (which lies entirely within the shaded plane) is represented by the bold curve. The nearest retarded point to p on the mirror's world sheet, p^*, is the only point whose perturbation contributes to the flux at p. This figure has been dimensionally reduced; each point represents a circle.

$$\langle T_{uu}(t,r)\rangle_{conf} = -\frac{1}{1440\pi^2}\frac{1}{rv}\left\{\frac{q}{m^3 n}\frac{d^2\alpha}{d\chi^2} - \frac{4}{m^3 n^2 r}(npr-q^2)\frac{d\alpha}{d\chi} + \frac{2s^2}{mn^3r^2}\times\left[q(\alpha+f)+p\frac{df}{d\chi}\right]\right\}. \tag{5.7}$$

The notation is as follows: the advanced and retarded times, v and u, are defined as in Eq. (4.2), and we write

$$m=-\frac{u}{2},\quad n=\frac{1}{2v}(R^2-b^2)=-\frac{1}{2}(u+b^2/v),$$
$$p=\frac{1}{8}(u^2-b^2),\quad q=\frac{1}{8}(u^2+b^2),$$
$$s=-\frac{1}{2}(v+b^2/v); \tag{5.8}$$

χ is the pseudo-angle along the (unperturbed) mirror trajectory (i.e., $\tau=b\chi$ is the mirror's proper time). Equation (4.1), giving the trajectory of the perturbed mirror, is now written $R=b[1+f(\chi)]$, and

$$\alpha=f''(\chi)-f(\chi) \tag{5.9}$$

is a measure of the non-uniformity of the acceleration A, which is given by

$$A=(1+\alpha)/b. \tag{5.10}$$

In Eq. (5.7), χ refers to the pseudo-angle at the point p^*, which is the nearest retarded point to (r,t), as in Fig. 2. The corresponding expression for the minimally coupled flux is too long to reproduce here, and is also deferred to Appendix B.

The limit $b\rightarrow\infty$ corresponds to a slowly accelerating, nearly plane mirror. This was the case studied by Ford and Vilenkin [23]. It is straightforward to show that in this limit our result (5.7) for the conformal flux (and also our result for the minimally coupled flux) reduces to the expressions they give.

To obtain a more intuitive grasp of the physical meaning of the complex expression (5.7), we can evaluate the flux F radiated at retarded time $u=-b$—i.e. when the mirror is near its minimum radius $r_M\approx b$—as measured by a stationary observer at radius $r\gg b$ and the same retarded time. Using Eq. (4.3), taking the appropriate limit of Eq. (5.7), and noting that T_{vv} does not contribute to the flux to leading order (as discussed at the end of Sec. IV) we find

$$F=-\frac{\hbar}{720\pi^2}\left(\frac{R_0}{r}\right)^2\left\{A\frac{d^2A}{d\tau^2}+2A^3\left(A-\frac{1}{R_0}\right)\right\}, \tag{5.11}$$

where $R_0=b[1+f(0)]$ is the proper radius of the mirror at the time of emission $\chi=0$, and we have restored Planck's constant to display the correct dimensionality. We recall that this perturbative result is correct to linear order in deviations from uniform acceleration $(A-1/R)$, $dA/d\tau$, $d^2A/d\tau^2$, but the acceleration A itself is arbitrary.

VI. CONCLUDING REMARKS

Our chief interest in this paper has been in the quantum flux radiated by the mirror and we have explicitly computed only those components of the stress-energy tensor from which it arises. However, the remaining components can be derived straightforwardly (though with some labor) from our expression (4.7) with the methods of Appendix B. It would be useful to have these to round out the picture.

The relevant Green's functions for an unperturbed, uniformly accelerating spherical mirror have a simple closed form, and this has enormously simplified our perturbative calculation. This simplification is bought at a price: we are limited to spherical mirrors whose acceleration A and proper radius R are nearly reciprocals. We cannot decouple the plane limit $R\rightarrow\infty$ from the limit of small acceleration, and cannot disentangle curvature (Casimir) effects from the effects of acceleration.

It is evident that much remains to be done before we can claim anything approaching a comprehensive understanding of the quantum dynamics of three-dimensional mirrors.

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APPENDIX A: EVALUATION OF THE INTEGRAL (3.7) FOR δW

To verify Eq. (4.7), one needs to evaluate $L_{\delta n}(p_1)W_0(\cdot,p_2)$, where the integral operator $L_{\delta n}$ is defined by Eq. (3.5) and W_0 by Eq. (2.7). Because of the spherical symmetry of the mirror, there is no loss of generality in taking p_1 to be in the z-t plane. Then the integration over the azimuthal cylindrical coordinate ϕ is trivial.

The problem is essentially solved by the following lemma. Let $F(p)=F(\rho,z,t)$ be any axisymmetric function. Then we have the identity

$$\int_{\Sigma_0} F(p)\frac{\partial}{\partial n}G_{ret}(p_1,p)d\Sigma$$

$$=-\mathrm{sgn}(R^2-b^2)\frac{z_1^*}{z_1}F(p_1^*)+\frac{1}{2}\mathrm{sgn}(z_1)\frac{R_1^2-b^2}{z_1^2}$$

$$\times\int_{-\infty}^{t_1^*}dt\left[\frac{\partial}{\partial z}F(\rho=\sqrt{b^2+t^2-z^2},z,t)\right]_{z=Z_1(t)} \quad \text{(A1)}$$

where p_1^* is defined as in Sec. IV and the linear function $z=Z_1(t)$ is the solution of $\xi(p_1,p)=0$ where

$$\xi(p_1,p)\equiv t_1t-z_1z+\frac{1}{2}(R_1^2+b^2). \quad \text{(A2)}$$

Geometrically, $z=Z_1(t)$ represents the line in the z-t plane through p_1^* and orthogonal to the radius vector joining p_1 to the origin $z=t=0$.

To prove Eq. (A1), let us note that the presence of G_{ret} will effectively confine the integration to a 2-space $\mathcal{S}$, formed by the intersection of the unperturbed mirror's history

$$\Sigma_0: \quad \sigma(0,p)=\frac{1}{2}b^2 \quad \text{(A3)}$$

with the past light cone of p_1, given by

$$\sigma(p_1,p)=0, \quad \text{(A4)}$$

where $\sigma(p_1,p_2)=\frac{1}{2}(p_1p_2)^2$ is the usual geodesic biscalar [c.f. Eq. (2.4) and see Fig. 2]. Taking the difference between Eqs. (A3) and (A4), we see that $\mathcal{S}$ can equivalently be regarded as the intersection of the mirror Σ_0 with the 3-plane $\xi(p_1,p)=0$, where

$$\xi(p_1,p)=\sigma(p_1,p)-\sigma(0,p)+\frac{1}{2}b^2 \quad \text{(A5)}$$

is the same as Eq. (A2).

G_{ret} in the integral (A1) is a (distributional) function of $\sigma(p_1,p)$ and $\sigma(p_1,\tilde{p})$ [see Eq. (2.3)], where $\tilde{p}$ is the image point of p. To obtain its normal derivative on the mirror $R=b$, it is convenient to introduce four dimensional polar coordinates for p,

$$\rho=R\cosh\chi\sin\theta, \quad z=R\cosh\chi\cos\theta, \quad t=R\sinh\chi, \quad \text{(A6)}$$

so that the normal derivative (‘‘outward’’ from Σ_0 as viewed from the point p outside Σ_0) corresponds to $-\partial/\partial R$. We easily find

$$2\sigma(p_1,p)=R^2+2R(t_1\sinh\chi-z_1\cosh\chi\cos\theta)+R_1^2, \quad \text{(A7)}$$

$$\frac{\partial}{\partial R}\sigma(p_1,p)|_{R=b}=\frac{1}{b}\left[\xi(p_1,p)-\frac{1}{2}(R_1^2-b^2)\right]. \quad \text{(A8)}$$

A similar calculation for the image contribution yields values for $2\sigma(p_1,\tilde{p})$ and $(\partial/\partial R)\sigma(p_1,\tilde{p})$ numerically equal to Eq. (A7) and (A8) on the mirror, but for the normal derivative the sign is opposite.

The element of 3-area $d\Sigma$ on the mirror

$$\Sigma_0: \quad \rho=\sqrt{b^2+t^2-z^2}, \quad \text{(A9)}$$

employing (t,z,ϕ) as intrinsic coordinates, is $d\Sigma=b\,d\phi\,dz\,dt$. Hence the integral of an axisymmetric function $H(\rho,z,t)$ takes the explicit form

$$\int_{\Sigma_0} d\Sigma\ \ H(\rho,z,t)=2\pi b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dt\,dz\ \theta(b^2+t^2-z^2)$$

$$\times H(\rho=\sqrt{b^2+t^2-z^2},z,t) \quad \text{(A10)}$$

in which the step function θ takes into account that, for fixed t, the range of z over Σ_0 is restricted by Eq. (A9).

The distributional factors $\delta'[2\sigma(p_1,p)]$ and $\delta[2\sigma(p_1,p)]$, which arise from $\partial G_{ret}(p_1,p)/\partial n$ in the integral over Σ_0, are handled as follows. For points p restricted to Σ_0, Eqs. (A3) and (A5) show that $\sigma(p_1,p)=\xi(p_1,p)$, given explicitly in Eq. (A2) in terms of the intrinsic coordinates (t,z,ϕ) of Σ_0. Taking the partial derivatives with respect to z tangentially along Σ_0 (i.e. holding ϕ and t fixed in ξ), we immediately find

$$\frac{\partial}{\partial z}\delta[2\sigma(p_1,p)]=-2zz_1\delta'[2\sigma(p_1,p)]. \quad \text{(A11)}$$

Thus, δ' can be eliminated in favor of the tangential derivative $\partial\delta/\partial z$, which can be converted through integration by parts in Eq. (A10) to

$$\delta[2\sigma(p_1,p)]=\delta[2\xi(p_1,p)]=\frac{1}{2|z_1|}\delta(z-Z_1(t)), \quad \text{(A12)}$$

by virtue of Eq. (A2).

Putting all this together leads straightforwardly to the quoted result (A1).

APPENDIX B: DERIVATION OF EQ. (5.7) FOR THE CONFORMAL FLUX

We briefly outline how the formula (5.7) for $\langle T_{uu}\rangle_{conf}$ is derived from Eq. (4.7) via Eq. (5.4).

δW involves the integral

$$z_1z_2\int_{t_1^*}^{t_2^*}\frac{f(t^*)}{(t^*-t_0)^3}dt^*. \quad \text{(B1)}$$

Because of the prefactor z_1z_2, it is convenient to change the variable of integration from t (the coordinate of a point p on the profile of Σ_0 in the z-t plane) to z (the coordinate of a point p on the line $v=v_0$ joining p_1 and p_2, and having the same retarded time, $u=u^*$). Since $t^*=\frac{1}{2}(u^*+v^*)=\frac{1}{2}(u-b^2/u)$ for a point p^* on Σ_0 and $z=\frac{1}{2}(v_0-u)$, the formal transformation is

$$t^*-t_0=-\frac{z}{uv_0}(b^2+uv_0),\quad t_0=\frac{1}{2v_0}(v_0^2-b^2). \tag{B2}$$

We thus find

$$\frac{1}{2}f(t^*)\frac{dt^*}{(t^*-t_0)^3}=A(z)dz,\quad A(z)=\frac{mq}{[z(z+s)]^3}f(t^*), \tag{B3}$$

where m, q and s are defined in Eq. (5.8), and here $v=v_0$.

Writing, for any function $F(z)$,

$$\overline{F}=\frac{1}{2}\{F(z_1)+F(z_2)\},\quad \Delta F=F(z_2)-F(z_1), \tag{B4}$$

the expression (4.7) for δW now reduces to

$$\frac{\pi^2v_0(\Delta z)^2}{(z_1+s)(z_2+s)}\,\delta W(p_1,p_2)=\frac{z_1z_2}{\Delta z}\int_{z_1}^{z_2}A(z)dz-\overline{z^2A(z)}. \tag{B5}$$

It is now straightforward, though tedious, to carry out the operations in Eq. (5.4). The following general identities are of help in this regard. Define, for an arbitrary $F(z)$,

$$\epsilon=\frac{1}{\Delta z}\int_{z_1}^{z_2}F(z)dz-\overline{F}. \tag{B6}$$

Then

$$\frac{\epsilon}{(\Delta z)^2}=-\frac{1}{12}\overline{F''}+\frac{1}{120}(\Delta z)^2\overline{F^{(4)}}+\dots \tag{B7}$$

$$\frac{\Delta F}{\Delta z}=\overline{F'}-\frac{1}{12}(\Delta z)^2\overline{F'''}+\frac{1}{120}(\Delta z)^4\overline{F^{(5)}}+\dots \tag{B8}$$

where primes and subscripts in parentheses denote derivatives with respect to z. If

$$J=\frac{1}{(\Delta z)^2}\left\{\frac{z_1z_2}{\Delta z}\int_{z_1}^{z_2}F(z)dz-\overline{z^2F}\right\}, \tag{B9}$$

then

$$\frac{\partial^2J}{\partial z_1z_2}=-2\frac{\epsilon}{(\Delta z)^2}\left(1+\frac{6z_1z_2}{(\Delta z)^2}\right)+\frac{2}{(\Delta z)^2}+\left(\frac{\Delta(zF)}{\Delta z}-\overline{F}-\frac{1}{2}\frac{\Delta(z^2F')}{\Delta z}\right). \tag{B10}$$

Taking coincidence limits $z_2\to z_1$ gives

$$\left[\frac{\partial^2J}{\partial z_1\partial z_2}\right]=-\frac{1}{60}(z^2F^{(4)}+10zF'''+20F''), \tag{B11}$$

$$\left[\overline{\frac{\partial J}{\partial z}}\right]=-\frac{1}{24}(z^2F'''+8zF''+12F'), \tag{B12}$$

$$[J]=-\frac{1}{12}[z^2F''+6(zF)'], \tag{B13}$$

$$\left[\overline{\frac{\partial^2J}{\partial z^2}}\right]=\frac{3}{2}\left[\frac{\partial^2J}{\partial z_1\partial z_2}\right]. \tag{B14}$$

A fairly long calculation then leads to the result (5.7) for the conformal flux. For the minimally coupled flux, we find

$$\delta\langle T_{uu}\rangle_{min}=\frac{1}{240\pi^2vr}\left\{\frac{q}{m^3n}\frac{d^2\alpha}{d\chi^2}+a_1\frac{d\alpha}{d\chi}+a_2\alpha+a_3\frac{1}{m}\left(f-\frac{df}{d\chi}\right)+a_4mf\right\} \tag{B15}$$

as the perturbation of the uniform acceleration result (5.5). Here,

$$a_1=\frac{1}{2mn}+\frac{1}{mr}+\frac{r}{m^3n^2}\left(\frac{7}{2}m^2-n^2-\frac{5}{2}mn\right), \tag{B16}$$

$$a_2=-\frac{1}{2mn}+\frac{3}{2}\frac{1}{n^2}+\frac{3}{2}\frac{1}{nr}+\frac{1}{r^2}+\frac{16r}{n^3}-\frac{16r}{mn^2}-\frac{1}{mr}, \tag{B17}$$

$$a_3=-\frac{1}{2n}+\frac{3}{2}\frac{m}{nr}-2\frac{m^2}{nr^2}-\frac{1}{r}+\frac{m}{r^2}+\frac{3}{2}\frac{m}{n^2}-\frac{m^2}{n^2r}-\frac{2m^2}{n^3}+\frac{16r(m-n)}{n^3}-\frac{45m(m-n)r}{n^4}, \tag{B18}$$

$$a_4=\frac{2}{n^3}+\frac{1}{n^2r}+\frac{2}{nr^2}+\frac{60(m-n)r}{n^5}. \tag{B19}$$

APPENDIX C: EVALUATION OF T_{vv}

For arbitrary points p_1 and p_2 in the z-t plane, the expression (4.7) for δW can be recast in the form

$$-\frac{2\pi^2}{\Delta u}(\overline{v}\Delta u-\overline{u}\Delta v)^3\delta W(p_1,p_2)$$
$$=(R_1^2-b^2)(R_2^2-b^2)\left\{\frac{z_1z_2}{\Delta z}\int_{z_1}^{z_2}H(z)dz-\overline{z^2H}\right\}$$
$$+\frac{1}{2}\Delta v\{(u_1u_2\overline{v}+b^2\overline{u})\Delta(z^2H)$$

$$-(u_1u_2\Delta v-b^2\Delta u)\overline{z^2H}\},\tag{C1}$$

which generalizes Eq. (B5) to the case where $\Delta v=v_2-v_1\neq 0$. The integral of $H(z)$ is to be taken along the straight-line segment joining p_1 and p_2, and we have defined

$$H(z)=\frac{u(u^2+b^2)}{[u(t^*-t_0)]^3}f(t^*).\tag{C2}$$

The denominator involves a quadratic function of retarded time, having the explicit form

$$u(t^*-t_0)=\frac{1}{2}(u^2-cu-b^2),\tag{C3}$$

where

$$c=\frac{(\overline{v^2}-b^2)\Delta u-(\overline{u^2}-b^2)\Delta v-\frac{1}{2}\Delta u\Delta v\Delta z}{\bar{v}\Delta u-\bar{u}\Delta v}.\tag{C4}$$

By assigning different values to the ratio $\Delta v/\Delta u$ as $\Delta u\to 0$, we approach the coincidence limit $p_1=p_2$ in all possible directions in the z-t plane. [This is subject to the restriction that t_0, given by Eq. (4.6), should be kept outside the interval (t_1^*,t_2^*) to keep the integral of H in Eq. (C1) mathematically well defined.] Both of the components T_{uu} and T_{vv} can thus be found from the corresponding directional derivatives.

We calculated T_{vv} with the aid of the computer algebra package MAPLE. Even so, the calculation is not straightforward. The main obstacle is the evaluation of the integral term in Eq. (C1). After taking the appropriate v_1 and v_2 derivatives of Eq. (C1), and then the partial coincidence limit $v_2\to v_1$, we express the integrand of the first term in a Laurent series in (u_2-u_1) and $(u-u_1)$, both to fifth order. The result is expressed in ratios of powers of $(u-u_1)$ and (u_2-u_1), with coefficients of these ratios being functions of u_1 and v_1 alone. The integration is then trivially performed. The first two terms in the integrated Laurent series diverge in the coincidence limit $u_2\to u_1$. However, these are exactly cancelled by terms arising from the v derivatives of the second term of Eq. (C1). Taking the (trivial) coincidence limit of the remaining terms we find the following expression for T_{vv}:

$$T_{vv}=-\frac{1}{45\pi^2}\frac{q^2}{v^3n^3r^3m}\left(q(\alpha+f)+p\frac{df}{d\chi}\right),\tag{C5}$$

where our notation is defined in Eq. (5.8). We have verified that the expressions (C5) for T_{vv} and (5.7) for T_{uu} satisfy the conservation identity (4.8).

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