Charged vacuum condensate near a superconducting cosmic string

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A charged superconducting cosmic string produces an extremely large electric field in its vicinity. This leads to vacuum instability and to the formation of a charged vacuum condensate which screens the electric charge of the string. We analyze the structure of this condensate using the Thomas-Fermi method. [S0556-2821(99)05318-7]

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I. INTRODUCTION

Cosmic strings are linear defects that could be formed at a phase transition in the early universe. (For a review see [1].) Witten [2] has shown that strings predicted in some grand unified models behave as superconducting wires. Such strings moving through magnetized cosmic plasmas can develop large currents and can give rise to a variety of astrophysical effects. In particular, they have been suggested as possible sources of ultrahigh energy cosmic rays [3].

Currents developed by oscillating strings in a magnetic field are not homogeneous along the strings because different portions of the string cross the magnetic field lines in different directions. This results in charge accumulation, and portions of the string can develop a charge per unit length λ comparable to the current, $\lambda \sim J$. (Here and below we use units in which $\hbar = c = 1$.) The electric field near the string is given by

$$E = \frac{2\lambda}{r}. \tag{1}$$

It can become extremely strong in the immediate vicinity of the string, and then quantum effects, such as vaccum polarization and pair production, must be taken into account. One can expect that the created particles of charge opposite to that of the string will accumulate in bound states and form a condensate screening the electric field near the string to below the critical value. It was noted earlier that such screening would lead to a drastic modification of string electrodynamics [4] and would have a significant effect on the propagation of high-energy particles emitted from the charged portions of the string [5]. The purpose of the present paper is to give a quantitative description of the screening condensate near a charged superconducting string.

The superconducting current in the strings is carried by charged particles which acquire a mass M at the string-forming phase transition but remain massless inside the strings. These massless charge carriers move along the strings at the speed of light. The string current is bounded by the critical value

$$J_c \sim eM, \tag{2}$$

at which the characteristic energy of the charge carriers becomes comparable to M, so that they have enough energy to jump out of the string. The mass M is model-dependent but is limited by the string symmetry breaking scale η, $M \lesssim \eta$.

In a cosmological setting, the string charges and currents vary on astronomical time and length scales, and for our purposes we can regard them as constant. For a string segment with $J < \lambda$, we can always find a Lorentz frame where $J = 0$. The electric field close to the string will then be well approximated by that of an infinite straight string. We shall consider, therefore, an infinite straight string with a constant charge per unit length λ and vanishing current, $J = 0$. For sufficiently large λ, charged particles (for definiteness electrons) have bound states localized near the string with negative energies smaller than $-m$, where m is the electron mass. The vacuum then becomes unstable with respect to production of electron-positron pairs. For a positively charged string, positrons are repelled away, while electrons form a vacuum condensate surrounding the string. We shall determine the electric field and the charge distribution in this condensate using the Thomas-Fermi method [6,7] in which the condensate is approximately treated as an ideal gas obeying the Fermi-Dirac statistics.

In the next section we shall review the derivation of the relativistic Thomas-Fermi equation and specify the boundary conditions appropriate for the case of cylindrical symmetry. Approximate analytic solutions of this equation are given in Sec. III, and its numerical solutions are presented in Sec. IV. The conclusions of the paper are summarized and discussed in Sec. V.

II. THOMAS-FERMI EQUATION

The density of electrons in a degenerate Fermi gas is related to the Fermi momentum p_F by

$$n_e(r) = \frac{p_F^3}{3\pi^2}. \tag{3}$$

The relativistic relation between the Fermi energy ϵ_F and Fermi momentum is

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$$p_F = [(\epsilon_F - V(r))^2 - m^2]^{1/2}, \tag{4}$$

where m is the electron mass, $-e$ is its charge, $V(r) = -e\varphi(r)$ and $\varphi(r)$ is the self-consistent electrostatic potential for an electron, taking into account both the field of the string and the average field produced by other electrons of the condensate. The condensate is formed of electrons occupying quantum states in the negative energy continuum, $\epsilon < -m$. We therefore set the Fermi energy to be $\epsilon_F = -m$. The electron density (3) is then given by

$$n_e(r) = \frac{1}{3\pi^2}[V^2(r) + 2mV(r)]^{3/2}. \tag{5}$$

Introducing the total charge density ρ_T which is composed of the electron charge and external string charge,

$$\rho_T = \rho_s - en_e \tag{6}$$

and using the Poisson equation

$$\triangle V(r) = 4\pi e\rho_T(r), \tag{7}$$

we find a self-consistent non-linear differential equation

$$\triangle V(r) = -4\pi e\left[\frac{e}{3\pi^2}(V^2(r) + 2mV(r))^{3/2} - \rho_s(r)\right]. \tag{8}$$

This equation has been used in [8,9] to study the electron condensate around supercharged nuclei. In the case of a string, the problem has cylindrical symmetry and

$$\triangle V(r) = V''(r) + \frac{1}{r}V'(r). \tag{9}$$

We shall approximate the string charge distribution as a uniform distribution in a cylinder of radius δ,

$$\rho_s(r) = \rho_0\theta(\delta - r). \tag{10}$$

The linear charge density of the string is given by $\lambda = \pi\delta^2\rho_0$. The charge carriers are typically concentrated in a tube of radius $r \sim M^{-1}$; hence, we should have $\delta \sim M^{-1}$.

It is easily seen from Eq. (5) that the density of electrons, $n_e(r)$, is different from zero only in the region of space where $V(r) < -2m$. Therefore, the condensate has a finite radius $r = R_c$. For $r > R_c$, the solution of Eq. (8) is just the usual logarithmic potential of a linear charge,

$$V(r) = 2e\lambda_0 \ln\frac{r}{R_*}. \tag{11}$$

Here, λ_0 is the total charge per unit length of string, including both the charge carriers in the core and the condensate, and R_* is the cutoff radius indicating the distance at which the approximation of an infinite straight string breaks down. R_* is given by the smallest of the following three length scales: (i) the typical distance between the strings in a cosmic string network, (ii) the characteristic curvature radius of string, (iii) the typical wavelength of the current-charge oscillations along the string.

The boundary condition for Eq. (8) at $r = 0$ is

$$V'(0) = 0, \tag{12}$$

while at $r = R_c$ we have

$$V(R_c) = -2m, \quad V'(R_c) = \frac{2}{R_c \ln(R_*/R_c)}. \tag{13}$$

Note that we have three rather than two boundary conditions, as a second-order differential equation would normally require. The third condition is needed to determine the condensate radius R_c.

We expect R_c to be microscopic, while R_* will typically be astrophysically large. Hence, the logarithm in Eq. (13) is $\ln(R_*/R_c) \sim 10^2$. In numerical calculations below we choose R_* so that $\ln(R_*/R_c) \approx 30$; our results are not sensitive to this choice.

The Thomas-Fermi approximation is adequate when the characteristic scale of variation of the condensate density $n_e(r)$ is large compared to the electron wavelength $1/p(r)$. The corresponding condition is

$$\left|\frac{d}{dr}\left[\frac{1}{p(r)}\right]\right| \ll 1. \tag{14}$$

We shall see that this condition is satisfied in most of the condensate region $0 < r < R_c$, provided that the charge density λ is sufficiently large.

III. ANALYTIC APPROXIMATIONS

The Thomas-Fermi equation (8) can be solved analytically in the limit when the magnitude of the potential $V(r)$ is large, $|V(r)| \gg 2m$. We can then neglect $2mV(r)$ compared to $V^2(r)$, and outside the string core Eq. (8) reduces to

$$V''(r) + \frac{1}{r}V'(r) = -\frac{4e^2}{3\pi}|V(r)|^3. \tag{15}$$

This has a solution

$$V(r) = -C/r \tag{16}$$

with

$$C = (3\pi/4e^2)^{1/2} \approx 18. \tag{17}$$

The corresponding electric field is

$$E(r) = C/er^2. \tag{18}$$

We note that the solutions (16) and (18) do not depend on the string charge density λ. As r decreases, the electric field (18) grows faster than that of the vacuum solution (1). It cannot, therefore, be extended all the way to the string but has to be matched with Eq. (1) at some radius R_s below which the vacuum solution takes over. The matching radius at which the two electric fields become comparable is

$$R_s \sim C/e\lambda \sim 200\lambda^{-1}. \tag{19}$$

We shall call it the screening radius. For $r \ll R_s$, the screening is unimportant and the electric field is given by Eq. (1). The screening radius is always large compared to the string thickness $\delta \sim M^{-1}$, provided that λ is smaller than the critical value (2),
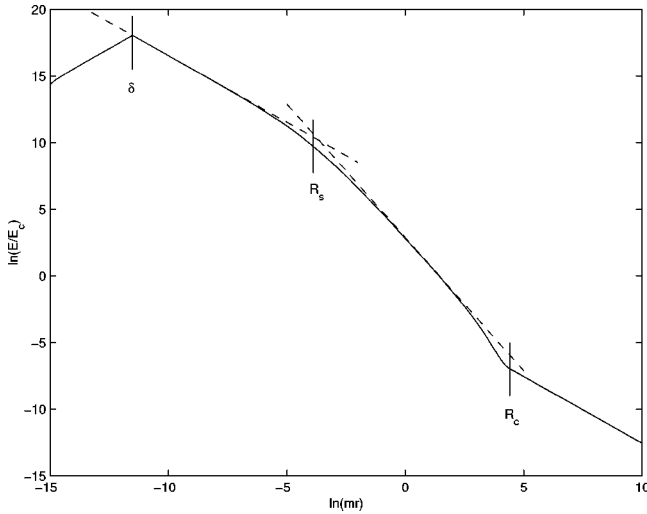
$$\lambda \lesssim eM. \tag{20}$$

FIG. 1. The electric field E in units of the critical field E_c $=m^2/e$ is shown as a function of the distance from the string r for $\lambda=4\times10^3m$ and $\delta=10^{-5}m^{-1}$ (solid line). Dotted lines indicate the analytic approximations (1) and (18) in the appropriate regimes. The core radius δ, the screening radius R_s and the condensate radius R_c are also indicated.

The potential corresponding to the vacuum solution (1) at $\delta<r\ll R_s$ is

$$V(r)=-2e\lambda[\ln(R_s/r)+B],   \qquad (21)$$

where $B\sim1$ is a numerical constant. The potential at the string core is thus

$$V(0)\approx-2e\lambda\ln(R_s/\delta).   \qquad (22)$$

The condition $|V(r)|\gg m$ implies $r\ll C/m$, and thus the solution (18) is valid in the range $C/e\lambda\ll r\ll C/m$. This range exists only if λ is sufficiently large, $\lambda\gg m/e$. Combined with the condition (20) this implies $M\gg m/e^2$. In models of astrophysical interest, the charge carrier mass M is
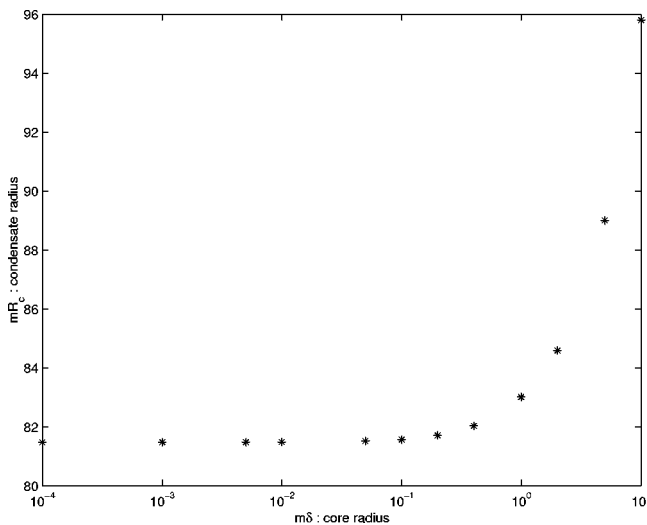

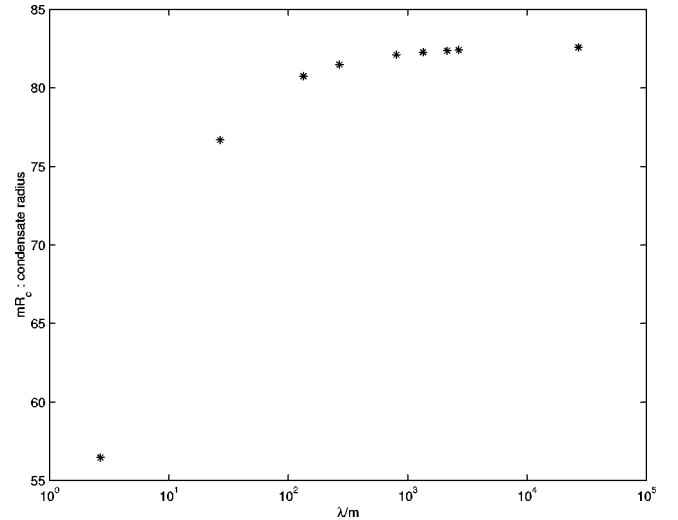FIG. 2. The condensate radius R_c vs the core radius δ for λ $=270m^{-1}$.

FIG. 3. The condensate radius R_c vs the linear charge density of the string, λ for $\delta=0.001m^{-1}$.

very large (so that the strings can develop large currents and charges), and this condition is satisfied with a large margin.

At $r\sim C/m$, the potential $V(r)$ becomes comparable to $-m$ signalling that we are close to the condensate boundary [see Eq. (13)]. Hence, we can estimate the condensate radius as

$$R_c\sim C/m.   \qquad (23)$$

The condition of validity of the Thomas-Fermi approximation (14), when applied to the solution (16), gives $C\gg1$. This is satisfied with a reasonable accuracy [see Eq. (17)].

IV. NUMERICAL CALCULATION

We obtained numerical solutions to the Thomas-Fermi equation for Eq. (8) for $V(r)$ with the boundary conditions (12) and (13) using the relaxation method. The resulting electric field is plotted in Fig. 1, together with the analytic approximations (1) and (18). The agreement between the analytic and numerical solutions is excellent in the appropriate ranges of the radius r.

We have verified that the shape of $V(r)$ outside the string core is not sensitive to the value of the core radius δ. In particular, the condensate radius R_c approaches a constant value independent of δ (see Fig. 2). This is very fortunate, since a realistic value of the core radius would be too small to resolve in our calculations. Figure 2 suggests that it is sufficient to choose $\delta\ll m^{-1}$. We used $\delta=10^{-3}m^{-1}$ in most of the calculations described below.

The condensate radius R_c is plotted in Fig. 3, as a function of the linear charge density of the string, λ. We see that at large λ, R_c approaches a constant value,

$$R_c=82m^{-1},   \qquad (24)$$

in agreement with Eq. (23). The screened linear charge density of the string λ_0, which determines the electric field outside R_c, is also found to be independent of λ:
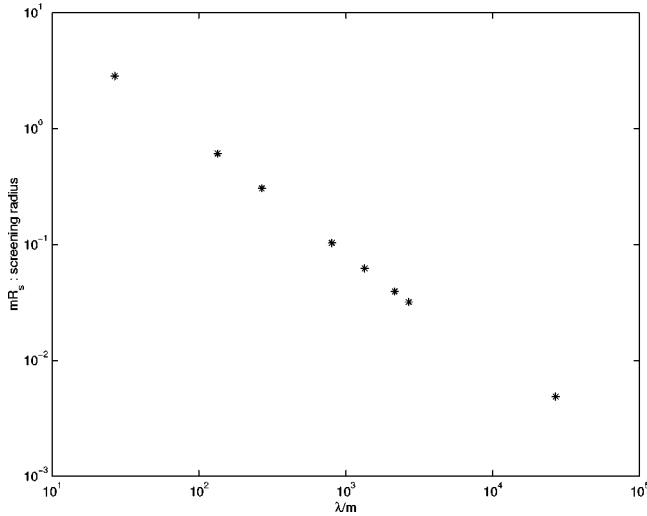
$$\lambda_0\approx5.34em.   \qquad (25)$$

FIG. 4. The screening radius R_s vs the linear charge density of the string λ for $\delta = 0.001 m^{-1}$.

The electric field at the condensate boundary is

$$E_0 = 2\lambda_0/R_c \approx 10^{-1} em^2. \qquad (26)$$

Note that this is considerably smaller than the critical field, $E_c = m^2/e$, which signals the onset of intensive pair production [10]. In our case, $E_0 \approx 10^{-3} E_c$. We shall return to this point later in Sec. V.

The effective linear charge density $\lambda_{eff}(r)$ inside the condensate can be found as

$$\lambda_{eff}(r) = 2\pi \int_0^r \rho(r') r' dr' = \frac{r}{2e} \frac{dV}{dr} \qquad (27)$$

where we have used Eq. (7). As r grows, $\lambda_{eff}(r)$ decreases and we can define the effective screening radius R_s as the radius at which half of the string charge is screened,

$$\lambda_{eff}(R_s) = \lambda/2. \qquad (28)$$

At the boundary of the condensate we must have $\lambda_{eff}(R_c) = \lambda_0$. The screening radius R_s is plotted in Fig. 4 for several values of λ. We see that, although the condensate radius R_c is independent of λ, the screening radius gets smaller rather rapidly as λ is increased. A numerical fit to the data in the Fig. 4 gives

$$R_s = 80\lambda^{-1}, \qquad (29)$$

in agreement with the order-of-magnitude estimate (19).

V. SUMMARY AND DISCUSSION

We have found that a superconducting cosmic string having a sufficiently large charge per unit length, $\lambda \gg m/e$, is surrounded by an electron condensate of radius $R_c \sim 100 m^{-1}$, where m is the electron mass. In the immediate vicinity of the string, the effect of the condensate is unimportant and the electric field is given by the vacuum solution, $E \approx 2\lambda/r$. Screening due to the condensate becomes significant at $r \sim R_s \sim 100\lambda^{-1}$, and for $R_s \ll r \ll R_c$ the electric field has the form $E \approx (3\pi)^{1/2}/2e^2 r^2$. Outside the condensate, at $r > R_c$, the field is given by $E \approx em/r \approx 10^{-2} em^2 (R_c/r)$.

As we already mentioned, the electric field at the condensate boundary is well below the critical field, $E_0 \sim 10^{-3} E_c$, where $E_c = m^2/e$. The rate of pair production per unit volume in a homogeneous electric field is [10]

$$dN/dVdt \approx (eE/\pi)^2 \exp(-\pi E_c/E), \qquad (30)$$

which indicates that the outer parts of the condensate where $E \ll E_c$ will be filled up very slowly. The characteristic time of pair production, $\tau \sim (eE)^{-1/2} \exp(\pi E_c/E)$, is greater than the age of the universe for $E \lesssim 4 \times 10^2 E_c$. For astrophysical strings, we expect the condensate radius to be given by the distance from the string at which such values of the electric field are reached. From Eq. (16) we find

$$R_c \sim \frac{\sqrt{C}}{20} m^{-1} \approx 0.2 m^{-1}. \qquad (31)$$

As the charge density of the string λ is increased, the potential near the string core becomes more and more negative. As a result, particles more massive than electrons develop condensates. From Eq. (22), particles of mass μ develop states with $\epsilon < -\mu$ at $\lambda \sim \mu/e \ln(R_s/\delta) \sim \mu$. The condensates of different particle species will have the form of coaxial cylinders, with condensates of more massive particles being closer to the string.

Finally, we would like to mention some open questions. In this paper we studied vacuum condensation of fermions. Charged Bose particles, such as Higgs and gauge bosons will also form vacuum condensates, and the properties of these bosonic condensates may differ from the fermionic case. Another important problem is the nature of modifications introduced by vacuum screening in string electrodynamics and in the propagation of charged particles emitted by the strings. We hope to return to some of these issues in future publications.

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