

Kaluza-Klein and H dyons in string theory

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Kaluza-Klein monopole and H -monopole solutions, which are T dual to each other, are the well-known solutions of string theory compactified on T^6 . Since string theory in this case has an S -duality symmetry, we explicitly construct the corresponding dyonic solutions by expressing the $D=4$ string effective action in a manifestly $SL(2,R)$ invariant form with an $SL(2,R)$ invariant constraint. The Schwarz-Sen charge spectrum, the Bogomol'nyi-Prasad-Sommerfield saturated mass formula as well as the stability of these states are discussed briefly. [S0556-2821(99)08118-7]

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I. INTRODUCTION

It is well known that the five dimensional pure Einstein gravity admits a solitonic solution known as the Kaluza-Klein (KK) monopole solution first obtained by Gross, Perry, and Sorkin (GPS) [1,2]. Since in this construction one of the spatial coordinates is compactified, this solution can also be viewed as a four-dimensional black hole with a magnetic charge. The $U(1)$ gauge field corresponding to the magnetic charge originates in this case from the isometry of the compact dimension. As the five-dimensional pure gravity is contained as a special case of dimensionally reduced string theory, obviously, string theory in four or five dimensions also admits a KK monopole solution. String theory in four dimensions admits another kind of monopole solution known as the H -monopole solution [3,4]. The $U(1)$ gauge field corresponding to the magnetic charge in this case arises from the dimensional reduction of the second rank antisymmetric tensor B_{MN} contained in the string theory spectrum. These two solutions are in fact T dual [5] to each other. In the original ten-dimensional theory they represent the five-brane solutions compactified on a circle and the T duality [6] relates the type IIA (IIB) KK monopole to the type IIB (IIA) Neveu-Schwarz five-branes. Many interesting dynamical properties of KK monopoles as well as the world volume theory have been studied in Refs. [7–14]. The world volume theories of KK monopole are related to various supersymmetric gauge theories in $5+1$ dimensions and can be used to study their properties.

Any string theory in four dimensions has been conjectured [15] to possess an exact $SL(2,Z)$ symmetry also known as the S -duality symmetry as a part of the complete nonperturbative U duality [16] symmetry. Many evidences in favor of this conjecture have been given in Ref. [17]. One such evidence is the prediction of the existence of dyonic solutions corresponding to both the KK monopole and the H -monopole solutions of string theory. The proof of existence of these dyonic excitations (in both heterotic and type II string theory) has been given in Refs. [18,19] by arguing that the degeneracies of the dyonic states match precisely with those of the elementary string states. In this paper, we

explicitly construct the dyonic solutions starting from the known monopole solutions of type II string theory in a simplified setting. We first express the four dimensional string effective action in a manifestly $SL(2,R)$ invariant form alongwith an $SL(2,R)$ invariant constraint on the field-strengths. Then we use this symmetry to rotate the monopole solutions and obtain the corresponding dyonic solutions. A different kind of dyonic solutions of both the KK and H monopoles have been discussed in Refs. [20,21], but the electric and the magnetic charges considered there correspond to different gauge fields instead of the same gauge field. We also obtain the Schwarz-Sen electric-magnetic charge spectrum [22] as well as the Bogomol'nyi-Prasad-Sommerfield (BPS) saturated Arnowitt-Deser-Misner (ADM) masses for these dyonic solutions. We then discuss how the stability of these states can be understood from the mass formula. Unlike the dyonic solutions discussed in Ref. [21], the dyonic black hole we obtain has finite Hawking temperature but zero entropy.

This paper is organized as follows. In Sec. II, we briefly discuss the KK monopole solution of GPS and mention how it can be regarded as a string theory solution in $D=4$. H -monopole solution is also discussed in brief. In Sec. III, we construct the dyonic solutions by applying $SL(2,R)$ transformation on the monopole solutions. The Schwarz-Sen charge spectrum and the BPS saturated mass formula are obtained. We also discuss the stability of the dyonic states. Finally, our conclusions are presented in Sec. IV.

II. KK- AND H-MONOPOLE SOLUTIONS IN STRING THEORY

The KK monopole was originally obtained by Gross, Perry, and Sorkin [1,2] as a solution of pure Einstein gravity in five dimensions (with one of the spatial dimensions compactified) where the action has the form

$$S_5 = \int d^5x \sqrt{-GR}. \quad (1)$$

If we denote the compact dimension as x^4 , then the five-dimensional metric can be decomposed in terms of four-dimensional metric as usual by the following KK ansatz:

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$$G_{MN} = \begin{pmatrix} G_{\mu\nu} + A_\mu A_\nu G_{44} & A_\mu G_{44} \\ A_\nu G_{44} & G_{44} \end{pmatrix}, \quad (2)$$

where $M, N = 0, 1, \dots, 4$ and $\mu, \nu = 0, 1, 2, 3$. Also all the four-dimensional fields here are independent of x^4 . Now the five-dimensional action (1) reduces to the four-dimensional action as follows:

$$S_4 = \int d^4x \sqrt{-\bar{G}} \left[R - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} - \frac{1}{4} e^{-\sqrt{3}\tilde{\phi}} F_{\mu\nu}^2 \right], \quad (3)$$

where the metric $\bar{G}_{\mu\nu} = G_{44}^{1/2} G_{\mu\nu}$, the scalar $\tilde{\phi} = -(\sqrt{3}/2) \log G_{44}$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The equations of motion obtained from action (3) has a magnetically charged black hole solution which in the extremal limit takes the form [23–25]

$$ds^2 = - \left(1 + \frac{P}{r} \right)^{-1/2} dt^2 + \left(1 + \frac{P}{r} \right)^{1/2} [dr^2 + r^2 d\Omega_2^2],$$

$$e^{2\tilde{\phi}} = \left(1 + \frac{P}{r} \right)^{\sqrt{3}} \quad \text{or} \quad e^{2\tilde{\phi}/\sqrt{3}} = \left(1 + \frac{P}{r} \right) = G_{44}^{-1}. \quad (4)$$

Here ds^2 is written in terms of the canonical metric $\bar{G}_{\mu\nu}$ and in terms of $G_{\mu\nu}$ it is given by

$$ds^2 = -dt^2 + \left(1 + \frac{P}{r} \right) [dr^2 + r^2 d\Omega_2^2]$$

$$= -dt^2 + G_{44}^{-1} [dr^2 + r^2 d\Omega_2^2], \quad (5)$$

where $d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ and P is the magnetic charge of the black hole. The solution here is written in terms of spherical polar coordinates r, θ, φ denoted as 1,2,3, respectively. Thus $F_{23} = P \sin \theta$. This is the KK monopole solution of GPS in four dimensions. Since the $U(1)$ gauge field in this case has the form $A_3 = P(1 - \cos \theta)$, the above solution can be written in terms of Taub-NUT metric in five dimensions as in Ref. [2]. Let us now try to understand how the above solution arises in string theory. The low energy effective action of any string theory in $D=5$ has the following form in common:

$$S_5^{(st)} = \int d^5x \sqrt{-G} e^{-2\Phi} \left[R + 4 \partial_M \Phi \partial^M \Phi - \frac{1}{12} H_{MNP} H^{MNP} \right]. \quad (6)$$

Here G_{MN} , Φ and B_{MN} are, respectively, the five-dimensional metric, dilaton, and Kalb-Ramond antisymmetric tensor field, with the field strength $H_{MNP} = \partial_M B_{NP} + \partial_N B_{PM} + \partial_P B_{MN}$. The rest of the fields which arise from the dimensional reduction are set to zero. If we now further set $H_{MNP} = 0$, then with the same KK ansatz (2) of the metric, we can write the reduced action in the form

$$S_4^{(st)} = \int d^4x \sqrt{-G} e^{-2\phi} \left[R + 4 \partial_\mu \phi \partial^\mu \phi - \partial_\mu \sigma \partial^\mu \sigma \right. \\ \left. - \frac{1}{4} e^{-2\sigma} F_{\mu\nu}^{(1)} F^{(1)\mu\nu} \right], \quad (7)$$

where the four-dimensional dilaton $\phi = \Phi + \frac{1}{2} \sigma$ and the scalar σ is given by $G_{44} = e^{-2\sigma}$. We have also renamed the gauge field A_μ as $A_\mu^{(1)}$ for later convenience. Thus we notice that the four-dimensional string action contains two scalars ϕ and σ instead of one, $\tilde{\phi}$, as in pure gravity case (3). The solution of the equations of motion following from Eq. (7) is also given in terms of two scalars as follows [25]:

$$ds^2 = -dt^2 + e^{2\phi + \sigma} [dr^2 + r^2 d\Omega_2^2],$$

$$e^{2\phi} = e^\sigma = \left(1 + \frac{P^{(1)}}{r} \right)^{1/2} \quad (8)$$

where we have renamed the KK monopole charge P as $P^{(1)}$. Note that the solution ds^2 is written in terms of string metric $G_{\mu\nu}$. In terms of canonical metric $\bar{G}_{\mu\nu} = e^{-2\phi} G_{\mu\nu}$, the solution reduces to

$$ds^2 = -e^{-2\phi} dt^2 + e^\sigma [dr^2 + r^2 d\Omega_2^2]. \quad (9)$$

We find that the solution (8) and (9) are precisely identical with Eqs. (5) and (4) since $\sigma = (1/\sqrt{3})\tilde{\phi}$. Note further that since $\phi = \frac{1}{2} \sigma$ is the solution, the five dimensional dilaton Φ is trivial (but the four-dimensional dilaton is not) as expected. Thus we note that although we started out with different actions namely, Eqs. (3) and (7), we end up with the same KK monopole solutions and this clarifies how KK monopole arises as a solution in string theory. We now discuss the H -monopole solution in string theory. Note from Eq. (6) that if instead of setting H_{MNP} to zero, we keep the component $B_{\mu 4} = A_\mu^{(2)}$, the reduced action would take the form with $A_\mu^{(1)} = 0$, as

$$\tilde{S}_4^{(st)} = \int d^4x \sqrt{-G} e^{-2\phi} \left[R + 4 \partial_\mu \phi \partial^\mu \phi \right. \\ \left. - \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{4} e^{2\sigma} F_{\mu\nu}^{(2)} F^{(2)\mu\nu} \right]. \quad (10)$$

The equations of motion following from Eq. (10) has the solution

$$ds^2 = -dt^2 + e^{2\phi - \sigma} [dr^2 + r^2 d\Omega_2^2],$$

$$e^{2\phi} = e^{-\sigma} = \left(1 + \frac{P^{(2)}}{r} \right)^{1/2}. \quad (11)$$

Here $P^{(2)}$ is the magnetic charge associated with the H monopole, i.e., $F_{23}^{(2)} = P^{(2)} \sin \theta$. In the canonical metric the solution can be written as

$$ds^2 = -e^{-2\phi} dt^2 + e^{-\sigma} [dr^2 + r^2 d\Omega_2^2]. \quad (12)$$

Note that since here $2\phi = -\sigma$ is the solution, the five-dimensional dilaton Φ is not trivial and therefore this solution is strictly a string theory solution which can not be obtained from pure gravity. This can also be understood since the gauge field $A_\mu^{(2)}$ originates in this case from the dimensional reduction of B_{MN} which is a field contained only in string theory spectrum. It should be mentioned here that the solution (11) can be obtained from Eq. (8) by the following transformations:

$$\begin{aligned} G_{\mu\nu} &\rightarrow G_{\mu\nu}, & \sigma &\rightarrow -\sigma, & \phi &\rightarrow \phi, \\ A_\mu^{(1)} &\rightarrow A_\mu^{(2)}, & P^{(1)} &\rightarrow P^{(2)}. \end{aligned} \quad (13)$$

Although neither the actions (7) nor (10) possess this symmetry, the full string theory action indeed has this symmetry, the T -duality symmetry [which is $O(1,1)$ symmetry in this case]. Thus the string theory admits both the KK- and H -monopole solutions which are related to each other by the T -duality transformations.

III. DYONIC SOLUTIONS

Starting from the monopole solutions discussed in the previous section, we, in this section, will construct the corresponding dyonic solutions having both magnetic and electric charges. In order to obtain these solutions we will use the $SL(2,R)$ symmetry of the four-dimensional string effective action. The relevant four dimensional string action containing both the gauge fields $A_\mu^{(1)}$ and $A_\mu^{(2)}$ as well as the antisymmetric tensor field $B_{\mu\nu}$ has the form

$$\begin{aligned} S^{(st)} = \int d^4x \sqrt{-G} e^{-2\phi} &\left[R + 4\partial_\mu \phi \partial^\mu \phi - \partial_\mu \sigma \partial^\mu \sigma \right. \\ &- \frac{1}{4} e^{-2\sigma} F_{\mu\nu}^{(1)} F^{(1)\mu\nu} - \frac{1}{4} e^{2\sigma} F_{\mu\nu}^{(2)} F^{(2)\mu\nu} \\ &\left. - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right]. \end{aligned} \quad (14)$$

Note that when both the gauge fields $A_\mu^{(1)}$ and $A_\mu^{(2)}$ are non-zero, the reduced form of the field strength $H_{\mu\nu\lambda}$ is given as,¹

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} - \frac{1}{2} (A_\mu^{(1)} F_{\nu\lambda}^{(2)} + A_\mu^{(2)} F_{\nu\lambda}^{(1)}) + \text{cyc. in } \mu\nu\lambda. \quad (15)$$

So, the action (14) is invariant under the T -duality transformation (13) along with

$$A_\mu^{(2)} \rightarrow A_\mu^{(1)}, \quad B_{\mu\nu} \rightarrow B_{\mu\nu}, \quad P^{(2)} \rightarrow P^{(1)}. \quad (16)$$

¹The antisymmetric tensor field $B_{\mu\nu}$ here is a modified form of the dimensionally reduced B_{MN} [26].

²Here the equations of motion are obtained after rewriting the action in the Einstein metric, where Einstein metric $\bar{G}_{\mu\nu} = e^{-2\phi} G_{\mu\nu}$.

Note that H^2 term is invariant under the transformations (13) and (16) by itself. The transformations (13) and (16) constitute the complete T -duality or $O(1,1)$ symmetry of the theory. When $H_{\mu\nu\lambda} = 0$ the equations of motion² following from Eq. (14) are as given below:

$$\nabla^2 \phi + \frac{1}{8} e^{-2\phi} [e^{-2\sigma} (F^{(1)})^2 + e^{2\sigma} (F^{(2)})^2] = 0, \quad (17)$$

$$\nabla^2 \sigma + \frac{1}{4} e^{-2\phi} [e^{-2\sigma} (F^{(1)})^2 - e^{2\sigma} (F^{(2)})^2] = 0, \quad (18)$$

$$\nabla_\mu (e^{-2\phi - 2\sigma} F^{(1)\mu\nu}) = 0, \quad (19)$$

$$\nabla_\mu (e^{-2\phi + 2\sigma} F^{(2)\mu\nu}) = 0, \quad (20)$$

$$\begin{aligned} R_{\mu\nu} = 2\partial_\mu \phi \partial_\nu \phi + \partial_\mu \sigma \partial_\nu \sigma + \frac{1}{2} e^{-2\phi} (e^{-2\sigma} F_{\mu\rho}^{(1)} F_{\nu}^{(1)\rho} \\ + e^{2\sigma} F_{\mu\rho}^{(2)} F_{\nu}^{(2)\rho}) - \frac{1}{8} \bar{G}_{\mu\nu} e^{-2\phi} [e^{-2\sigma} (F^{(1)})^2 \\ + e^{2\sigma} (F^{(2)})^2]. \end{aligned} \quad (21)$$

The equations of motion (17)–(21) can be easily solved by assuming the form of the metric to be static, spherically symmetric which becomes flat asymptotically. In fact, the supersymmetric BPS saturated solution of the above equations of motion has already been obtained by Cvetic and Youm [21] which has the following form:

$$\begin{aligned} ds^2 = -(f_1 f_2)^{-1/2} dt^2 + (f_1 f_2)^{1/2} [dr^2 + r^2 d\Omega_2^2], \\ e^{2\phi} = (f_1 f_2)^{1/2}, \quad e^{-\sigma} = \left(\frac{f_2}{f_1} \right)^{1/2}, \end{aligned} \quad (22)$$

where $f_i = (1 + P^{(i)}/r)$ with $i=1,2$. Here $P^{(i)}$'s are the magnetic charges associated with the gauge fields $A_\mu^{(i)}$. The solution is indeed invariant under the T -duality transformations (13) and (16). Note from Eq. (22) that $P^{(2)}=0$ corresponds to KK monopole solution (8) whereas $P^{(1)}=0$ corresponds to H -monopole solution (11) as discussed in the previous section.

We would like to point out that for $H_{\mu\nu\lambda} \neq 0$ the action (14) as well as the equations of motion following from it has a larger symmetry than what has already been noted in Eqs. (13) and (16). In fact, apart from the T -duality symmetry, it also has an S -duality symmetry and we will use this symmetry to obtain the dyonic solutions.

We first note that the action (14) can be written in an $O(1,1)$ invariant form [27] as follows:

$$\begin{aligned} S^{(st)} = \int d^4x \sqrt{-G} e^{-2\phi} &\left[R + 4\partial_\mu \phi \partial^\mu \phi + \frac{1}{8} \text{tr} \partial_\mu M \partial^\mu M^{-1} \right. \\ &\left. - \frac{1}{4} \mathcal{F}_{\mu\nu}^T M^{-1} \mathcal{F}^{\mu\nu} - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right], \end{aligned} \quad (23)$$

where

$$M = \begin{pmatrix} e^{2\sigma} & 0 \\ 0 & e^{-2\sigma} \end{pmatrix}$$

is an $O(1,1)$ matrix satisfying $M^T \eta M = \eta$, with

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\mathcal{F}_{\mu\nu} = \begin{pmatrix} F_{\mu\nu}^{(1)} \\ F_{\mu\nu}^{(2)} \end{pmatrix}.$$

The action (23) is invariant under a global $O(1,1)$ transformation

$$\begin{aligned} M &\rightarrow \Omega M \Omega^T, & \mathcal{F}_{\mu\nu} &\rightarrow \Omega \mathcal{F}_{\mu\nu}, & G_{\mu\nu} &\rightarrow G_{\mu\nu}, \\ \phi &\rightarrow \phi, & \text{and } B_{\mu\nu} &\rightarrow B_{\mu\nu}, \end{aligned} \quad (24)$$

where Ω is an $O(1,1)$ matrix. Note that in this particular case $\Omega = \eta$. By writing Eq. (23) in Einstein metric,

$$\begin{aligned} \bar{S}^{(st)} = \int d^4x \sqrt{-\bar{G}} &\left[R - 2\partial_\mu \phi \partial^\mu \phi + \frac{1}{8} \text{tr} \partial_\mu M \partial^\mu M^{-1} \right. \\ &\left. - \frac{1}{4} e^{-2\phi} \mathcal{F}_{\mu\nu}^T M^{-1} \mathcal{F}^{\mu\nu} - \frac{1}{12} e^{-4\phi} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right], \end{aligned} \quad (25)$$

we find that the equations of motion derived from Eq. (25) can also be obtained from an alternative action [28,17]

$$\begin{aligned} S^{(\text{alt})} = \int d^4x \sqrt{-\bar{G}} &\left[R - \frac{1}{2\lambda_2^2} \partial_\mu \lambda \partial^\mu \bar{\lambda} + \frac{1}{8} \text{tr} \partial_\mu M \partial^\mu M^{-1} \right. \\ &\left. - \frac{1}{4} \lambda_2 \mathcal{F}_{\mu\nu}^T M^{-1} \mathcal{F}^{\mu\nu} - \frac{1}{4} \lambda_1 \mathcal{F}_{\mu\nu}^T \eta \tilde{\mathcal{F}}^{\mu\nu} \right], \end{aligned} \quad (26)$$

where we have defined

$$H^{\mu\nu\lambda} = -\frac{1}{\sqrt{-\bar{G}}} e^{4\phi} \epsilon^{\mu\nu\lambda\rho} \partial_\rho a \quad (27)$$

with a , a pseudoscalar called an axion. Equation (27) in fact follows from the equation of motion of $H_{\mu\nu\lambda}$ in Eq. (25). Also, λ is a complex scalar defined as

$$\lambda = a + i e^{-2\phi} = \lambda_1 + i \lambda_2 \quad (28)$$

and $\tilde{\mathcal{F}}_{\mu\nu} = \frac{1}{2} \sqrt{-\bar{G}} \epsilon_{\mu\nu\lambda\rho} \mathcal{F}^{\lambda\rho}$. The equations of motion following from Eq. (26) are

$$\frac{1}{\lambda_2^2} \nabla_\mu \partial^\mu \lambda + \frac{i}{\lambda_2^3} \partial_\mu \lambda \partial^\mu \bar{\lambda} - \frac{1}{4} \mathcal{F}_{\mu\nu}^T \eta \tilde{\mathcal{F}}^{\mu\nu} - \frac{i}{4} \mathcal{F}_{\mu\nu}^T M^{-1} \mathcal{F}^{\mu\nu} = 0, \quad (29)$$

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{4\lambda_2^2} (\partial_\mu \lambda \partial_\nu \bar{\lambda} + \partial_\nu \lambda \partial_\mu \bar{\lambda}) + \frac{1}{2} \lambda_2 \mathcal{F}_{\mu\rho}^T M^{-1} \mathcal{F}_\nu^\rho \\ &\quad - \frac{1}{8} \lambda_2 \bar{G}_{\mu\nu} \mathcal{F}_{\rho\sigma}^T M^{-1} \mathcal{F}^{\rho\sigma} - \frac{1}{8} \text{tr} \partial_\mu M \partial_\nu M^{-1}, \end{aligned} \quad (30)$$

$$\nabla_\mu (\lambda_2 M^{-1} \mathcal{F}^{\mu\nu} + \lambda_1 \eta \tilde{\mathcal{F}}^{\mu\nu}) = 0 \quad (31)$$

and the σ equation is as given in Eq. (18). It can now be checked that the above equations of motion are invariant under the global $SL(2,R)$ transformations as given below [29,28,17]:

$$\lambda \rightarrow \frac{a\lambda + b}{c\lambda + d}, \quad \bar{G}_{\mu\nu} \rightarrow \bar{G}_{\mu\nu}, \quad M \rightarrow M,$$

$$\mathcal{F}_{\mu\nu} \rightarrow (c\lambda_1 + d) \mathcal{F}_{\mu\nu} - c\lambda_2 M \eta \tilde{\mathcal{F}}_{\mu\nu}, \quad (32)$$

where the global $SL(2,R)$ transformation matrix

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $ad - bc = 1$. However, we note that the action (26) is not $SL(2,R)$ invariant. Let us next rewrite the action in an $SL(2,R)$ invariant [16,30] fashion.³ In order to do this we define

$$\mathcal{M} = \begin{pmatrix} e^{-2\phi} + a^2 e^{2\phi} & a e^{2\phi} \\ a e^{2\phi} & e^{2\phi} \end{pmatrix} \quad (33)$$

and

$$\mathcal{H}_{\mu\nu} = \begin{pmatrix} \mathcal{F}_{\mu\nu} \\ \mathcal{G}_{\mu\nu} \end{pmatrix}, \quad (34)$$

where

$$\mathcal{G}_{\mu\nu} = \hat{\mathcal{F}}_{\mu\nu} - a \mathcal{F}_{\mu\nu}. \quad (35)$$

Here $\hat{\mathcal{F}}_{\mu\nu}$ is a pair of new fields we introduce which will be related to the known fields through a constraint as we will see below. The action (26) can now be written in a manifestly $SL(2,R)$ invariant form as follows:

$$\begin{aligned} S = \int d^4x \sqrt{-\bar{G}} &\left[R + \frac{1}{8} \text{tr} \partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1} \right. \\ &\left. + \frac{1}{4} \text{tr} \partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1} - \frac{1}{4} \mathcal{H}_{\mu\nu}^T \mathcal{M} \mathcal{M}^{-1} \mathcal{H}^{\mu\nu} \right], \end{aligned} \quad (36)$$

where the $SL(2,R)$ transformations are given as

$$\bar{G}_{\mu\nu} \rightarrow \bar{G}_{\mu\nu}, \quad M \rightarrow M,$$

³An $SL(2,R)$ invariant four dimensional string effective action has also been constructed in [31] using a different method.

$$\mathcal{M} \rightarrow \Lambda \mathcal{M} \Lambda^T, \quad \mathcal{H}_{\mu\nu} \rightarrow (\Lambda^T)^{-1} \mathcal{H}_{\mu\nu}, \quad (37)$$

where Λ is an $SL(2, R)$ matrix. In order to see the equivalence between Eqs. (26) and (36), we note that the Bianchi identities and the equations of motion of the field strengths and the gauge fields following from Eq. (36) are

$$\nabla_\mu \tilde{\mathcal{F}}^{\mu\nu} = 0, \quad (38)$$

$$\nabla_\mu \tilde{\mathcal{G}}^{\mu\nu} = 0 \Rightarrow \nabla_\mu (\tilde{\mathcal{F}}^{\mu\nu} - a \tilde{\mathcal{G}}^{\mu\nu}) = 0, \quad (39)$$

and

$$\nabla_\mu (e^{2\phi} M^{-1} \hat{\mathcal{F}}^{\mu\nu}) = 0, \quad (40)$$

$$\nabla_\mu (e^{-2\phi} M^{-1} \mathcal{F}^{\mu\nu} + a e^{2\phi} M^{-1} \hat{\mathcal{F}}^{\mu\nu}) = 0. \quad (41)$$

Now it can be checked that if we impose the constraint

$$\hat{\mathcal{F}}^{\mu\nu} = e^{-2\phi} M \eta \tilde{\mathcal{F}}^{\mu\nu} \quad (42)$$

then Eqs. (39) and (41) reduce to the equations of motion (31) derived from Eq. (26), whereas, Eq. (40) reduces to the Bianchi identity (38). All other equations of motion can also be shown to remain unaffected. Thus, we conclude that the action (26) is equivalent to action (36) subject to the constraint (42). Now the constraint (42) can also be written in an $SL(2, R)$ invariant form as

$$\Lambda = \begin{pmatrix} e^{-\phi_0} \cos \alpha + a_0 \sin \alpha e^{\phi_0} & -\sin \alpha e^{-\phi_0} + a_0 \cos \alpha e^{\phi_0} \\ e^{\phi_0} \sin \alpha & e^{\phi_0} \cos \alpha \end{pmatrix}. \quad (44)$$

Here α is an arbitrary parameter and will be fixed later. The transformation of the field strengths from Eq. (37) then is given as

$$\mathcal{F}_{\mu\nu} \rightarrow e^{\phi_0} \cos \alpha \mathcal{F}_{\mu\nu} - e^{\phi_0} \sin \alpha \mathcal{G}_{\mu\nu}, \quad (45)$$

$$\begin{aligned} \mathcal{G}_{\mu\nu} \rightarrow & (e^{-\phi_0} \sin \alpha - a_0 e^{\phi_0} \cos \alpha) \mathcal{F}_{\mu\nu} \\ & + (e^{-\phi_0} \cos \alpha + a_0 e^{\phi_0} \sin \alpha) \mathcal{G}_{\mu\nu}. \end{aligned} \quad (46)$$

Note that the initial configuration, i.e., the monopole solution (22) has only $\mathcal{F}_{\mu\nu}$ where $F_{23}^{(i)} = P^{(i)} \sin \theta$, for $i = 1, 2$. So, the magnetic charges $P^{(1)}, P^{(2)}$ are integers measured in some basic units. In other words, $P^{(1)} = mP$ and $P^{(2)} = nP$, where m, n are integers and P is the charge unit which is set to 1 from now on. Now as we make the $SL(2, R)$ transformations (45), (46), they will no longer remain integers. So, we modify $P^{(1)}$ and $P^{(2)}$ by $\Delta_1^{1/2}$ and $\Delta_2^{1/2}$ and demand that the charges remain integers after the transformation. Δ_1 and Δ_2 will be fixed soon. Thus from Eqs. (45) and (46) we obtain

$$\cos \alpha = e^{-\phi_0} \Delta_1^{-1/2} p_1 = e^{-\phi_0} \Delta_2^{-1/2} p_2, \quad (47)$$

$$\tilde{\mathcal{H}}_{\mu\nu} = M \eta \Sigma \mathcal{M} \mathcal{H}_{\mu\nu}, \quad (43)$$

where

$$\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is the $SL(2, R)$ metric satisfying $\Lambda \Sigma \Lambda^T = \Lambda^T \Sigma \Lambda = \Sigma$. We would like to point out that the action (36) contains an $SL(2, R)$ doublet of field strengths $\mathcal{F}_{\mu\nu}$ and $\mathcal{G}_{\mu\nu}$, which makes it manifestly $SL(2, R)$ invariant. On the other hand, action (26) does not contain the $SL(2, R)$ doublet of field strengths, but the equations of motion following from it contains the $SL(2, R)$ doublet $\mathcal{F}_{\mu\nu}$ and $e^{-2\phi} M \eta \tilde{\mathcal{F}}_{\mu\nu}$, which can be seen from (42). That is why the equations of motion of Eq. (26) are $SL(2, R)$ invariant even though the action itself is not.

The $SL(2, R)$ transformation matrix which takes the asymptotic value (the subscript 0 will always denote the asymptotic value) of the complex moduli $\lambda_0 = i$ (corresponding to $a_0 = \phi_0 = 0$) to an arbitrary value λ_0 has the form [32,33]

$$\sin \alpha = (q_1 + a_0 p_1) e^{\phi_0} \Delta_1^{-1/2} = (q_2 + a_0 p_2) e^{\phi_0} \Delta_2^{-1/2}, \quad (48)$$

where (p_i, q_i) are integers measuring the number of units of magnetic and ‘‘electric’’ charges of the dyonic solution. Note that the ‘‘electric’’ charge mentioned here is an auxiliary electric charge corresponding to $\mathcal{G}_{\mu\nu}$. The true electric charge of the theory is given later in Eq. (59) which is non-integral. Thus the dyonic solution is characterized by two pairs of integers corresponding to the magnetic and ‘‘electric’’ charges associated with $\mathcal{F}_{\mu\nu}$ and $\mathcal{G}_{\mu\nu}$. Equations (47) and (48) determines the value of Δ_1 and Δ_2 as

$$\Delta_i = e^{-2\phi_0} p_i^2 + e^{2\phi_0} (q_i + a_0 p_i)^2 = (p_i, q_i) \mathcal{M}_0 \begin{pmatrix} p_i \\ q_i \end{pmatrix}. \quad (49)$$

As is clear, i is not summed over in the right-hand side of Eq. (49). Since the charges transform as

$$\begin{pmatrix} p_i \\ q_i \end{pmatrix} \rightarrow (\Lambda^T)^{-1} \begin{pmatrix} p_i \\ q_i \end{pmatrix},$$

we find that Eq. (49) is $SL(2,Z)$ invariant. Note from Eq. (49) that in order to maintain the charge vector to be integer valued the $SL(2,R)$ transformation would have to be restricted to $SL(2,Z)$ i.e. integer valued, but then the asymptotic value of the dilaton (ϕ) and the axion (a) can not be maintained to a fixed value. We here construct the dyonic solution for a fixed but arbitrary asymptotic value of the background fields. The transformed value of the complex moduli and the field strengths are given as

$$\lambda' = \frac{a_0 \Delta_1 A + p_1 q_1 e^{-2\phi_0} (A-1) + i \Delta_1 A^{1/2} e^{-2\phi_0}}{p_1^2 e^{-2\phi_0} + A e^{2\phi_0} (a_0 p_1 + q_1)^2} \quad (50)$$

$$= \frac{a_0 \Delta_2 A + p_2 q_2 e^{-2\phi_0} (A-1) + i \Delta_2 A^{1/2} e^{-2\phi_0}}{p_2^2 e^{-2\phi_0} + A e^{2\phi_0} (a_0 p_2 + q_2)^2} \quad (51)$$

and

$$\mathcal{F}'_{23} = \begin{pmatrix} p_1 \sin \theta \\ p_2 \sin \theta \end{pmatrix}, \quad (52)$$

$$\mathcal{G}'_{23} = \begin{pmatrix} q_1 \sin \theta \\ q_2 \sin \theta \end{pmatrix}. \quad (53)$$

In Eqs. (50) and (51) the function A is defined as

$$A = \left[\left(1 + \frac{\Delta_1^{1/2}}{r} \right) \left(1 + \frac{\Delta_2^{1/2}}{r} \right) \right]^{-1/2}. \quad (54)$$

Note that asymptotically $A \rightarrow 1$ and therefore, from Eqs. (50) and (51) we have $\lambda \rightarrow \lambda_0$ as expected. Finally, the canonical metric for the dyonic solution is given as

$$ds^2 = - \left[\left(1 + \frac{\Delta_1^{1/2}}{r} \right) \left(1 + \frac{\Delta_2^{1/2}}{r} \right) \right]^{-1/2} dt^2 + \left[\left(1 + \frac{\Delta_1^{1/2}}{r} \right) \left(1 + \frac{\Delta_2^{1/2}}{r} \right) \right]^{1/2} [dr^2 + r^2 d\Omega_2^2] \quad (55)$$

and

$$e^{-\sigma} = \left[\frac{1 + \Delta_2^{1/2}/r}{1 + \Delta_1^{1/2}/r} \right]^{1/2}. \quad (56)$$

Thus starting from the magnetic monopole solution (22) we have obtained the dyonic solution carrying both magnetic and electric charges given by the field configurations (50)–(56). The magnetic charge $\Delta_2^{1/2} = 0$ corresponds to the KK dyonic solution whereas $\Delta_1^{1/2} = 0$ corresponds to the H dyonic solution. The BPS saturated ADM mass of the dyonic solution can be easily calculated [34] from Eq. (55) which is given by

$$M^2 = \frac{1}{16} (\Delta_1^{1/2} + \Delta_2^{1/2})^2. \quad (57)$$

We note from above that the mass formula in Eq. (57) can be written in a manifestly $SL(2,Z)$ and $O(1,1,Z)$ invariant form as [17,22]

$$M^2 = \frac{1}{16} (p_i, q_i) \mathcal{M}_0 (M_0 + \eta)_{ij} \begin{pmatrix} p_j \\ q_j \end{pmatrix}. \quad (58)$$

M_0 in Eq. (58) is actually an identity matrix since $e^\sigma \rightarrow 1$ asymptotically as can be seen from Eq. (56). This is the Schwarz-Sen mass formula in this case. The electric, magnetic charge spectrum for the dyonic solutions can also be obtained from Eqs. (35) and (42) as

$$(\mathcal{Q}_i^{(\text{mag})}, \mathcal{Q}_i^{(el)}) = [p_i, e^{2\phi_0} (M_0 \eta)_{ij} (q_j + a_0 p_j)]. \quad (59)$$

This charge spectrum has been obtained by Schwarz and Sen in Ref. [22] and they satisfy the Dirac-Schwinger-Zwanziger-Witten quantization rule [35].

Now in order to understand the stability [36,17] of the dyonic solution, we first note from Eq. (58) that the masses of such solution are characterized by two pairs of integers (p_1, q_1, p_2, q_2) and they satisfy as usual for a BPS state the triangle inequality of the form

$$M_{(p_1, q_1, p_2, q_2)} + M_{(p'_1, q'_1, p'_2, q'_2)} \geq M_{(p_1 + p'_1, q_1 + q'_1, p_2 + p'_2, q_2 + q'_2)}. \quad (60)$$

The equality holds when

$$(p_1 + p_2)(q'_1 + q'_2) = (p'_1 + p'_2)(q_1 + q_2). \quad (61)$$

Hence we notice that when sum of the magnetic charges $(p_1 + p_2)$ is relatively prime to the sum of the electric charges $(q_1 + q_2)$, the dyonic state will be stable since it will be prevented from decaying into lower mass state by the inequality (60).

Finally, we note from Eq. (58) that unlike the dyonic solution considered in Ref. [21], the solution considered here has zero area of the event horizon, i.e., zero entropy but finite Hawking temperature

$$T_H = \frac{1}{4\pi \sqrt{\Delta_1^{1/2} \Delta_2^{1/2}}}. \quad (62)$$

IV. CONCLUSION

To summarize, we have briefly discussed the KK magnetic monopole solution in five-dimensional pure Einstein gravity with one of the spatial dimensions compactified and then showed how this solution arises in string theory. We have also discussed another kind of monopole solution known as the H -monopole solution in string theory. These two solutions are related to each other by a T -duality transformation in string theory. Next, we considered the magnetic monopole solution when both KK gauge field and the gauge field originating from the dimensional reduction of Kalb-Ramond antisymmetric tensor field are present. Then by including the H^2 term we have shown that the full string

theory effective action can be expressed in a manifestly $SL(2,R)$ invariant form with an $SL(2,R)$ invariant constraint. By using this symmetry we have explicitly constructed the corresponding dyonic solution. We have obtained the BPS saturated ADM mass formula and the Schwarz-Sen electric-magnetic charge spectrum for this solution. The dyonic solution is characterized by two pairs of integers. By using the mass formula we have shown that when sum of the magnetic charges is relatively prime to the sum of the “electric” charges the dyonic solution is stable. The stability can be understood from a triangle inequality

relation satisfied by the masses of the dyonic states. We have mentioned that unlike the dyonic solution considered in Ref. [21], the solution we described has zero entropy, but finite Hawking temperature.

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