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Regular and irregular boundary conditions in the AdS-CFT correspondence

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We expand on the recent proposal of Klebanov and Witten for formulating the AdS-CFT correspondence using irregular boundary conditions. The proposal is shown to be correct to any order in perturbation theory. [S0556-2821(99)50218-X]

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I. INTRODUCTION

The celebrated AdS conformal field theory (CFT) correspondence relates field theories on anti–de Sitter (AdS) space with CFTs living on the AdS horizon. The main prediction of this duality is that CFT correlation functions of conformal operators can be calculated by evaluating the AdS action on shell as a functional of prescribed boundary values.

For example, using a scalar field theory on AdS space, CFT correlators of conformal fields of scaling dimensions $\Delta \geqslant d/2$ have been calculated ([1–6] and references therein) Until recently, no prescription was known to include operators with scaling dimension Δ, $d/2-1<\Delta<d/2$. Here, $d/2-1$ is the unitary bound on the conformal dimension of scalar operators. Recently, Klebanov and Witten [7] proposed a method to do just that. They used the fact that a scalar field on AdS space can obey two types of boundary conditions [8]. The regular one, which can always be imposed, leads to the CFT correlators with $\Delta \geqslant d/2$, whereas the irregular one would lead to $d/2-1<\Delta \leqslant d/2$. Group theoretical results [9] indicate that the respective boundary fields are conjugate to each other. Klebanov and Witten proposed to use a Legendre transform of the action, expressed as a functional of the irregular boundary value, as the generating functional. They also demonstrated the correctness of this proposal for CFT two point functions.

In this article, we would like to expand on their proposal and demonstrate its correctness to all orders in perturbation theory. A second result of our analysis is that a different Green's function must be used for internal lines in second or higher order graphs.

The outline of the article shall be as follows. In the remainder of this section motivating arguments about the origin of the irregular boundary conditions will be given. In Sec. II we will for completeness repeat the formalism using regular boundary conditions. Then, in Sec. III Klebanov and Witten's proposal to include irregular boundary conditions shall be analyzed and shown to be correct to any order in perturbation theory.

To start, consider an interacting scalar field, whose action is given by

$$I=\frac{1}{2}\int_{\Omega}\mathbf{d}x\,(D_{\mu}\phi D^{\mu}\phi+m^{2}\phi^{2})+I_{int}, \tag{1}$$

where I_{int} denotes the interaction terms and $\mathbf{d}x=d^{d+1}x\sqrt{g(x)}$ is the invariant volume integral measure. The equation of motion following from the action (1) is given by

$$(D_{\mu}D^{\mu}-m^{2})\phi(x)=B(x), \tag{2}$$

where[1]

$$B(x)=\frac{\delta I_{int}}{\delta\phi(x)}.$$

Using as AdS representation the conventional upper half space $\mathbf{x}\in\mathbb{R}^{d}$, $x_0>0$ with the metric

$$ds^{2}=(x_{0})^{-2}dx^{\mu}dx^{\mu}, \tag{3}$$

the solution to Eq. (2) can be written in the form

$$\phi(x)=\int d^{d}y\left[\frac{x_{0}}{(x-\mathbf{y})^{2}}\right]^{(d/2)\pm\alpha}f_{\mp}(\mathbf{y})+\int_{\Omega}\mathbf{d}y\,G(x,y)B(y), \tag{4}$$

where $\alpha=\sqrt{d^{2}/4+m^{2}}$ and $G(x,y)$ is a standard Green's function satisfying

$$(D_{\mu}D^{\mu}-m^{2})G(x,y)=\frac{\delta(x-y)}{\sqrt{g(x)}}. \tag{5}$$

The free field solution with the lower sign exists classically for $\alpha<d/2$, but the unitary bound restricts it further to $\alpha<1$ [8]. The functions f_{-} and f_{+} are called regular and irregular boundary values and are conformal fields of scaling dimensions $d/2-\alpha$ and $d/2+\alpha$, respectively.

In the AdS-CFT correspondence the fields obeying regular boundary conditions give rise to CFT correlation functions of operators with conformal dimensions Δ restricted by $\Delta\geqslant d/2$. Hence, the use of irregular boundary conditions enables one to obtain correlation functions for operators with scaling dimensions $d/2-1<\Delta<d/2$.

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[1]The functional variation is done covariantly, cf. [10].

II. REGULAR BOUNDARY CONDITIONS

Let us start by rewriting the expression (4) as

$$\phi(x)=\phi^{(0)}(x)+\int_\Omega \mathbf{d}x\, G(x,y)B(y), \tag{6}$$

where the Green's function $G(x)$ is given by [11,3]

$$G(x,y)=-(x_0y_0)^{d/2}\int \frac{d^dk}{(2\pi)^d}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \times\begin{cases} I_\alpha(kx_0)K_\alpha(ky_0) & \text{for } x_0<y_0,\\ I_\alpha(ky_0)K_\alpha(kx_0) & \text{for } x_0>y_0,\end{cases} \tag{7}$$

$$=-\frac{c_\alpha}{2}\xi^{-[(d/2)+\alpha]}F(d/2,d/2+\alpha;1+\alpha;\xi^{-2}), \tag{8}$$

where F is the hypergeometric function,

$$\xi=\frac{1}{2x_0y_0}\left\{\frac{1}{2}[(x-y)^2+(x-y^*)^2]+\sqrt{(x-y)^2(x-y^*)^2}\right\}$$

[y^* denotes the vector $(-y_0,\mathbf{y})$], and

$$c_\alpha=\frac{\Gamma(d/2+\alpha)}{\pi^{d/2}\Gamma(1+\alpha)}. \tag{9}$$

Moreover, the free field solution $\phi^{(0)}$ shall be written as

$$\phi^{(0)}(x)=\int d^dy\,\mathcal{K}_\alpha(x,\mathbf{y})\phi_-^{(0)}(\mathbf{y}) =\int d^dy\,\mathcal{K}_{-\alpha}(x,\mathbf{y})\phi_+^{(0)}(\mathbf{y}). \tag{10}$$

The bulk-boundary propagators occurring in Eq. (10) are given by

$$\mathcal{K}_{\pm\alpha}(x,\mathbf{y})=\pm\alpha c_{\pm\alpha}\left[\frac{x_0}{(x-\mathbf{y})^2}\right]^{(d/2)\pm\alpha}, \tag{11}$$

where $c_{\pm\alpha}$ is given by Eq. (9), and their Fourier transforms read

$$\mathcal{K}_{\pm\alpha}(x,\mathbf{k})=\frac{\pm 2\alpha}{\Gamma(1\pm\alpha)}e^{i\mathbf{k}\cdot\mathbf{x}}\left(\frac{k}{2}\right)^{\pm\alpha}x_0^{d/2}K_\alpha(kx_0). \tag{12}$$

Equations (12) and (10) imply that the boundary functions $\phi_+^{(0)}$ and $\phi_-^{(0)}$ are related by

$$\phi_+^{(0)}(\mathbf{k})=-\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\left(\frac{k}{2}\right)^{2\alpha}\phi_-^{(0)}(\mathbf{k}). \tag{13}$$

Obviously, the free field $\phi^{(0)}$ can be written as a sum of two series, whose leading powers are $x_0^{d/2-\alpha}$ and $x_0^{d/2+\alpha}$, respectively. Thus, one finds by direct comparison with Eqs. (10) and (12) that the small x_0 behavior of $\phi^{(0)}$ is

$$\phi^{(0)}(x)\overset{x_0\to 0}{\approx} x_0^{(d/2)-\alpha}\phi_-^{(0)}(\mathbf{x})+x_0^{(d/2)+\alpha}\phi_+^{(0)}(\mathbf{x}), \tag{14}$$

where subleading terms have been dropped. Moreover, the Green's function (8) goes like

$$G(x,y)\overset{x_0\to 0}{\approx}-\frac{1}{2\alpha}x_0^{(d/2)+\alpha}\mathcal{K}_\alpha(\mathbf{x},y). \tag{15}$$

Hence, the interaction contributes only to the ϕ_+ part of the asymptotic boundary behavior, i.e., one can write

$$\phi(x)\overset{x_0\to 0}{\approx} x_0^{(d/2)-\alpha}\phi_-(\mathbf{x})+x_0^{(d/2)+\alpha}\phi_+(\mathbf{x}), \tag{16}$$

where

$$\phi_-(\mathbf{x})=\phi_-^{(0)}(\mathbf{x}), \tag{17}$$

$$\phi_+(\mathbf{x})=\phi_+^{(0)}(\mathbf{x})-\frac{1}{2\alpha}\int_\Omega \mathbf{d}y\,\mathcal{K}_\alpha(\mathbf{x},y)B(y). \tag{18}$$

Identical relations hold for the Fourier transformed expressions.

Now consider the on-shell action, treated as a functional of the regular boundary values ϕ_-. Integrating Eq. (1) by parts yields

$$I=\frac{1}{2}\int d^dx\,x_0^{-d}n^\mu\phi\partial_\mu\phi-\frac{1}{2}\int_\Omega \mathbf{d}x\,\phi(x)B(x)+I_{int}.$$

The first term must be regularized, which is done by writing

$$x_0^{-d}n^\mu\phi\partial_\mu\phi=-x_0^{-d}\phi\left[\left(\frac{d}{2}-\alpha\right)x_0^{(d/2)-\alpha}\phi_- +\left(\frac{d}{2}+\alpha\right)x_0^{(d/2)+\alpha}\phi_+ +\cdots\right] =-x_0^{-d}\left(\frac{d}{2}-\alpha\right)\phi^2-2\alpha\phi_-\phi_+ +\cdots,$$

where the ellipses indicate contributions from subleading terms and other terms which vanish for $x_0=0$. The first term in the last line is cancelled by a covariant counterterm. Hence, the renormalized on-shell action is

$$I[\phi_-]=-\alpha\int\frac{d^dk}{(2\pi)^d}\phi_-(\mathbf{k})\phi_+(-\mathbf{k}) -\frac{1}{2}\int_\Omega \mathbf{d}x\,\phi(x)B(x)+I_{int} =I^{(0)}[\phi_-] -\frac{1}{2}\int_\Omega \mathbf{d}x\,\mathbf{d}y\,B(x)G(x,y)B(y)+I_{int}, \tag{19}$$

where Eqs. (18), (17), (13), (6), and (10) have been used. The term $I^{(0)}$ in Eq. (19) is given by

$$I^{(0)}[\phi_-]=\alpha\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\int\frac{d^dk}{(2\pi)^d}\left(\frac{k}{2}\right)^{2\alpha}\phi_-(\mathbf{k})\phi_-(-\mathbf{k})$$
$$=-\alpha^2c_\alpha\int d^dx\,d^dy\,\frac{\phi_-(\mathbf{x})\phi_-(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2\alpha}} \tag{20}$$

and thus yields the correct two point function of scalar operators of conformal dimension $\Delta=d/2+\alpha$, if one uses the AdS-CFT correspondence formula

$$\mathrm{e}^{-I[\phi_-]}=\left\langle\exp\left[\alpha\int d^dx\,\mathcal{O}(\mathbf{x})\phi_-(\mathbf{x})\right]\right\rangle. \tag{21}$$

The other two terms have to be expressed as a perturbative series in terms of $\phi^{(0)}$. However, by virtue of Eqs. (10) and (17) this naturally yields a perturbative series in terms of the boundary function ϕ_-.

III. IRREGULAR BOUNDARY CONDITIONS

The treatment of irregular boundary conditions follows an idea by Klebanov and Witten [7]. Consider the expression

$$\frac{\delta I[\phi_-]}{\delta\phi_-(\mathbf{k})}=2\alpha\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\left(\frac{k}{2}\right)^{2\alpha}\phi_-(-\mathbf{k})+\int_\Omega\mathbf{d}x\,B(x)\frac{\delta\phi(x)}{\delta\phi_-(\mathbf{k})}-\int_\Omega\mathbf{d}x\,\mathbf{d}y\,\mathbf{d}z\,\frac{\delta^2I_{int}}{\delta\phi(x)\delta\phi(z)}G(x,y)B(y)\frac{\delta\phi(z)}{\delta\phi_-(\mathbf{k})}.$$

Using Eq. (13) and the formula

$$\frac{\delta\phi(x)}{\delta\phi_-(\mathbf{k})}=\mathcal{K}_\alpha(x,-\mathbf{k})+\int_\Omega\mathbf{d}y\,\mathbf{d}z\,G(x,y)\frac{\delta^2I_{int}}{\delta\phi(y)\delta\phi(z)}\frac{\delta\phi(z)}{\delta\phi_-(\mathbf{k})},$$

one finds

$$\frac{\delta I[\phi_-]}{\delta\phi_-(\mathbf{k})}=-2\alpha\phi_+^{(0)}(-\mathbf{k})+\int_\Omega\mathbf{d}x\,\mathcal{K}_\alpha(x,-\mathbf{k})B(x)=-2\alpha\phi_+(-\mathbf{k}),$$

or, after an inverse Fourier transformation,

$$\frac{\delta I[\phi_-]}{\delta\phi_-(\mathbf{x})}=-2\alpha\phi_+(\mathbf{x}). \tag{22}$$

This expression holds to any order in perturbation theory. This fact was obtained in [7] using graph arguments. Furthermore, it shows first that ϕ_+ can be regarded as the conjugate field of ϕ_- and secondly that the functional

$$J[\phi_-,\phi_+]=I[\phi_-]+2\alpha\int d^dx\,\phi_-(\mathbf{x})\phi_+(\mathbf{x}) \tag{23}$$

has a minimum with respect to a variation of ϕ_-.

Klebanov and Witten's idea [7] is to formulate the AdS-CFT correspondence by the formula

$$\mathrm{e}^{-J[\phi_+]}=\left\langle\exp\left[\alpha\int d^dx\,\mathcal{O}(\mathbf{x})\phi_+(\mathbf{x})\right]\right\rangle. \tag{24}$$

Here, the functional $J[\phi_+]$ is a Legendre transform of the action I, i.e., it is the minimum value of the expression (23), expressed in terms of ϕ_+.

In the following, Klebanov and Witten's result about the correctness of the two point function [7] shall be confirmed and interactions included. The minimum of J is easiest found from Eqs. (19) and (23), giving

$$J[\phi_+]=\alpha\int\frac{d^dk}{(2\pi)^d}\phi_-(\mathbf{k})\phi_+(-\mathbf{k})-\frac{1}{2}\int_\Omega\mathbf{d}x\,\phi(x)B(x)+I_{int}$$
$$=-\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\int\frac{d^dk}{(2\pi)^d}\left(\frac{k}{2}\right)^{-2\alpha}\phi_+(\mathbf{k})\phi_+(-\mathbf{k})-\frac{1}{2}\int_\Omega\mathbf{d}x\,\phi(x)B(x)+I_{int}+\frac{1}{2}\int d^dx\int_\Omega\mathbf{d}y\,\mathcal{K}_{-\alpha}(y,\mathbf{x})\phi_+(\mathbf{x})B(y)$$
$$=-\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\int\frac{d^dk}{(2\pi)^d}\left(\frac{k}{2}\right)^{-2\alpha}\phi_+(\mathbf{k})\phi_+(-\mathbf{k})-\frac{1}{2}\int_\Omega\mathbf{d}x\,\mathbf{d}y\;\;B(x)G(x,y)B(y)+I_{int}$$
$$-\frac{1}{4\alpha}\int d^dz\int_\Omega\mathbf{d}x\,\mathbf{d}y\,\mathcal{K}_\alpha(x,\mathbf{z})\mathcal{K}_{-\alpha}(y,\mathbf{z})B(x)B(y). \tag{25}$$

Here Eqs. (17), (13), (18), (12), and (6) have been used. The first term in Eq. (25) can be inversely Fourier transformed, which yields

$$J^{(0)} = -\alpha^2 c_{-\alpha} \int d^d x\, d^d y \frac{\phi_+(\mathbf{x})\phi_+(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d-2\alpha}}. \tag{26}$$

According to the correspondence formula (24), this yields the correct two point function of conformal operators $\mathcal{O}$ of scaling dimension $\Delta = d/2 - \alpha$.

Then, the second and fourth term in Eq. (25) can be combined by defining the Green's function

$$\tilde{G}(x,y) = G(x,y) + \frac{1}{2\alpha}\int d^d z\, \mathcal{K}_\alpha(x,\mathbf{z})\mathcal{K}_{-\alpha}(y,\mathbf{z}). \tag{27}$$

This modified Green's function $\tilde{G}$ also satisfies Eq. (5), because the second term in Eq. (27) does not contribute to the discontinuity. Moreover, using Eqs. (7) and (12) one finds

$$\tilde{G}(x,y) = -(x_0 y_0)^{d/2} \int \frac{d^d k}{(2\pi)^d} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \left[\frac{2K_\alpha(kx_0)K_\alpha(ky_0)}{\Gamma(\alpha)\Gamma(1-\alpha)} + \begin{cases} K_\alpha(ky_0)I_\alpha(kx_0) & \text{for } x_0<y_0, \\ K_\alpha(kx_0)I_\alpha(ky_0) & \text{for } x_0>y_0, \end{cases} \right]$$

$$= -(x_0 y_0)^{d/2} \int \frac{d^d k}{(2\pi)^d} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \begin{cases} K_\alpha(ky_0)I_{-\alpha}(kx_0) & \text{for } x_0<y_0, \\ K_\alpha(kx_0)I_{-\alpha}(ky_0) & \text{for } x_0>y_0, \end{cases}$$

which differs from Eq. (7) only by interchanging α and $-\alpha$. Hence, the result (8) can be taken over, yielding

$$\tilde{G}(x,y) = -\frac{c_{-\alpha}}{2}\xi^{-[(d/2)-\alpha]} F(d/2, d/2-\alpha; 1-\alpha; \xi^{-2}). \tag{28}$$

Thus, inserting Eq. (27) into Eq. (25) yields

$$J[\phi_+] = J^{(0)}[\phi_+] - \frac{1}{2}\int_\Omega \mathbf{d}x\, \mathbf{d}y\, B(x)\tilde{G}(x,y)B(y) + I_{int}. \tag{29}$$

Moreover, one can see from Eq. (28) that for small x_0 $\tilde{G}$ behaves as

$$\tilde{G}(x,y) \overset{x_0\to 0}{\approx} \frac{1}{2\alpha} x_0^{(d/2)-\alpha} \mathcal{K}_{-\alpha}(\mathbf{x},y). \tag{30}$$

Hence, writing

$$\phi(x) = \int d^d y\, \mathcal{K}_{-\alpha}(x,\mathbf{y})\phi_+(\mathbf{y}) + \int_\Omega \mathbf{d}y\, \tilde{G}(x,y)B(y), \tag{31}$$

the interaction contributes only to ϕ_-. This in turn means that, expressing I_{int} and B as a perturbative series and using Eq. (31), the functional J is naturally expressed in terms of the irregular boundary value ϕ_+. Moreover, it has the expected form, in that it is obtained from Eq. (19) by replacing α with $-\alpha$ and ϕ_- with ϕ_+. An important point is that the Green's function $\tilde{G}$ must be used for the calculation of internal lines.

Finally, by a calculation similar to that of the derivation of Eq. (22) one finds

$$\frac{\delta J[\phi_+]}{\delta\phi_+(\mathbf{x})} = 2\alpha\phi_-(\mathbf{x}). \tag{32}$$

This is a final confirmation of the fact that the fields ϕ_- and ϕ_+ are conjugate to each other.

In conclusion, we have expanded on Klebanov and Witten's recent idea for formulating the AdS-CFT correspondence using irregular boundary conditions, showing it to give the expected answers to any order in perturbation theory.

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