Small eigenvalues of the staggered Dirac operator in the adjoint representation and random matrix theory

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The low-lying spectrum of the Dirac operator is predicted to be universal, within three classes, depending on symmetry properties specified according to random matrix theory. The three universal classes are the orthogonal, unitary and symplectic ensembles. Lattice gauge theory with staggered fermions has verified two of the cases so far, unitary and symplectic, with staggered fermions in the fundamental representation of SU(3) and SU(2). We verify the missing case here, namely orthogonal, with staggered fermions in the adjoint representation of SU(N_c), $N_c=2,3$. [S0556-2821(99)05715-X]

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Random matrix theory (RMT) has been successful in predicting spectral properties of QCD-like theories in the so-called microscopic scaling regime, defined by $1/\Lambda_{QCD} \ll L \ll 1/m_\pi$, with L the linear extent of the system [1,2]. Assuming spontaneous chiral symmetry breaking in the underlying theory, this is the regime dominated by the soft pions associated with the chiral symmetry breaking [3]. Up to a scale, given by the chiral condensate (at infinite volume) $\Sigma=\langle\bar{\psi}\psi\rangle$, the distribution of the low lying eigenvalues, the so-called microscopic spectral density

$$\rho_S(z)=\lim_{V\to\infty}\frac{1}{V}\rho\left(\frac{z}{V\Sigma}\right) \tag{1}$$

with $\rho(\lambda)$ the usual (macroscopic) spectral density and, in particular, the rescaled distribution of the lowest eigenvalue $P_{\rm min}(z)$ with $z=V\Sigma\lambda_{\rm min}$ are universal, dependent only on the symmetry properties, the number of dynamical quark flavors and the number of exact zero modes, i.e., the topological sector, but not the potential in RMT [4]. Recently, these properties have been derived directly from the effective, finite-volume partition functions of QCD of Leutwyler and Smilga [3], without the detour through RMT [5].

The lattice fermion action should have a chiral symmetry for the predictions of random matrix theory to apply. Until recently, staggered fermions were the only fermions regularized on the lattice that retained a chiral symmetry.[1] The RMT predictions have been nicely verified for staggered fermions in the fundamental representations of SU(2) [8] and SU(3) [9] gauge groups in quenched QCD, and for SU(2) also with dynamical fermions [10]. These represent two out of the three different cases predicted by RMT, chiral symplectic and chiral unitary. Staggered fermions are not always in the same universality class as continuum fermions in the context of random matrix theory. Staggered and continuum fermions belong to the unitary ensemble when the fermions are in the fundamental representation of SU(N_c), $N_c\geq 3$. But staggered fermions in the fundamental representation of SU(2) belong to the symplectic ensemble since their entire spectrum is twofold degenerate [11] in contrast to continuum fermions which belong to the orthogonal ensemble [2]. Another example where the staggered fermions and continuum fermions are not in the same universality class is when the fermions are in the adjoint representation of SU(N_c). Here the situation is just reversed: continuum fermions are in the symplectic ensemble [2] with a twofold degeneracy of the entire spectrum[2] while staggered fermions are in the orthogonal ensemble — they are real, since adjoint gauge fields are real and since the ''Dirac matrices'' for staggered fermions are just phases, ± 1. We therefore use this example with

TABLE I. The chiral condensate, Σ, from fits of the distribution of the lowest eigenvalue to the RMT predictions. The last column gives the confidence level of the fit.

Group	β	L	N	Σ	Q
SU(2)	1.8	4	5000	1.796(18)	0.921
SU(2)	2.0	4	5000	1.676(17)	0.640
SU(2)	2.0	6	2000	1.722(30)	0.496
SU(2)	2.2	4	5000	1.489(16)	0.840
SU(2)	2.2	6	2000	1.529(28)	0.344
SU(2)	2.4	6	2000	1.212(21)	0.852
SU(2)	2.4	8	1000	1.264(30)	0.874
SU(3)	5.1	4	2500	4.724(76)	0.786
SU(3)	5.1	6	1500	4.671(91)	0.035

[1] A new lattice regularization of massless fermions with good chiral properties, and even an index theorem, has recently been developed [6]. The RMT predictions for these overlap fermions have been verified for examples in all three ensembles, including the classification into different topological sectors, in [7].

[2] To compare with RMT predictions only one of each degenerate pair of eigenvalues is kept and the adjoint fermion is thereby considered as a Majorana fermion.

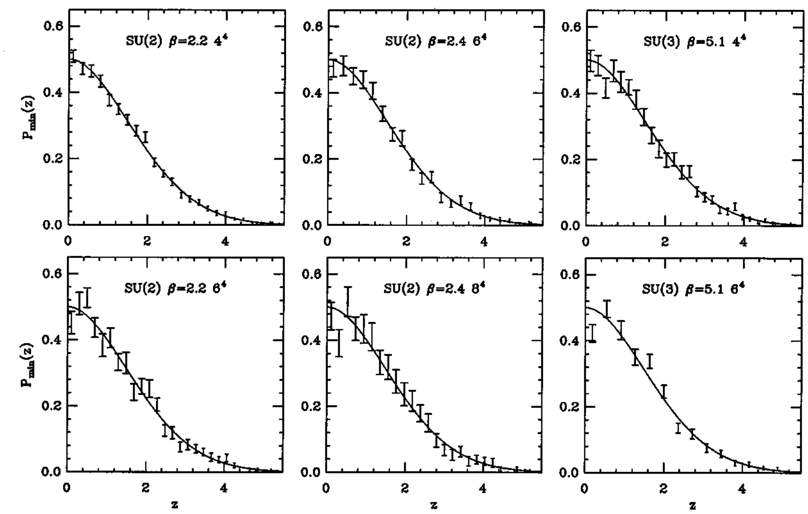

FIG. 1. The distribution of the rescaled lowest eigenvalue, $P_{\mathrm{min}}(z)$, together with the RMT prediction using the best estimate for the chiral condensate, which was found to be independent of the lattice size at fixed coupling.

staggered fermions in the adjoint representation of $\mathrm{SU}(N_c)$ to test the missing case.

In this Brief Report, we consider staggered fermions in the adjoint representation of SU(2) and SU(3) in the quenched approximation. Since staggered fermions (at finite lattice spacing) do not obey an index theorem and thus have no exact zero modes [12], the RMT predictions for $\nu=0$ are expected to apply. This has been found in the studies with staggered fermions in the fundamental representation [8–10], and our results will support it for staggered fermions in the adjoint representation. The rescaled distribution of the lowest eigenvalue, $z=V\Sigma\lambda_{\mathrm{min}}$, should thus be [13]

$$P_{\mathrm{min}}(z)=\frac{2+z}{4}e^{-z/2-z^2/8}. \qquad (2)$$

This distribution is quite distinct from all other cases: it starts at a finite, non-zero value at $z=0$. Given the chiral condensate, Σ, Eq. (2) is a parameter free prediction. If Σ is not otherwise known, it can be determined from a one-parameter fit of the distribution of the lowest eigenvalue to Eq. (2).

We have computed the low lying eigenvalues of the staggered Dirac operator in the adjoint representation with the Ritz functional method [14], applied to $D^\dagger D=-D^2$, for several SU(2) gauge field ensembles and for two SU(3) ensembles. The distribution of the lowest eigenvalue, approximated by a histogram with jackknife errors, was fit to the predicted form, Eq. (2), with Σ, the infinite volume value of $\langle\bar{\psi}\psi\rangle$, as the only free parameter. The results of the fit, together with the number of configurations considered, gauge coupling and lattice size, are given in Table I. In all cases we obtained good fits to the predicted form. For most gauge couplings we considered two different lattice sizes. The values for Σ obtained from the two different lattice sizes agree well, within statistical errors. Some distributions, together with the fitted analytical predictions, are shown in Fig. 1.

The consistency of the extracted value for the chiral condensate, Σ, from the two different lattice sizes makes it evident that we could have used the value obtained from one lattice size to get a parameter free description of the results from the other lattice size. Given the scale Σ the microscopic spectral density ρ_S, Eq. (1), is predicted by results of RMT [15]. The comparison for the same systems as in Fig. 1 is

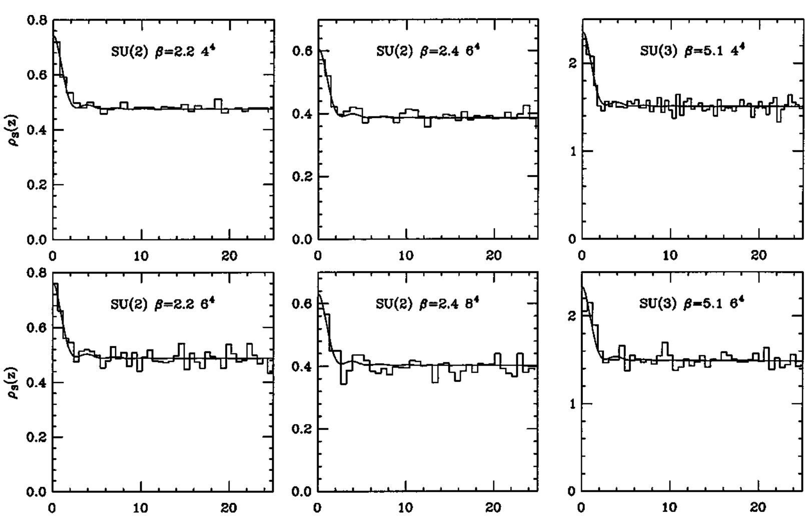

FIG. 2. The microscopic spectral density, ρ_S, together with the RMT prediction [15] using the best estimate for the chiral condensate, which was found to be independent of the lattice size at fixed coupling.

shown in Fig. 2. The agreement is quite nice, extending over the entire region covered by the eigenvalues that we determined. Note that for the chiral orthogonal ensemble the oscillations in the RMT prediction for ρ_S, except the first one coming from the lowest eigenvalue, are very small. Obviously, we would need much more statistics to resolve additional ''wiggles'' in our data.

In conclusion, adjoint staggered fermions are argued to belong to the chiral orthogonal ensemble of RMT. We found that the RMT predictions indeed describe the spectrum of low lying eigenvalues of the staggered Dirac operator in the adjoint representation very well for both SU(2) and SU(3) in our quenched simulations.

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