Relation Tr $\gamma_5 = 0$ and the index theorem in lattice gauge theory

Kazuo Fujikawa
Department of Physics, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan
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The relation Tr $\gamma_5 = 0$ implies the contribution to the trace from unphysical (would-be) species doublers in lattice gauge theory. This statement is also true for Pauli-Villars regularization in continuum theory. If one insists on Tr $\gamma_5 = 0$, one thus inevitably includes unphysical states in Hilbert space. If one truncates the trace to the contribution from physical species only, one obtains $\widetilde{\mathrm{Tr}}\, \gamma_5 = n_+ - n_-$ which is equal to the Pontryagin index. A smooth continuum limit of $\widetilde{\mathrm{Tr}}\, \gamma_5 = \mathrm{Tr}\, \gamma_5[1-(a/2)D] = n_+ - n_-$ for the Dirac operator D satisfying the Ginsparg-Wilson relation leads to a natural treatment of a chiral anomaly in the continuum path integral. In contrast, the continuum limit of Tr $\gamma_5 = 0$ is not defined consistently. It is shown that the nondecoupling of heavy fermions in the anomaly calculation is crucial to understand the consistency of the customary lattice calculation of the anomaly where Tr $\gamma_5 = 0$ is used. We also comment on a closely related phenomenon in the analysis of the photon phase operator where the notion of index and the modification of index by a finite cutoff play a crucial role. [S0556-2821(99)07517-7]

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I. INTRODUCTION

Recent developments in the treatment of fermions in lattice gauge theory led to a better understanding of chiral symmetry not only in lattice theory [1–7] but possibly also in continuum theory [8]. These developments are based on a Hermitian lattice Dirac operator $\gamma_5 D$ which satisfies the so-called Ginsparg-Wilson relation [1]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D. \tag{1.1}$$

An explicit example of the operator satisfying Eq. (1.1) and free of species doubling was given by Neuberger [2]. The operator was also discussed as a fixed-point form of block transformations [3]. Relation (1.1) led to interesting analyses of the notion of the index in lattice gauge theory [4–9]. Here γ_5 is a Hermitian chiral Dirac matrix.

The index relation is generally written as [4,5]

$$\mathrm{Tr}\, \gamma_5(1-\tfrac{1}{2}aD) = n_+ - n_-, \tag{1.2}$$

which is confirmed by [8]

$$\begin{aligned}\mathrm{Tr}[\gamma_5(1-\tfrac{1}{2}aD)] &= \sum_n \{\phi_n^\dagger \gamma_5 \phi_n - \tfrac{1}{2}\phi_n^\dagger \gamma_5 aD\phi_n\} \\ &= \sum_{\lambda_n=0} \phi_n^\dagger\gamma_5\phi_n + \sum_{\lambda_n\neq 0}\phi_n^\dagger\gamma_5\phi_n \\ &\quad - \sum_n \tfrac{1}{2}a\lambda_n\phi_n^\dagger\phi_n \\ &= \sum_{\lambda_n=0}\phi_n^\dagger\gamma_5\phi_n \\ &= n_+ - n_- = \text{ index},\end{aligned} \tag{1.3}$$

where $n_\pm$ stand for the number of normalizable zero modes in

$$\gamma_5 D\phi_n = \lambda_n \phi_n \tag{1.4}$$

for the *Hermitian* operator $\gamma_5 D$ with simultaneous eigenvalues $\gamma_5\phi_n = \pm\phi_n$. We also used the relation

$$\phi_n^\dagger\gamma_5\phi_n = \frac{a}{2}\lambda_n\phi_n^\dagger\phi_n = \frac{a}{2}\lambda_n \tag{1.5}$$

for $\lambda_n \neq 0$, which is derived by sandwiching relation (1.1) between $\phi_n^\dagger\gamma_5$ and ϕ_n. It should be emphasized that relation (1.3) is derived *without* using Tr $\gamma_5 = 0$. The inner product $\phi_n^\dagger\phi_n = (\phi_n, \phi_n) \equiv \Sigma_x a^4 \phi_n^\star(x)\phi_n(x)$ is defined by summing over all the lattice points which are not explicitly written in ϕ_n. See the Appendix for further notational details.

An advantage of gauge theory defined on a finite lattice is that one can analyze some subtle aspects of chiral symmetry in continuum theory in a well-defined finite setting. The purpose of the present paper is to study some of those aspects of chiral symmetry in the hope that this analysis also deepens our understanding of lattice regularization. In the path-integral treatment of chiral anomaly in continuum, the relation

$$\mathrm{Tr}\, \gamma_5 = n_+ - n_- \tag{1.6}$$

in a suitably regularized sense plays a fundamental role [10,11]. On the other hand, it is expected that the relation

$$\mathrm{Tr}\, \gamma_5 = 0 \tag{1.7}$$

holds on a finite lattice. As Chiu pointed out [12], relation (1.7) leads to an interesting *constraint*

$$\mathrm{Tr}\, \gamma_5 = n_+ - n_- + N_+ - N_- = 0, \tag{1.8}$$

where $N_\pm$ stand for the number of eigenstates $\gamma_5 D\phi_n = \pm(2/a)\phi_n$, with $\gamma_5\phi_n = \pm\phi_n$, respectively. It is important to recognize that Tr $\gamma_5 = 0$ means that this relation holds for *any* sensible basis set with any background gauge field in a given theory, which may be used to define the trace. Consequently, the seemingly trivial relation Tr $\gamma_5=0$ in fact carries important physical information. In this paper we show that Tr $\gamma_5=0$ implies the inevitable contribution from unphysical (would-be) species doublers in lattice theory or an unphysical bosonic spinor in Pauli-Villars regularization. In other words, Tr $\gamma_5=0$ cannot hold in the physical Hilbert space consisting of physical states only, and the continuum limit of Tr $\gamma_5=0$ is not defined consistently, as is seen in Eq. (1.8). It is shown that the failure of the decoupling of heavy fermions in the anomaly calculation is crucial to understand the consistency of the customary lattice calculation of anomaly where Tr $\gamma_5=0$ is used. (The continuum limit in this paper stands for the so-called ''naive'' continuum limit with $a\to 0$, and the lattice size is gradually extended to infinity for any finite a in the process of taking the limit $a\to 0$.) We then discuss the possible implications of our analysis on the treatment of chiral anomalies in continuum theory. We also briefly comment on an analogous phenomenon in the analysis of photon phase operator, where the notion of index plays a crucial role.

II. CONSISTENCY OF THE RELATION Tr $\gamma_5=0$

In Sec. I we have seen that the consistency of the relation Tr $\gamma_5=0$ requires the presence of the $N_\pm$ states for an operator $\gamma_5 D$ satisfying Eq. (1.1) on a finite lattice. We thus want to analyze the nature of the $N_\pm$ states in more detail. For this purpose, we start with the conventional Wilson operator D_W:

$$
\begin{aligned}
D_W(n,m) &\equiv i\gamma^\mu C_\mu(n,m)+B(n,m)-\frac{1}{a}m_0\delta_{n,m},\\
C_\mu(n,m) &= \frac{1}{2a}[\delta_{m+\mu,n}U_\mu(m)-\delta_{m,n+\mu}U_\mu^\dagger(n)],\\
B(n,m) &= \frac{r}{2a}\sum_\mu [2\delta_{n,m}-\delta_{m+\mu,n}U_\mu(m)-\delta_{m,n+\mu}U_\mu^\dagger(n)],\\
U_\mu(m) &= \exp[iagA_\mu(m)],
\end{aligned}
\tag{2.1}
$$

where we added a constant mass term to D_W for later convenience. Our matrix convention is that γ^μ are anti-Hermitian, $(\gamma^\mu)^\dagger=-\gamma^\mu$, and thus $\mathcal{C}\equiv\gamma^\mu C_\mu(n,m)$ is Hermitian:

$$
\mathcal{C}^\dagger=\mathcal{C}. \tag{2.2}
$$

Since the operator $\mathcal{C}$ forms the basis for any fermion operator on a lattice, we start with an analysis of $\mathcal{C}$.

A. Operator $\mathcal{C}$ and Tr $\gamma_5=0$

It was noted elsewhere [8] that Tr $\gamma_5=0$ implies species doubling for the operator $\mathcal{C}$. The basic reasoning is based on the index relation

$$
\dim\ker\left(\frac{1-\gamma_5}{2}\right)\mathcal{C}\left(\frac{1+\gamma_5}{2}\right)-\dim\ker\left(\frac{1+\gamma_5}{2}\right)\mathcal{C}\left(\frac{1-\gamma_5}{2}\right)=0, \tag{2.3}
$$

where we understand $[(1-\gamma_5)/2]\mathcal{C}[(1+\gamma_5)/2]$ as standing for the two-component operator b in

$$
\mathcal{C}=\begin{pmatrix}0 & b^\dagger\\ b & 0\end{pmatrix}. \tag{2.4}
$$

This form of $\mathcal{C}$ is deduced by noting $\mathcal{C}^\dagger=\mathcal{C}$ and $\gamma_5\mathcal{C}+\mathcal{C}\gamma_5=0$ in the representation where γ_5 is diagonal. The operator $b(m,n)$ projects a two-component spinor on a finite lattice to another two-component spinor on the same lattice, and thus it is a square matrix in the coordinate representation. For a general finite dimensional square matrix M, the index theorem $\dim\ker M-\dim\ker M^\dagger=0$ holds [8], where $\dim\ker M$, for example, stands for the number of normalizable modes in $Mu_n=0$. In the present context, $\dim\ker[(1-\gamma_5)/2]\mathcal{C}[(1+\gamma_5)/2]=\dim\ker b$ stands for the number of normalizable zero modes in

$$
\mathcal{C}\phi_n=0, \tag{2.5}
$$

with $[(1+\gamma_5)/2]\phi_n=\phi_n$. Thus the index relation (2.3) shows that possible zero modes with $\gamma_5\phi_n=\pm\phi_n$ are always paired. The eigenstates with nonzero eigenvalues in

$$
\mathcal{C}\phi_n=\lambda_n\phi_n \tag{2.6}
$$

give a vanishing contribution to the trace Tr γ_5 since

$$
\phi_n^\dagger\gamma_5\phi_n=0, \tag{2.7}
$$

by noting $\gamma_5\mathcal{C}+\mathcal{C}\gamma_5=0$. The index relation (2.3) is thus equivalent to Tr $\gamma_5=0$.

If one recalls the Atiyah-Singer index theorem [10,13] written in the same notation as Eq. (2.3),

$$
\dim\ker\left(\frac{1-\gamma_5}{2}\right)\not{D}\left(\frac{1+\gamma_5}{2}\right)-\dim\ker\left(\frac{1+\gamma_5}{2}\right)\not{D}\left(\frac{1-\gamma_5}{2}\right)=\nu, \tag{2.8}
$$

where ν stands for the Pontryagin index (i.e., an integral of anomaly) and $\not{D}\equiv\gamma^\mu(\partial_\mu-igA_\mu)$, one sees that a smooth continuum limit of the lattice index relation (2.3) for a general background gauge field configuration is *inconsistent* with the absence of species doublers.

In the present $\mathcal{C}$, a very explicit construction of species doublers is known. For a square lattice one can explicitly show that the simplest lattice fermion action

$$
S=\bar\psi i\mathcal{C}\psi \tag{2.9}
$$

is invariant under the transformation [14]

$$
\psi'=\mathcal{T}\psi,\qquad \bar\psi'=\bar\psi\mathcal{T}^{-1}, \tag{2.10}
$$

where $\mathcal{T}$ stands for any one of the 16 operators

$$1, \quad T_1T_2, \quad T_1T_3, \quad T_1T_4,$$

$$T_2T_3, \quad T_2T_4, \quad T_3T_4, \quad T_1T_2T_3T_4, \tag{2.11}$$

and

$$T_1, \quad T_2, \quad T_3, \quad T_4,$$

$$T_1T_2T_3, \quad T_2T_3T_4, \quad T_3T_4T_1, \quad T_4T_1T_2. \tag{2.12}$$

The operators T_μ are defined by

$$T_\mu \equiv \gamma_\mu \gamma_5 \exp(i\pi x^\mu/a), \tag{2.13}$$

and satisfy the relation

$$T_\mu T_\nu + T_\nu T_\mu = 2\delta_{\mu\nu}, \tag{2.14}$$

with $T_\mu^\dagger = T_\mu = T_\mu^{-1}$ for anti-Hermitian γ_μ. We denote the 16 operators by $\mathcal{T}_n$, $n=0\sim 15$, in the following with $\mathcal{T}_0=1$. By recalling that the operator T_μ adds the momentum π/a to the fermion momentum k_μ, we cover the entire Brillouin zone

$$-\frac{\pi}{2a} \leqslant k_\mu < \frac{3\pi}{2a} \tag{2.15}$$

by operation (2.10) starting with the free fermion defined in

$$-\frac{\pi}{2a} \leqslant k_\mu < \frac{\pi}{2a}. \tag{2.16}$$

The operators in Eq. (2.11) commute with γ_5, whereas those in Eq. (2.12) anticommute with γ_5 and thus change the sign of chiral charge, reproducing the 15 species doublers with correct chiral charge assignment: $\Sigma_{n=0}^{15}(-1)^n\gamma_5=0$.

In a smooth continuum limit, the operaor $\mathcal{C}$ produces $\not{D}$ for each species doubler with alternating chiral charge. The relation Tr $\gamma_5=0$ or Eq. (2.3) for the operator $\mathcal{C}$ is consistent for any background gauge field because of the presence of these species doublers, which are degenerate with the physical species in the present case.

B. Wilson operator D_W and Tr $\gamma_5=0$

The consistency of Tr $\gamma_5=0$ is analyzed by means of topological properties which are specified by Eq. (2.8), and thus it is best described in the nearly continuum limit. To be more precise, one may define the near-continuum configurations by the momentum k_μ carried by the fermion

$$-\frac{\pi}{2a}\epsilon \leqslant k_\mu \leqslant \frac{\pi}{2a}\epsilon \tag{2.17}$$

for sufficiently small a and ϵ combined with the operation $\mathcal{T}_n$ in Eqs. (2.11) and (2.12). To identify each species doubler clearly in the near-continuum configurations, we also keep r/a and m_0/a finite for $a\rightarrow$ small [14], and the gauge fields are assumed to be sufficiently smooth. For these configurations, we can approximate the operator D_W by

$$D_W = i\not{D} + M_n + O(\epsilon^2) + O(agA_\mu) \tag{2.18}$$

for each species doubler, where the mass parameters M_n stand for $M_0=-m_0/a$ and one of

$$\frac{2r}{a}-\frac{m_0}{a}, \quad (4,-1), \quad \frac{4r}{a}-\frac{m_0}{a}, \quad (6,1),$$
$$\frac{6r}{a}-\frac{m_0}{a}, \quad (4,-1), \quad \frac{8r}{a}-\frac{m_0}{a}, \quad (1,1) \tag{2.19}$$

for $n=1-15$. Here we denoted (multiplicity, chiral charge) in the brackets for species doublers. In Eq. (2.18) we used the relation valid for configurations (2.17), for example,

$$\begin{aligned} D_W(k) &= \sum_\mu \gamma^\mu \frac{\sin ak_\mu}{a} + \frac{r}{a}\sum_\mu (1-\cos ak_\mu) - \frac{m_0}{a} \\ &= \gamma^\mu k_\mu[1+O(\epsilon^2)] + \frac{r}{a}O(\epsilon^2) - \frac{m_0}{a} \end{aligned} \tag{2.20}$$

in the momentum representation with vanishing gauge field.

In these near-continuum configurations, the topological properties are specified by the operator $\not{D}$ in D_W. We can thus evaluate Tr γ_5 by using the basis set defined by

$$\not{D}\phi_n = \lambda_n \phi_n, \tag{2.21}$$

which formally *diagonalize* the effective operator D_W in Eq. (2.18) describing the low-energy excitations of each species doubler. We then obtain

$$\mathrm{Tr}\,\gamma_5 = \sum_{n=0}^{15} (-1)^n \lim_{L\rightarrow large} \sum_{l=1}^{L} \phi_l^\dagger \gamma_5 \phi_l = 0, \tag{2.22}$$

where $\phi_l^\dagger\gamma_5\phi_l=0$ for $\lambda_l\neq 0$ because of $\gamma_5\not{D}+\not{D}\gamma_5=0$, and Eq. (2.22) states the cancellation of zero-mode contributions $\Sigma_{\lambda_l=0}\phi_l^\dagger\gamma_5\phi_l$ among various species. We are assuming that our near-continuum configurations (2.18) are accurate in the treatment of these zero modes. An argument to support our identification of the near continuum configurations will be given in Sec. II C.

Tr $\gamma_5=0$ is thus consistent even for a topologically nontrivial gauge background because of the presence of the would-be species doublers. This property is related to the well-known fact that one can safely ignore the Jacobian factor for global chiral transformation $\delta\psi=i\epsilon\gamma_5\psi$ and $\delta\bar{\psi}=\bar{\psi}i\epsilon\gamma_5$ for the theory defined by $S=\bar{\psi}D_W\psi$.

C. Overlap Dirac operator and Tr $\gamma_5=0$

The operator D introduced by Neuberger [2], which satisfies relation (1.1), has an explicit expression

$$aD = 1-\gamma_5\frac{H}{\sqrt{H^2}} = 1 + D_W\frac{1}{\sqrt{D_W^\dagger D_W}}, \tag{2.23}$$

where $D_W = -\gamma_5 H$ is the Wilson operator. For the near-continuum configurations specified above in Eq. (2.17), one can approximate

$$D = \sum_{n=0}^{15} (1/a)\left[1 + (i\not{D} + M_n)\frac{1}{\sqrt{\not{D}^2 + M_n^2}}\right]|n\rangle\langle n|,$$

$$\gamma_5 D = \sum_{n=0}^{15} (-1)^n \gamma_5 (1/a)\left[1 + (i\not{D} + M_n)\frac{1}{\sqrt{\not{D}^2 + M_n^2}}\right]|n\rangle\langle n|, \tag{2.24}$$

$$\gamma_5 = \sum_{n=0}^{15} (-1)^n \gamma_5 |n\rangle\langle n|.$$

Here we explicitly write the projection $|n\rangle\langle n|$ for each species doubler. The operators in Eqs. (2.24) preserve the Ginsparg-Wilson relation (1.1). We can again use the basis set in Eq. (2.21), which formally diagonalizes the basic operator D in Eqs. (2.24), to define the trace operation. We thus obtain

$$\mathrm{Tr}\,\gamma_5 = \sum_{n=0}^{15} (-1)^n \lim_{L\to large} \sum_{l=1}^{L} \phi_l^\dagger \gamma_5 \phi_l = 0 \tag{2.25}$$

by assuming that our effective operators (2.24) are accurate in describing the excitations near the zero modes $\not{D}\phi_l = 0$, which are relevant for topological considerations. Again the presence of the would-be species doublers makes the relation $\mathrm{Tr}\,\gamma_5 = 0$ consistent for any topologically nontrivial background gauge field. A justification of our effective description [Eqs. (2.24)] will be given later.

The above expression of D also shows that

$$D\phi_l = 0, \qquad D\phi_l = \frac{2}{a}\phi_l \tag{2.26}$$

for the physical species and the unphysical species doublers, respectively, if one uses the zero modes $\not{D}\phi_l = 0$. Note that $M_0 < 0$ and the rest of $M_n > 0$ in Eqs. (2.19) and (2.24) [2]. We also note that ϕ_l can be a simultaneous eigenstate of γ_5 only for $\not{D}\phi_l = 0$. That is, $N_\pm$ states with the eigenvalue $2/a$ in fact correspond to topological excitations associated with species doublers; this means that the multiplicities of these $N_\pm$ are quite high due to the 15 species doublers, although they satisfy the sum rule $n_+ + N_+ = n_- + N_-$. This sum rule is a direct consequence of Eqs. (2.25) and (2.26) by noting that $\phi_l^\dagger \gamma_5 \phi_l = 0$ for $\lambda_l \neq 0$.

The calculation of the index (1.2) may proceed as

$$\begin{aligned}
\mathrm{Tr}\,\gamma_5\left(1 - \frac{a}{2}D\right) &= -\mathrm{Tr}\,\gamma_5 \frac{a}{2} D \\
&= -\frac{1}{2}\mathrm{Tr}\sum_{n=0}^{15} (-1)^n \gamma_5 \left[1 + (i\not{D} + M_n)\frac{1}{\sqrt{\not{D}^2 + M_n^2}}\right] \\
&= -\frac{1}{2}\mathrm{Tr}\sum_{n=0}^{15} (-1)^n \gamma_5 (i\not{D} + M_n)\frac{1}{\sqrt{\not{D}^2 + M_n^2}} \\
&= -\frac{1}{2}\sum_{n=0}^{15} (-1)^n \sum_l \phi_l^\dagger \gamma_5 M_n \frac{1}{\sqrt{\not{D}^2 + M_n^2}}\phi_l \\
&= -\frac{1}{2}\sum_{n=0}^{15} (-1)^n M_n \frac{1}{\sqrt{M_n^2}} \sum_{\lambda_l = 0} \phi_l^\dagger \gamma_5 \phi_l \\
&= -\frac{1}{2}\sum_{n=0}^{15} (-1)^n M_n \frac{1}{\sqrt{M_n^2}} (n_+ - n_-) \\
&= -\frac{1}{2}\left(-1 + \sum_{n=1}^{15} (-1)^n\right)(n_+ - n_-) = n_+ - n_-,
\end{aligned} \tag{2.27}$$

where we used $\gamma_5\not{D} + \not{D}\gamma_5 = 0$ and the fact that $\phi_l^\dagger \gamma_5 \phi_l = 0$ for $\lambda_l \neq 0$ in $\not{D}\phi_l = \lambda_l \phi_l$. We also used the fact that $M_0 < 0$ and $M_n > 0$ for $n = 1 \sim 15$ [2]. Index (2.27) is defined for $\not{D}$, while index (1.3) is defined for $\gamma_5 D$, and both agree with the Pontryagin index, as seen in Eq. (2.30) below.

In the above calculation [Eq. (2.27)], we used the relation $\mathrm{Tr}\,\gamma_5 = 0$ twice: In the second line, this relation requires the presence of the physical species as well as the species doublers. As a result, we have the contribution to the final index from both of the physical species and 15 species doublers, although the species doublers with $\lambda_l = 2/a$ should saturate the index (and anomaly) in the expression [12]

$$-\mathrm{Tr}(a/2)\gamma_5 D=n_+-n_-, \tag{2.28}$$

as noted in Eq. (A10) in the Appendix. Our analysis of the global topological property on the basis of effective operators (2.24) is thus consistent.

The above calculational scheme of index (2.27) in fact corresponds to the evaluation of the local index (i.e., anomaly) performed in Ref. [8]. By using the plane-wave basis, one has (in the limit $a\rightarrow 0$ with r/a and m_0/a kept fixed)

$$\begin{aligned}
\mathrm{tr}\,\gamma_5\left(1-\frac{a}{2}D\right)(x)\\
&=-\frac{1}{2}\sum_{n=0}^{15}(-1)^n\,\mathrm{tr}\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}\\
&\quad\times e^{-ikx}\gamma_5(i\not{D}+M_n)\frac{1}{\sqrt{\not{D}^2+M_n^2}}e^{ikx}\\
&=\frac{1}{2}\mathrm{tr}\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}e^{-ikx}\gamma_5\frac{1}{\sqrt{\not{D}^2/M_0^2+1}}e^{ikx}\\
&\quad-\frac{1}{2}\sum_{n=1}^{15}(-1)^n\,\mathrm{tr}\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}e^{-ikx}\\
&\quad\times\gamma_5\frac{1}{\sqrt{\not{D}^2/M_n^2+1}}e^{ikx},
\end{aligned} \tag{2.29}$$

which gives rise to the anomaly for all $|M_n|\rightarrow\infty$ in the continuum limit:

$$\begin{aligned}
\mathrm{tr}\,\gamma_5\left(1-\frac{a}{2}D\right)(x)&=\lim_{M\rightarrow\infty}\mathrm{tr}\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}e^{-ikx}\gamma_5 f\left(\frac{\not{D}^2}{M^2}\right)e^{ikx}\\
&=\frac{g^2}{32\pi^2}\mathrm{tr}\,\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}.
\end{aligned} \tag{2.30}$$

Here we defined $f(x)=1/\sqrt{x+1}$, which satisfies

$$\begin{aligned}
&f(0)=1,\quad f(\infty)=0,\\
&f'(x)x|_{x=0}=f'(x)x|_{x=\infty}=0.
\end{aligned} \tag{2.31}$$

The right-hand side of Eq. (2.30) is known to be independent of the choice of $f(x)$ which satisfies the mild condition (2.31) [11].

A direct evaluation of the anomaly without using Eq. (2.27) is of course possible. We briefly sketch the procedure here, since it justifies our analysis based on the effective expressions in Eq. (2.24) [and partly Eq. (2.18) also]. For an operator $O(x,y)$ defined on the lattice, one may define

$$O_{mn}\equiv\sum_{x,y}\phi_m^*(x)O(x,y)\phi_n(y), \tag{2.32}$$

and the trace

$$\begin{aligned}
\mathrm{Tr}\,O&=\sum_n O_{nn}\\
&=\sum_n\sum_{x,y}\phi_n^*(x)O(x,y)\phi_n(y)\\
&=\sum_x\left(\sum_{n,y}\phi_n^*(x)O(x,y)\phi_n(y)\right).
\end{aligned} \tag{2.33}$$

The local version of the trace (or anomaly) is then defined by $\mathrm{Tr}\,O(x,x)\equiv\Sigma_{n,y}\phi_n^*(x)O(x,y)\phi_n(y)$. For the operator of our interest, we have

$$\begin{aligned}
\mathrm{tr}\left(-\frac{1}{2}\gamma_5 D_W\frac{1}{\sqrt{D_W^\dagger D_W}}\right)(x)\\
&=-\frac{1}{2}\sum_{n=0}^{15}\mathrm{tr}\int_{-\pi/2a}^{\pi/2a}\frac{d^4k}{(2\pi)^4},\\
&\quad\times e^{-ikx}\mathcal{T}_n^{-1}\gamma_5 D_W\frac{1}{\sqrt{D_W^\dagger D_W}}\mathcal{T}_n e^{ikx}
\end{aligned} \tag{2.34}$$

where we used the plane-wave basis defined in Eq. (2.16) combined with the operation $\mathcal{T}_n$. We also used a shorthand notation $Oe^{ikx}=\Sigma_y O(x,y)e^{iky}$.

We first take the $a\rightarrow 0$ limit of this expression with all M_n, $n=0$–15, kept fixed, and then take the limit $|M_n|\rightarrow\infty$ later. For fixed M_n (to be precise, for fixed m_0/a and r/a), one can confirm that the above integral (2.34) for the domain $(\pi/2a)\epsilon\leqslant|k_\mu|\leqslant\pi/2a$ vanishes (at least) linearly in a for $a\rightarrow 0$, if one takes into account the trace with γ_5. See also Refs. [7,9]. In the remaining integral

$$\begin{aligned}
&-\frac{1}{2}\sum_{n=0}^{15}(-1)^n\,\mathrm{tr}\\
&\quad\times\int_{-(\pi/2a)\epsilon}^{(\pi/2a)\epsilon}\frac{d^4k}{(2\pi)^4}e^{-ikx}\gamma_5\mathcal{T}_n^{-1}D_W\frac{1}{\sqrt{D_W^\dagger D_W}}\mathcal{T}_n e^{ikx},
\end{aligned} \tag{2.35}$$

one may take the limit $a\rightarrow 0$ [and $(\pi/2a)\epsilon\rightarrow\infty$], keeping ϵ arbitrarily small. By taking Eq. (2.20) into account, one thus recovers expression (2.29). One can arrive at the same conclusion by using an auxiliary regulator $h(\not{C}^2/m^2)$ in the integrand in Eq. (2.34) to make the intermediate steps better defined [8]. The domain in Eq. (2.17) with arbitrarily small but finite ϵ thus correctly describes the topological aspects of the continuum limit in the present prescription.

Here we went through the details of the anomaly calculation to show that the interpretation of the $N_\pm$ states in Eq. (A8) as topological excitations related to species doublers, as is shown in Eq. (2.26), is also consistent with the local anomaly calculation. As for a general analysis of chiral anomaly in the overlap operator, see Ref. [15].

At this stage it is instructive to consider an operator defined by

$$D \equiv \frac{1}{a}\left[1+(i\not{D}+M_0)\frac{1}{\sqrt{\not{D}^2+M_0^2}}\right] \tag{2.36}$$

instead of D in Eq. (2.24). This D is regarded as an $M_n \to \infty$, $n \neq 0$, limit of the effective operator D [Eq. (2.24)] in the Lagrangian level, and it satisfies (a continuum version of) the Ginsparg-Wilson relation (1.1) without any species doubler. The relation $\mathrm{Tr}\,\gamma_5 = 0$ is thus expected to be inconsistent. In fact we have an index related to the chiral Jacobian [5]

$$\begin{aligned}\mathrm{Tr}\,\gamma_5\left(1-\frac{a}{2}D\right) &= \lim_{L\to large}\sum_{l=1}^{L}\phi_l^\dagger\gamma_5\left(1-\frac{a}{2}D\right)\phi_l\\ &= \sum_{\lambda_l=0}\phi_l^\dagger\gamma_5\phi_l = n_+ - n_-\end{aligned} \tag{2.37}$$

by noting $\phi_l^\dagger\gamma_5\phi_l = 0$ for $\lambda_l \neq 0$ in $\not{D}\phi_l = \lambda_l\phi_l$. On the other hand, if one incorrectly uses $\mathrm{Tr}\,\gamma_5 = 0$, one obtains

$$\begin{aligned}\mathrm{Tr}\,\gamma_5\left(1-\frac{a}{2}D\right) &= -\frac{1}{2}\mathrm{Tr}\left(\gamma_5(i\not{D}+M_0)\frac{1}{\sqrt{\not{D}^2+M_0^2}}\right)\\ &= -\frac{1}{2}\lim_{L\to large}\sum_{l=1}^{L}\phi_l^\dagger\gamma_5 M_0\frac{1}{\sqrt{\not{D}^2+M_0^2}}\phi_l\\ &= \frac{1}{2}(n_+ - n_-)\end{aligned} \tag{2.38}$$

by noting that $\gamma_5\not{D}+\not{D}\gamma_5 = 0$, $\phi_l^\dagger\gamma_5\phi_l = 0$ for $\lambda_l \neq 0$, and $M_0 < 0$. One thus loses half of the index or anomaly. In this example, the evaluation of $\mathrm{Tr}\,\gamma_5$ is somewhat subtle, but $\mathrm{Tr}\,\gamma_5 = 0$ is definitely inconsistent since the calculation in the last line in Eq. (2.38) is well defined. In fact, the relations

$$\mathrm{Tr}\,\gamma_5 = n_+ - n_- \quad \text{and} \quad \mathrm{Tr}\left(-\frac{a}{2}\gamma_5 D\right) = 0 \tag{2.39}$$

are consistent for the present operator D, since the species doublers at $\gamma_5 D\phi_l = \pm(2/a)\phi_l$ are missing. A more rigorously regularized Jacobian for the present example is given by formula (3.3), to be discussed below.

D. General lattice Dirac operator and Tr $\gamma_5 = 0$

We expect that our analysis of $\mathrm{Tr}\,\gamma_5 = 0$, namely, that its consistency, is ensured only by the presence of the would-be species doublers in the Hilbert space, works for a general lattice Dirac operator, since any lattice operator contains $\not{C}$ as an essential part. For the smooth near-continuum configurations, the lowest-dimensional operator $\not{C}$ is expected to specify the topological properties. From this viewpoint, the overlap Dirac operator D describes the topological properties such as the index theorem and $\mathrm{Tr}\,\gamma_5 = 0$ in a neater way than the Wilson operator D_W, mainly because the operator D projects all the species doublers to the vicinity of $2/a$: The behavior for small values of $\not{C}$ (i.e., for $|\not{C}| \ll 1/a$) is described in a more clear-cut way by D, and one can recognize clearly the topological $N_\pm$ states related to species doublers.

We here note that the Pauli-Villars regularization in continuum theory can be analyzed in a similar way. The Pauli-Villars regulator is defined in the path integral by introducing a bosonic spinor ϕ into the action:

$$S = \int d^4x[\bar{\psi}(i\not{D}-m)\psi + \bar{\phi}(i\not{D}-M)\phi]. \tag{2.40}$$

The Jacobian for the global chiral transformation then gives rise to the graded trace [11]

$$\mathrm{Tr}\,\gamma_5 = \mathrm{Tr}_\psi\,\gamma_5 - \mathrm{Tr}_\phi\,\gamma_5 = 0. \tag{2.41}$$

The relation $\mathrm{Tr}\,\gamma_5 = 0$ is thus consistent with any topologically nontrivial background gauge field because of the presence of the unphysical regulator ϕ. This ϕ is analogous to the species doublers in lattice regularization.

III. IMPLICATIONS OF THE PRESENT ANALYSIS

We have shown that the consistency of $\mathrm{Tr}\,\gamma_5 = 0$ for a topologically nontrivial background gauge field requires the presence of some unphysical states in the Hilbert space. Coming back to the original lattice theory defined by

$$S = \bar{\psi}D\psi, \tag{3.1}$$

with D satisfying relation (1.1), one obtains two times Eq. (1.3) as a Jacobian factor for the global chiral transformation [5] $\delta\psi = i\epsilon\gamma_5[1-(a/2)D]\psi$ and $\delta\bar{\psi} = \bar{\psi}i\epsilon[1-(a/2)D]\gamma_5$, which leaves action (3.1) invariant. One can rewrite Eq. (1.3) as

$$\mathrm{Tr}\,\gamma_5\left(1-\frac{a}{2}D\right) = \widetilde{\mathrm{Tr}}\,\gamma_5\left(1-\frac{a}{2}D\right) = \widetilde{\mathrm{Tr}}\,\gamma_5 = n_+ - n_-, \tag{3.2}$$

where the modified trace $\widetilde{\mathrm{Tr}}$ is defined by truncating the unphysical $N_\pm$ states with $\lambda_n = \pm 2/a$. Without the $N_\pm$ states, $\widetilde{\mathrm{Tr}}\,\gamma_5(a/2)D = 0$, since the eigenvalues λ_n of $\gamma_5 D$ with $\lambda_n \neq 0, \pm 2/a$ appear always pairwise at $\pm|\lambda_n|$. See the Appendix.

If one takes a smooth continuum limit of $\widetilde{\mathrm{Tr}}\,\gamma_5 = n_+ - n_-$ in Eq. (3.2), one recovers the result of the continuum path integral (1.6). If one considers that $\widetilde{\mathrm{Tr}}\,\gamma_5$ is too abstract, one may define it more concretely by

$$\begin{aligned}\mathrm{Tr}\,\gamma_5\left(1-\frac{a}{2}D\right)f\left(\frac{(\gamma_5 D)^2}{M^2}\right) &= \widetilde{\mathrm{Tr}}\,\gamma_5\left(1-\frac{a}{2}D\right)f\left(\frac{(\gamma_5 D)^2}{M^2}\right)\\ &= \widetilde{\mathrm{Tr}}\,\gamma_5 f\left(\frac{(\gamma_5 D)^2}{M^2}\right) = n_+ - n_-\end{aligned} \tag{3.3}$$

for *any* $f(x)$ which satisfies the mild condition in Eq. (2.31). See also Eqs. (1.3) and (1.5). This relation suggests that we can extract the local index (or anomaly) by

$$\mathrm{tr}\,\gamma_5\left(1-\frac{a}{2}D\right)f\left(\frac{(\gamma_5 D)^2}{M^2}\right)(x), \tag{3.4}$$

which is shown to be independent of the choice of $f(x)$ in the limit $a\to 0$ and leads to Eq. (2.30) [for $f(x)$ which goes to zero rapidly for $x\to\infty$] by using only the general properties of D [8]. If one constrains the momentum domain to Eq. (2.16) from the beginning, one may use the last expression in Eq. (3.3) to evaluate the anomaly for a more general class of $f(x)$. We thus naturally recover the result of the continuum path integral [11].

As for a more practical implication of our analysis of Tr $\gamma_5=0$ in lattice theory, one may say that any result which depends critically on the states $N_\pm$ is *unphysical*. It is thus necessary to define the scalar density (or mass term) and pseudoscalar density in the theory (3.1) by [16,6]

$$S(x)=\bar{\psi}_L\psi_R+\bar{\psi}_R\psi_L=\bar{\psi}\left(1-\frac{a}{2}D\right)\psi,$$
$$P(x)=\bar{\psi}_L\psi_R-\bar{\psi}_R\psi_L=\bar{\psi}\gamma_5\left(1-\frac{a}{2}D\right)\psi. \tag{3.5}$$

Here we defined two independent projection operators

$$P_\pm=\tfrac{1}{2}(1\pm\gamma_5),$$
$$\hat{P}_\pm=\tfrac{1}{2}(1\pm\hat{\gamma}_5), \tag{3.6}$$

with $\hat{\gamma}_5=\gamma_5(1-aD)$ which satisfies $\hat{\gamma}_5^2=1$ [6]. The left and right components are then defined by

$$\bar{\psi}_{L,R}=\bar{\psi}P_\pm, \quad \psi_{R,L}=\hat{P}_\pm\psi \tag{3.7}$$

which is based on the decomposition

$$D=P_+D\hat{P}_-+P_-D\hat{P}_+. \tag{3.8}$$

The physical operators $S(x)$ and $P(x)$ in Eqs. (3.5) do not contain the contribution from the unphysical states $N_\pm$ in Eq. (A8). In the spirit of this construction, the definition of the index by Eq. (3.3), which is independent of unphysical states $N_\pm$, is natural. In particular, all the unphysical species doublers (not only the topological ones at $2/a$) decouple from the anomaly defined by Eq. (3.4) in the limit $a\to 0$ with fixed M.

The customary calculation of the index (and also anomaly) by the relation [4–7,9]

$$\mathrm{Tr}\,\gamma_5\left(1-\frac{a}{2}D\right)=\mathrm{Tr}\left(-\frac{a}{2}\gamma_5 D\right)=n_+-n_- \tag{3.9}$$

by itself is of course consistent, since one simply includes the unphysical states $N_\pm$ in evaluating Tr $\gamma_5=0$, and consequently one obtains the index $\mathrm{Tr}[-(a/2)\gamma_5 D]$ from the unphysical states $N_\pm$ only. We after all know that the left-hand side of Eq. (3.9) is independent of $N_\pm$.

Rather, the major message of our analysis is that the continuum limit of $\mathrm{Tr}\gamma_5=0$ in Eq. (1.8) (unlike the relation $\widetilde{\mathrm{Tr}}\,\gamma_5=n_+-n_-$) *cannot* be defined in a consistent way when the (would-be) species doublers disappear from the Hilbert space. It is clear from the expression of Tr $\gamma_5=0$ in Eq. (1.8) that the $a\to 0$ limit of Tr $\gamma_5=0$ is not defined consistently. One may then ask how the calculation of local anomaly on the basis of Eq. (3.9) could be consistent in the limit $a\to 0$ if Tr $\gamma_5=0$ is inconsistent. A key to resolve this apparent paradox is the failure of the decoupling of heavy fermions in the evaluation of anomaly. The massive unphysical species doublers do not decouple from the anomaly—as is seen in Eq. (2.29), for example. If one insists on Tr $\gamma_5=0$ in the continuum limit, one is also insisting on the failure of the decoupling of these infinitely massive particles from Tr $\gamma_5=0$. The contributions of these heavy fermions to the anomaly and to Tr $\gamma_5=0$ precisely cancel, just as in the case of the evaluation of global index in Eq. (3.9). That is, the local anomaly itself is *independent* of these massive species doublers in the continuum limit, as is clear in Eq. (3.4). In this sense, Eq. (3.4) is the only logically consistent definition of local anomaly. It is an advantage of the finite lattice formulation that we can now clearly illustrate this subtle cancellation of the contributions of those ultraheavy regulators to Tr $\gamma_5=0$ and anomaly on the basis of Eq. (1.8). (In the case of the Wilson fermion operator D_W, an analogous cancellation takes place in Tr γ_5+pseudoscalar mass term induced by the chiral variation of the action.)

When one defines a chiral theory by recalling (3.8) [6,17],

$$S=\bar{\psi}P_+D\hat{P}_-\psi=\bar{\psi}_L D\psi_L, \tag{3.10}$$

one obtains the *covariant* gauge anomaly (or Jacobian)

$$\mathrm{tr}\,T^a\gamma_5\left(1-\frac{a}{2}D\right)=\sum_n \phi_n(x)^\dagger T^a\gamma_5\left(1-\frac{a}{2}D\right)\phi_n(x) \tag{3.11}$$

for the gauge transformation $\delta\psi_L(x)=i\alpha^a(x)T^a\hat{P}_-\psi_L$ and $\delta\bar{\psi}_L(x)=\bar{\psi}_L P_+(-i)\alpha^a(x)T^a$.

An analog of the $U(1)$ anomaly [Eq. (3.4)] is then defined for the gauge anomaly (3.11) by [using ϕ_n in Eq. (1.4)]

$$\sum_n \phi_n(x)^\dagger T^a\gamma_5\left(1-\frac{a}{2}D\right)f\left(\frac{(\gamma_5 D)^2}{M^2}\right)\phi_n(x)$$
$$=\sum_n f\left(\frac{\lambda_n^2}{M^2}\right)\phi_n(x)^\dagger T^a\left(\gamma_5-\frac{a}{2}\lambda_n\right)\phi_n(x), \tag{3.12}$$

which reduces to the lattice expression for $M\to\infty$ with $f(0)=1$. In practice, one first takes the continuum limit $a\to 0$ with M fixed and one obtains (see, for example, Ref. [8])

$$\mathrm{tr}\,T^a\gamma_5 f\left(\frac{\not{D}^2}{M^2}\right), \tag{3.13}$$

which is again known to be independent of the specific choice of $f(x)$ in the limit $M\to\infty$ [11]. For the overlap Dirac operator, one can show that the anomaly calculation in Eq. (3.11) by using tr $T^a\gamma_5=0$ corresponds effectively to a specific choice of $f(x)=1/\sqrt{1+x}$ in Eq. (3.13), just as in the case of $U(1)$ anomaly in Eq. (2.30).

The definition of the regularized Jacobian (3.12) may be regarded to correspond to the truncation of the states $N_\pm$ from the chiral action

$$S=\sum_{n\in N_+}\left(\frac{2}{a}\right)\overline{C}_nC_n+\sum_{0\le\lambda_n<2/a}\lambda_n\overline{C}_nC_n \qquad (3.14)$$

to

$$\tilde{S}=\sum_{0\le\lambda_n<2/a}\lambda_n\overline{C}_nC_n, \qquad (3.15)$$

and then taking the continuum limit $a\to 0$, which is logically more natural as the $N_\pm$ states are eliminated from the Hilbert space *before* taking the continuum limit.

Incidentally, action (3.14) is obtained from Eq. (3.10) by expanding

$$\bar{\psi}P_+=\sum_n \overline{C}_n\overline{v}_n, \qquad \hat{P}_-\psi=\sum_n C_nv_n \qquad (3.16)$$

with the choice of the basis sets

$$\{v_j\}=\{\phi_n|\gamma_5D\phi_n=0,\gamma_5\phi_n=-\phi_n\}\oplus\{\phi_n|\gamma_5D\phi_n=2/a\,\phi_n,\gamma_5\phi_n=+\phi_n\}\oplus\{\hat{P}_-\phi_n/\sqrt{(1+a\lambda_n/2)/2}\,|\gamma_5D\phi_n=\lambda_n\phi_n,2/a>\lambda_n>0\}, \qquad (3.17)$$

$$\{\overline{v_k^\dagger}\}=\{\phi_n|\gamma_5D\phi_n=0,\gamma_5\phi_n=+\phi_n\}\oplus\{\phi_n|\gamma_5D\phi_n=2/a\,\phi_n,\gamma_5\phi_n=+\phi_n\}\oplus\{P_+\phi_n/\sqrt{(1+a\lambda_n/2)/2}|\gamma_5D\phi_n=\lambda_n\phi_n,2/a>\lambda_n>0\} \qquad (3.18)$$

in terms of the eigenstates of $\gamma_5D\phi_n=\lambda_n\phi_n$ summarized in the Appendix.

Consequently, the path integral for a fixed background gauge field is defined by

$$Z=J\int\prod_{n\in N_+}d\overline{C}_ndC_n\prod_{0\le\lambda_n<2/a}d\overline{C}_n\prod_{0\le\lambda_m<2/a}dC_m\exp S,$$
$$\tilde{Z}=\tilde{J}\int\prod_{0\le\lambda_n<2/a}d\overline{C}_n\prod_{0\le\lambda_m<2/a}dC_m\exp\tilde{S}, \qquad (3.19)$$

with Jacobian factors J and $\tilde{J}$ which depend on the basis set. A (naive) continuum limit of the truncated expression $\tilde{Z}$ naturally gives rise to the covariant path integral formulation of chiral gauge theory [11]. In particular, the fermion number anomaly which is given by Eq. (3.3) gives rise to the fermion number violation in chiral gauge thoery. As is well known, this formulation of the continuum limit is consistent if the anomaly cancellation condition $\mathrm{tr}\,T^a\{T^b,T^c\}=0$ is satisfied, when combined with the argument of the robustness of lattice gauge symmetry [18]. See also Ref. [19].

An interesting analysis of the definition of chiral theory at a finite a was given by Lüscher recently [20]. The fermion number violation arises from the nontrivial index of the rectangular (*not* square) matrix in Eq. (3.10) [cf. Eq. (2.3)],

$$\dim\ker\hat{P}_-\gamma_5DP_+-\dim\ker P_+\gamma_5D\hat{P}_-=n_+-n_-, \qquad (3.20)$$

as is seen in the explicit construction of the basis vectors in Eqs. (3.17) and (3.18): For a general $n\times m$ matrix M, one can prove an index theorem

$$\dim\ker M-\dim\ker M^\dagger=m-n, \qquad (3.21)$$

which is a generalization of the case of a square matrix with $m=n$. For the operator $\hat{P}_-\gamma_5DP_+$ in Eq. (3.20), the dimensions of the column and row vectors are given, respectively, by using the projection operators as $\mathrm{Tr}\,\hat{P}_-$ and $\mathrm{Tr}\,P_+$, and thus $m-n=\mathrm{Tr}\,P_+-\mathrm{Tr}\,\hat{P}_-=\mathrm{Tr}\,\gamma_5[1-(a/2)D]=n_+-n_-$ [20]. Incidentally, an analogous analysis provides an alternative proof of the equivalence of $\mathrm{Tr}\,\gamma_5=0$ with the index relation (2.3).

IV. DISCUSSION AND CONCLUSION

Motivated by the recent interesting developments in lattice gauge theory, we analyzed the physical implications of the condition $\mathrm{Tr}\,\gamma_5=0$ in detail. We have shown that $\mathrm{Tr}\,\gamma_5=0$, whose validity is often taken for granted, is consistent only when one includes some unphysical states in the Hilbert space. The continuum $a\to 0$ limit of $\mathrm{Tr}\,\gamma_5=0$ is not defined consistently, as seen in Eq. (1.8). We have explained that the failure of the decoupling of heavy fermions in the anomaly calculation is a key to understand the consistency of the customary lattice calculation of anomaly where $\mathrm{Tr}\,\gamma_5=0$ is used. Our analysis is perfectly consistent with relation (1.6) in the continuum path integral, and even provides positive support for formula (3.3) and the related definition of anomaly (3.4) in lattice theory.

We here want to comment on an analysis of the photon phase operator [21], where a closely related phenomenon associated with the notion of index takes place [22]. The Maxwell field is expanded into an infinite set of harmonic oscillators, and thus the analysis of the photon phase operator is performed for a simple harmonic oscillator

$$H=\tfrac{1}{2}(p^2+\omega^2q^2)=\hbar\omega(a^\dagger a+\tfrac{1}{2}). \qquad (4.1)$$

The quantum requirement of the absence of the negative normed states leads to $a|0\rangle=0$, and thus the index relation

$$\dim\ker a-\dim\ker a^\dagger=1, \qquad (4.2)$$

since no states are annihilated by $a^\dagger$. On the other hand, the existence of the observable *Hermitian* phase operator φ requires a decomposition [21]

$$a=U(\varphi)\sqrt{N},\quad a^\dagger=\sqrt{N}U(\varphi)^{-1}, \tag{4.3}$$

with a *unitary* $U(\varphi)=e^{i\varphi}$ and $N=a^\dagger a$. These expressions suggest

$$\dim\ker a-\dim\ker a^\dagger=0, \tag{4.4}$$

in contradiction to relation (4.2), since the unitary factor $U(\varphi)$ does not influence the analysis of index. Index (4.2) thus provides a no-go theorem against the Hermitian photon phase operator and the resulting familiar phase-number uncertainty relation [22].

To circumvent the topological stricture (4.2), one may truncate the operator a to an $[(s+1)\times(s+1)]$-dimensional square matrix

$$a_s=\begin{pmatrix} 0 & 1 & 0 & 0 & .. & 0\\ 0 & 0 & \sqrt{2} & 0 & .. & 0\\ 0 & 0 & 0 & \sqrt{3} & .. & 0\\ . & . & . & . & . & .\\ 0 & 0 & 0 & 0 & .. & \sqrt{s}\\ 0 & 0 & 0 & 0 & .. & 0 \end{pmatrix}$$
$$=|0\rangle\langle 1|+|1\rangle\langle 2|\sqrt{2}\ \ldots .+|s-1\rangle\langle s|\sqrt{s} \tag{4.5}$$

and $a_s^\dagger=(a_s)^\dagger$. One then obtains a vanishing index for a finite-dimensional square matrix [22]

$$\dim\ker a_s-\dim\ker a_s^\dagger=0, \tag{4.6}$$

and one can in fact introduce a Hermitian phase operator ϕ [23] which satisfies the relation $a_s=e^{i\phi}\sqrt{a_s^\dagger a_s}$.

The parameter s or the state $|s\rangle$ stands for the cutoff parameter analogous to the $N_\pm$ states related to Tr $\gamma_5=0$ in lattice theory. A careful analysis of the uncertainty relation shows that the Hermitian operator ϕ, when used to analyze the data which is already in the quantum limit, leads to a substantial deviation from the minimum uncertainty relation at the characteristically quantum domain with *small* average photon numbers. This artificial deviation from the minimum uncertainty is caused by the presence of the unphysical cutoff introduced by $|s\rangle$, which fails to decouple from the low-energy quantities for arbitrarily large but finite s [22]. Also, a large s limit of Eq. (4.6) is not defined consistently, which is analogous to the ill-defined continuum limit of Tr $\gamma_5=0$ in Eq. (1.8).

It is expected that an analogous unphysical result will appear in lattice gauge theory if one analyzes the low energy quantity which critically depends on the unphysical states $N_\pm$. In fact, it is known that one *has to* eliminate the contribution of the $N_\pm$ states to the physical observables such as $S(x)$ and $P(x)$ in Eq. (3.5) [16,6].

APPENDIX: FINITE-DIMENSIONAL REPRESENTATIONS OF THE GINSPARG-WILSON ALGEBRA

In this appendix we recapitulate the finite dimensional representations of the basic algebraic relation (1.1). A construction of the operator $\gamma_5 D$, which satisfies the Ginsparg-Wilson relation on a finite lattice, by using a corresponding operator $\gamma_5 D$ on an infinite lattice has been discussed in Ref. [20]. We first define an operator

$$\Gamma_5\equiv\gamma_5(1-\tfrac{1}{2}aD), \tag{A1}$$

which is Hermitian and satisfies the basic relation

$$\Gamma_5\gamma_5 D+\gamma_5 D\Gamma_5=0. \tag{A2}$$

This relation suggests that if

$$\gamma_5 D\phi_n=\lambda_n\phi_n,\quad(\phi_n,\phi_n)=1 \tag{A3}$$

then

$$\gamma_5 D(\Gamma_5\phi_n)=-\lambda_n(\Gamma_5\phi_n). \tag{A4}$$

That is, the eigenvalues λ_n and $-\lambda_n$ are always paired if $\lambda_n\neq 0$ and $(\Gamma_5\phi_n,\Gamma_5\phi_n)\neq 0$.

We evaluate the norm of $\Gamma_5\phi_n$:

$$\begin{aligned}(\Gamma_5\phi_n,\Gamma_5\phi_n)&=\left[\phi_n,\left(\gamma_5-\frac{a}{2}\gamma_5 D\right)\left(\gamma_5-\frac{a}{2}\gamma_5 D\right)\phi_n\right]\\&=\left[\phi_n,\left(1-\frac{a}{2}\gamma_5(\gamma_5 D+D\gamma_5)\right.\right.\\&\quad\left.\left.+\frac{a^2}{4}(\gamma_5 D)^2\right)\phi_n\right]\\&=\left[\phi_n,\left(1-\frac{a^2}{4}(\gamma_5 D)^2\right)\phi_n\right]\\&=\left(1-\frac{a}{2}\lambda_n\right)\left(1+\frac{a}{2}\lambda_n\right).\end{aligned} \tag{A5}$$

That is, ϕ_n is a ‘‘highest’’ state

$$\Gamma_5\phi_n=\left(\gamma_5-\frac{a}{2}\gamma_5 D\right)\phi_n=0 \tag{A6}$$

if $[1-(a/2)\lambda_n][1+(a/2)\lambda_n]=0$ for the Euclidean SO(4)-invariant positive definite inner product (ϕ_n,ϕ_n). We thus conclude that the states ϕ_n with $\lambda_n=\pm 2/a$ are *not* paired by the operation $\Gamma_5\phi_n$ and are the simultaneous eigenstates of γ_5, $\gamma_5\phi_n=\pm\phi_n$ respectively. One can also show that these eigenvalues λ_n are the maximum or minimum of the possible eigenvalues of $\gamma_5 D$. This is based on relation (1.5), $|a\lambda_n/2|=|\phi_n^\dagger\gamma_5\phi_n|\leqslant\|\phi_n\|\,\|\gamma_5\phi_n\|=1$.

On the other hand, the relation Tr $\gamma_5=0$, which is expected to be valid on a finite lattice leads to [by using Eq. (1.5)]

$$\begin{aligned}\mathrm{Tr}\,\gamma_5&=\sum_n \phi_n^\dagger\gamma_5\phi_n\\&=\sum_{\lambda_n=0}\phi_n^\dagger\gamma_5\phi_n+\sum_{\lambda_n\neq 0}\phi_n^\dagger\gamma_5\phi_n\\&=\sum_{\lambda_n=0}\phi_n^\dagger\gamma_5\phi_n+\sum_{\lambda_n\neq 0}\frac{a}{2}\lambda_n\\&=n_+-n_-+\sum_{\lambda_n\neq 0}\frac{a}{2}\lambda_n=0.\end{aligned}\tag{A7}$$

In the last line of this relation, all the states except for the states with $\lambda_n=\pm 2/a$ cancel pairwise for $\lambda_n\neq 0$. We thus obtain a chirality sum rule $n_+-n_-+N_+-N_-=0$ [12] or,

$$n_++N_+=n_-+N_- \tag{A8}$$

where $N_\pm$ stand for the number of isolated (unpaired) states with $\lambda_n=\pm 2/a$ and $\gamma_5\phi_n=\pm\phi_n$, respectively. These relations show that the chirality asymmetry at vanishing eigenvalues is balanced by the chirality asymmetry at the largest eigenvalues with $|\lambda_n|=2/a$.

We note that all other states with $0<|\lambda_n|<2/a$, which appear pairwise with $\lambda_n=\pm|\lambda_n|$ (note that $\Gamma_5(\Gamma_5\phi_n)=[1-(a\lambda_n/2)^2]\phi_n\propto\phi_n$ for $|a\lambda_n/2|\neq 1$), satisfy the relations

$$\begin{aligned}&\phi_n^\dagger\Gamma_5\phi_n=0,\\&\phi_n^\dagger\gamma_5\phi_n=\frac{a\lambda_n}{2},\\&\phi_m^\dagger\gamma_5\phi_n=0\quad\text{for}\quad\lambda_m\neq\lambda_n,\quad\lambda_m\lambda_n>0.\end{aligned}\tag{A9}$$

These states ϕ_n cannot be the eigenstates of γ_5 as $|a\lambda_n/2|<1$. The states $N_\pm$ saturate the index theorem commonly written in the form [4–6]

$$\mathrm{Tr}\left(\frac{-1}{2}a\gamma_5 D\right)=n_+-n_-; \tag{A10}$$

that is, only the states $N_\pm$ contribute to the left-hand side.

Those properties we analyzed so far in this appendix hold both for non-Abelian and Abelian gauge theories. We did not specify precise boundary conditions, since our analysis is valid once nontrivial zero modes appear for a given boundary condition. For an Abelian theory, one needs to introduce the gauge field configuration with suitable boundary conditions, which carries a nonvanishing magnetic flux, to generate a nontrivial index n_+-n_- [20]. Our analysis of the index in this appendix is formal, since it is well known that the Ginsparg-Wilson relation (1.1) by itself does not uniquely specify the index or the coefficient of chiral anomaly for a given gauge field configuration [24].

To summarize the analyses of the present appendix, all the normalizable eigenstates ϕ_n of $\gamma_5 D$ on a finite lattice are categorized into the following three classes.

(i) $n_\pm$ states,

$$\gamma_5 D\phi_n=0,\quad \gamma_5\phi_n=\pm\phi_n. \tag{A11}$$

(ii) $N_\pm$ states,

$$\gamma_5 D\phi_n=\pm\frac{2}{a}\phi_n,\quad \gamma_5\phi_n=\pm\phi_n,\quad\text{respectively.} \tag{A12}$$

(iii) Remaining states with $0<|\lambda_n|<2/a$,

$$\gamma_5 D\phi_n=\lambda_n\phi_n,\quad \gamma_5 D(\Gamma_5\phi_n)=-\lambda_n(\Gamma_5\phi_n), \tag{A13}$$

and the sum rule $n_++N_+=n_-+N_-$ holds.

All the $n_\pm$ and $N_\pm$ states are the eigenstates of D, $D\phi_n=0$ and $D\phi_n=(2/a)\phi_n$, respectively. If one denotes the number of states in (iii) by $2N_0$, the total number of states N is given by $N=2(n_++N_++N_0)$, which is expected to be a constant independent of background gauge field configurations.

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