Dispersion relations and rescattering effects in B nonleptonic decays

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Recently, the final state strong interactions in nonleptonic B decays were investigated in a formalism based on hadronic unitarity and dispersion relations in terms of the off-shell mass squared of the B meson. We consider a heuristic derivation of the dispersion relations in the mass variables using the reduction Lehmann-Symanzik-Zimmermann formalism and find a discrepancy between the spectral function and the dispersive variable used in recent works. The part of the unitarity sum which describes final state interactions is shown to appear as a spectral function in a dispersion relation based on the analytic continuation in the mass squared of one final particle. As an application, by combining this formalism with Regge theory and $SU(3)$ flavor symmetry we obtain constraints on the tree and the penguin amplitudes of the decay $B^0 \to \pi^+ \pi^-$. [S0556-2821(99)04817-1]

I. INTRODUCTION

Final state strong interactions are known to play an important role in weak nonleptonic decays. In particular, the interplay between the strong and the weak phases is required by many signals of direct CP violation in B decays. The presence of final state rescattering is relevant also for the extraction of the Cabibbo-Kobayashi-Maskawa (CKM) phases from time dependent asymmetries, and can modify the magnitude of some processes suppressed in the standard model such as $B \to \pi K$ transitions, thus reducing their sensitivity to new physics.

The effect of the final state interactions (FSIs) was assumed until recently to be small in the very energetic decays such as those of B meson to light pseudoscalar mesons, where the final particles get apart very quickly and have no time to interact strongly by soft multigluon exchanges. A related argument was based on the fact that at the high energy scale imposed by the mass of the B meson there is a suppression of rescattering due to the specific form of the Regge amplitudes dominant at this scale. Recently, this qualitative picture was challenged by a more detailed dynamical approach [1–4]. The crucial remark made in Ref. [1] is that, contrary to conventional expectations, the soft final state interactions do not disappear at the center of mass energy set by m_B. The analysis made in Refs. [1–4] is based on hadronic unitarity and very general features of high energy soft interactions. Consider the weak decay $B \to P_1 P_2$, where P_i are pseudoscalar mesons, and denote by $A_{B \to P_1 P_2}$ the amplitude of this process. In the most general way, the unitarity of the S matrix allows one to express the discontinuity of $A_{B \to P_1 P_2}$ as

$$\text{Disc} A_{B \to P_1 P_2} = \frac{1}{2i} [\langle P_1 P_2 | \mathcal{T} | B \rangle - \langle P_1 P_2 | \mathcal{T}^\dagger | B \rangle]$$

$$= \frac{1}{2} \sum_I \langle P_1 P_2 | \mathcal{T}^\dagger | I \rangle \langle I | \mathcal{T} | B \rangle, \quad (1)$$

where $\mathcal{T}$ is the transition operator ($S = 1 - i\mathcal{T}$), which describes both the weak and strong interactions. To first order, the weak hamiltonian $\mathcal{H}_w$ can appear either in the first matrix element of the product in the right hand side of Eq. (1), or in the second. The intermediate hadronic states $\{I\}$ depend of course on the place of $\mathcal{H}_w$. If $\mathcal{H}_w$ is acting in the matrix element containing B, $\{I\}$ denote hadronic states produced by the weak decay of B and connected to the final state $\{P_1 P_2\}$ by a strong rescattering. It is this configuration of the sum (1) which describes the final state interactions in B nonleptonic decays. Alternatively, the operator $\mathcal{H}_w$ can be located in the first matrix element in the unitarity sum, in which case $\{I\}$ are states produced by the strong decay of B, and connected to $\{P_1 P_2\}$ through a weak interaction. When all the particles are on shell these terms vanish, since B is stable with respect to strong interactions. The above remarks, though rather trivial, will be useful below for the discussion of the dispersion relations.

According to general principles, the decay amplitude $A_{B \to P_1 P_2}$ can be obtained from its discontinuity by means of a dispersion relation. This approach was considered in Refs. [2–4], where, neglecting possible subtractions, a dispersion relation of the following form was used:

$$A(m_B^2, m_1^2, m_2^2) = \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{\text{Disc} A(s + i\epsilon, m_1^2, m_2^2)}{s - m_B^2 - i\epsilon}. \quad (2)$$

In this relation and the similar ones written below, the limit $\epsilon \to 0$ is implicitly assumed.

We use here the notation $A_{B \to P_1 P_2} = A(m_B^2, m_1^2, m_2^2)$ to show explicitly the dependence of the decay amplitude on the external masses. As for the discontinuity (already divided by $2i$), it was taken in Refs. [2–4] as the "rescattering part" of the unitarity sum (1) discussed above, evaluated for an off-shell B meson of mass squared equal to s.

The dispersion relation (2) is based obviously on the analytic continuation of the decay matrix element with respect to

the mass of the initial meson B. We recall that dispersion relations in the external mass variables were derived in the frame of axiomatic field theory [5–8], and were used in phenomenological calculations of form factors [9,10] (see also Ref. [11]). Heuristic derivations of such dispersion relations are based on the Lehmann-Symanzik-Zimmermann (LSZ) reduction formalism [8], combined with causality and hadronic unitarity. As emphasized in Ref. [11] one must be cautious in using such dispersion relations, since the heuristic conjectures might be violated, in particular through the appearance of anomalous thresholds.

Of interest to the present work is the fact that the specific form of the dispersion relation and of its discontinuity depends on which of the external particles is reduced in the LSZ formula. By treating the matrix element of the decay $B \to P_1 P_2$ in the frame of this formalism, one can prove that the spectral function of a dispersion relation with respect to s (the mass squared of the B meson), as in Eq. (2), is given *not* by the ''rescattering'' part of Eq. (1), as it was assumed in Refs. [2–4], but by the second class terms in this sum. Moreover, one can also prove that the terms describing the final state interaction in the unitarity sum (1) appear as spectral function in a dispersion relation in terms of the mass squared of one of the final mesons (P_1 or P_2). Therefore, the calculation of the whole decay amplitude from its discontinuity proceeds along a different line than that applied in Refs. [2–4].

In the present paper, we provide arguments for the assertions made above and consider some applications. In the next section, using the LSZ formalism, we discuss the heuristic derivation of dispersion relations for the decay amplitude $A_{B \to P_1 P_2}$, when either B or one of the final mesons P_1 or P_2 are off-mass shell. We do not attempt to give rigorous proofs, but only to establish the correspondence between the dispersion variable and the expression of the absorbtive part. In Sec. III we consider in more details the approximation of two particle unitarity combined with Regge theory for strong interactions, and in Sec. IV we discuss some applications: first we briefly indicate how the conclusions of Ref. [4] are modified by the use of the adequate dispersion relation. Then, by combining the formalism with $SU(3)$ flavor symmetry we derive constraints on the amplitudes of the $B^0 \to \pi^+ \pi^-$ decay.

II. DISPERSION RELATIONS IN THE EXTERNAL MASS VARIABLE

We consider the weak decay amplitude $A_{B \to P_1 P_2}$ defined as

$$A(p^2, k_1^2, k_2^2) = \langle P_1(k_1) P_2(k_2), \text{out} | \mathcal{H}_w(0) | B(p) \rangle, \quad (3)$$

where we indicated the dependence on the Lorentz invariants p^2, k_1^2, and k_2^2 (for the physical amplitude $p^2 = m_B^2$, $k_1^2 = m_1^2$, $k_2^2 = m_2^2$). By applying the well known LSZ formalism [8] we ''reduce'' the particle P_1, which gives

$$A(m_B^2, k_1^2, m_2^2) = \frac{i}{\sqrt{2\omega_1}} \int dx \; e^{ik_1 x} \theta(x_0)$$

$$\times \langle P_2(k_2) | [\eta_1(x), \mathcal{H}_w(0)] | B(p) \rangle. \quad (4)$$

In this relation, $\eta_1(x)$ denotes the source of the meson P_1, defined as $\mathcal{K}_x \phi_1(x) = \eta_1(x)$, where $\mathcal{K}_x$ is the Klein-Gordon operator and $\phi_1(x)$ the interpolating field ($\omega_1 = \sqrt{\mathbf{k}_1^2 + m_1^2}$, is the energy of the on shell particle P_1). We use here and in what follows the fact that the single particle states in and out are identical. As shown in Refs. [5–8], due to the factor $\theta(x_0)$ the amplitude (4) can be extended as an analytic function in the upper half of the complex plane of the time component k_{10}. A more detailed analysis [7], exploiting also the causality properties of the retarded commutator and the relation $k_1^2 = k_{10}^2 + \mathbf{k}_1^2$, shows that the amplitude $A(m_B^2, k_1^2, m_2^2)$ can be extended as an analytic function in the whole complex plane k_1^2, cut along a part of the real axis, where its discontinuity is [5]

$$\text{Disc} A(m_B^2, k_1^2, m_2^2) = \frac{1}{2\sqrt{2\omega_1}} \int dx \; e^{ik_1 x} \langle P_2(k_2) |$$

$$\times [\eta_1(x), \mathcal{H}_w(0)] | B(p) \rangle. \quad (5)$$

This discontinuity coincides actually with the imaginary part Im $A(m_B^2, k_1^2 + i\epsilon, m_2^2)$ of the decay amplitude on the upper edge of the cut (it will be shown below that this spectral function is real). The right-hand side of Eq. (5) is treated in the standard way by inserting a complete set of states in the two terms of the commutator. By performing the translation

$$\eta_1(x) = e^{iPx} \eta_1(0) e^{-iPx}, \quad (6)$$

we write Eq. (5) as

$$\text{Im } A(m_B^2, k_1^2 + i\epsilon, m_2^2) = \frac{1}{2\sqrt{2\omega_1}} \int dx \; e^{ik_1 x}$$

$$\times \sum_n [e^{i(k_2 - p_n)x} \langle P_2(k_2) | \eta_1 | n \rangle$$

$$\times \langle n | \mathcal{H}_w | B(p) \rangle$$

$$- e^{i(p_n - p)x} \langle P_2(k_2) | \mathcal{H}_w | n \rangle$$

$$\times \langle n | \eta_1 | B(p) \rangle], \quad (7)$$

where we denoted $\mathcal{H}_w = \mathcal{H}_w(0)$ and $\eta_1 = \eta_1(0)$. The trivial integral with respect to x gives

$$\text{Im } A(m_B^2, k_1^2 + i\epsilon, m_2^2) = \frac{1}{2\sqrt{2\omega_1}} \sum_n [\delta(k_1 + k_2 - p_n)$$

$$\times \langle P_2(k_2) | \eta_1 | n \rangle \langle n | \mathcal{H}_w | B(p) \rangle$$

$$- \delta(k_1 + p_n - p) \langle P_2(k_2) | \mathcal{H}_w | n \rangle$$

$$\times \langle n | \eta_1 | B(p) \rangle]. \quad (8)$$

The states contributing to the first sum have the four-momentum $p_n = k_1 + k_2 = p$ and the invariant mass $p_n^2 = m_B^2$, they correspond to what we called above the "rescattering" part of Eq. (1). It is easy to see that this sum is nonzero for k_1^2 in the allowed interval $0 < k_1^2 < (m_B - m_2)^2$ (as we mentioned anomalous thresholds might be present). As the second sum in Eq. (8) is concerned, it gives a vanishing contribution, since B is stable with respect to the strong interactions.

We recall that by the reduction formula we obtained the analytic continuation of the physical matrix element with respect to the variable k_1^2 (the mass squared of an off-shell meson P_1). Therefore, the decay amplitude can be calculated from its discontinuity by means of a dispersion relation in this external mass variable. As discussed above, the integral extends along a finite interval, so the dispersion relation reads

$$A(m_B^2, m_1^2, m_2^2) = \frac{1}{\pi} \int_0^{(m_B - m_2)^2} dz \frac{\text{Im } A(m_B^2, z + i\epsilon, m_2^2)}{z - m_1^2 - i\epsilon},$$
(9)

with the discontinuity given by the first sum in Eq. (8).

Let us see now what is the form of the dispersion relation obtained in the LSZ formalism, when the analytic continuation is done with respect to the mass squared of the B meson. To this end, we start again from the matrix element $\langle P_1(k_1) P_2(k_2), \text{out} | \mathcal{H}_w(0) | B(p) \rangle$, and apply the LSZ formula, reducing this time the initial meson B. We obtain

$$A(p^2, m_1^2, m_2^2) = \frac{i}{\sqrt{2\omega_B}} \int dx \, e^{-ipx} \theta(-x_0)$$

$$\times \langle P_1(k_1) P_2(k_2), \text{out} | [\mathcal{H}_w(0), \eta_B(x)] | 0 \rangle.$$
(10)

By exploiting the causality properties of the retarded commutator one can prove [7] that the amplitude can be extended as a real analytic function in the complex plane $s = p^2$, with the discontinuity across the real axis given by

$$\text{Im } A(p^2 + i\epsilon, m_1^2, m_2^2)$$

$$= \frac{1}{2\sqrt{2\omega_B}}$$

$$\times \sum_n [\delta(p - p_n) \langle P_1(k_1) P_2(k_2), \text{out} | \mathcal{H}_w | n \rangle \langle n | \eta_B | 0 \rangle$$

$$- \delta(p_n) \langle P_1(k_1) P_2(k_2), \text{out} | \eta_B | n \rangle \langle n | \mathcal{H}_w | 0 \rangle]. \quad (11)$$

We obtained this result in the standard way [5], replacing $i\theta(-x_0)$ in Eq. (10) by $1/2$, and inserting a complete set of states in the commutator. Actually, the second sum in Eq. (11) vanishes because the only intermediate states allowed have zero four-momentum. In the first term, the allowed particles are those connected to the final state through a weak process and to B through a strong transition (we recall that the last matrix element vanishes when B is on the mass

shell). The lowest two particle state entering the unitarity sum is $B^*\pi$ which defines the normal unitarity threshold.

The spectral function (11) enters a dispersion relation of the form (2) with respect to $s = p^2$. However, it is obvious that such a dispersion relation is not useful for estimating the rescattering effects in nonleptonic B decays. As discussed above, in this way one does not describe the strong interactions in the final state, but rather the strong interactions in the initial state.

The fact that a dispersion relation in the mass squared of the B meson cannot describe final state rescattering effects is understood by simple qualitative arguments: in order to make the analytic continuation in the variable s we must reduce the B meson. Hence, the source η_B and the weak Hamiltonian $\mathcal{H}_w$ enter different matrix elements in the unitarity sum, and terms describing the weak decay of B multiplied by strong scattering amplitudes cannot appear. Therefore the procedure applied in Refs. [2–4], based on the analytic continuation in s combined with a discontinuity containing rescattering terms is not consistent. As we pointed out above the unitarity sum (1) defines the spectral function in a dispersion relation with respect to the mass variable of one of the final mesons. The relations (8) and (9) are the main results we obtained in the frame of the standard LSZ formalism.

A few comments about the above formulas are of interest. First, it is clear that one can repeat the procedure by reducing the meson P_2 instead of P_1. The corresponding expressions can be obtained easily from those given above by permutting the indices 1 and 2. The expressions seem different, but of course the results should be the same when a complete set of states is inserted in the unitarity sum.

A more subtle question, which is also connected to the completeness of the set inserted in the unitarity sum is whether the discontinuity defined in Eq. (8) is real or complex. For T (or CP) conserving interactions, the reality of the spectral function was proved a long time ago [12,13]. It turns out that the absorbtive part remains real even if the relevant terms in the weak Hamiltonian are not CP conserving. We take into account the fact that these terms have the form

$$\mathcal{H}_w = \sum_j c_j \mathcal{O}_j,$$
(12)

where c_j are complex numbers and $\mathcal{O}_j$ are products of V and A currents. Consider the spectral function

$$\sigma(z) = \text{Im } A(m_B^2, z + i\epsilon, m_2^2),$$
(13)

defined by the first sum in Eq. (8), and assume that a complete set of in states is inserted in the unitarity sum. Following Refs. [12,13] (see also Ref. [11]) we can express the two matrix elements in this sum as

$$\langle P_2(k_2) | \eta_1 | n, \text{in} \rangle = \langle P_2(k_2) | (PT)^{-1} (PT) \eta_1$$

$$\times (PT)^{-1} (PT) | n, \text{in} \rangle$$

$$= \langle P_2(k_2) | \eta_1 | n, \text{out} \rangle^* \qquad (14)$$

and

$$\langle n, \text{in}|\mathcal{H}_w|B(p)\rangle$$
$$= \langle n, \text{in}|(PT)^{-1}(PT)\mathcal{H}_w(PT)^{-1}(PT)|B(p)\rangle$$
$$= \langle n, \text{out}|\mathcal{H}_w|B(p)\rangle^*. \tag{15}$$

We used here the transformation properties of the V and A currents under P and T transformations and the fact that under space-time reversal the particles conserve their momenta, and the in (out) states become out (in) states, respectively. Moreover, the intrinsic parities of the states and the operators have a product equal to $+1$, and the matrix element are replaced by their complex conjugates, given the antiunitary character of the operator T. By using the relations (14) and (15) in Eq. (8) we obtain

$$\sigma(z) = \frac{1}{2\sqrt{2}\omega_1} \sum_n \delta(k_1 + k_2 - p_n)\langle P_2(k_2)|\eta_1|n, \text{in}\rangle$$
$$\times \langle n, \text{in}|\mathcal{H}_w|B(p)\rangle$$
$$= \frac{1}{2\sqrt{2}\omega_1} \left[\sum_n \delta(k_1 + k_2 - p_n)\langle P_2(k_2)|\eta_1|n, \text{out}\rangle \right.$$
$$\left. \times \langle n, \text{out}|\mathcal{H}_w|B(p)\rangle \right]^*$$
$$= \sigma^*(z), \tag{16}$$

where the equivalence between the complete sets of in and out states in the definition of $\sigma(z)$ was taken into account. From Eq. (16) it follows that the discontinuity is manifestly real only if the intermediate states form a complete set. If the unitarity sum is truncated, this property is lost, since various terms have complex phases which do not compensate each other in an obvious way. As noticed in Ref. [12], in order to maintain the proper reality condition at all stages of approximation, it is convenient to write the sum over the complete set of states $|n\rangle$ as a combination $1/2|n, \text{in}\rangle + 1/2|n, \text{out}\rangle$. This prescription will be applied in Sec. IV when discussing the $B \rightarrow \pi\pi$ decay.

III. TWO-PARTICLE UNITARITY AND REGGE AMPLITUDES

In this section we write down the dispersion relation (9) in the approximation that only two particle states are kept in the unitarity sum (8). Denoting by $\{P_3P_4\}$ the two meson intermediate states in this sum, the off-shell imaginary part of the decay amplitude $A_{B \rightarrow P_1P_2}$ required in the dispersion relation (9) reads

Im $A_{B \rightarrow P_1P_2}(m_B^2, z + i\epsilon, m_2^2)$

$$= \frac{1}{2} \sum_{\{P_3P_4\}} \int \frac{d^3\mathbf{k}_3}{(2\pi)^3 2\omega_3} \frac{d^3\mathbf{k}_4}{(2\pi)^3 2\omega_4} (2\pi)^4$$
$$\times \delta^{(4)}(p - k_3 - k_4) A_{B \rightarrow P_3P_4}(m_B^2, m_3^2, m_4^2)$$
$$\times \mathcal{M}^*_{P_3P_4 \rightarrow P_1P_2}(s,t). \tag{17}$$

In the sum we include the two particle states $P_3P_4 = P_1P_2$ defining the elastic channel, as well as $P_3P_4 \neq P_1P_2$ responsible for the inelastic scattering. Let us note that the c.m. energy is set up by the mass of the B meson ($\sqrt{s} = m_B = 5.2$ GeV), and the weak decay amplitudes $A_{B \rightarrow P_3P_4}$ are on shell, and independent on the Mandelstam variable t (or the rescattering angle θ). Therefore Eq. (17) can be written as

Im $A_{B \rightarrow P_1P_2}(m_B^2, z + i\epsilon, m_2^2)$

$$= \sum_{\{P_3P_4\}} C^*_{P_3P_4; P_1P_2}(z) A_{B \rightarrow P_3P_4}(m_B^2, m_3^2, m_4^2), \tag{18}$$

where

$$C_{P_3P_4; P_1P_2}(z) = \frac{1}{2} \int \frac{d^3\mathbf{k}_3}{(2\pi)^3 2\omega_3} \frac{d^3\mathbf{k}_4}{(2\pi)^3 2\omega_4} (2\pi)^4$$
$$\times \delta^{(4)}(p - k_3 - k_4) \mathcal{M}_{P_3P_4 \rightarrow P_1P_2}(s,t). \tag{19}$$

These coefficients depend on the masses of all the particles participating in the rescattering process. To simplify the notation we indicate explicitly only the dependence on the off shell mass squared z of the particle P_1. Following Refs. [1–4] we adopt for the strong amplitudes $\mathcal{M}$ the parametrizations obtained from the Regge theory [14]

$$\mathcal{M}_{P_3P_4; P_1P_2}(s,t) = - \sum_{V=P,f,A_2,K_2^* \cdots} \gamma^V_{P_3P_4; P_1P_2}(t)$$
$$\times \frac{e^{-i[\pi\alpha_V(t)/2]}}{\sin[\pi\alpha_V(t)/2]} \left(\frac{s}{s_0}\right)^{\alpha_V(t)}$$
$$+ \sum_{V=\rho,K^* \cdots} i\gamma^V_{P_3P_4 \rightarrow P_1P_2}(t)$$
$$\times \frac{e^{-i[\pi\alpha_V(t)/2]}}{\cos[\pi\alpha_V(t)/2]} \left(\frac{s}{s_0}\right)^{\alpha_V(t)}, \tag{20}$$

where the first sum includes $C = 1$ trajectories and the second one $C = -1$ trajectories. As usual we take $s_0 \approx 1$ GeV^2 and linear trajectories

$$\alpha_V(t) = \alpha_0 + \alpha' t, \tag{21}$$

with the standard choices [14]

$$\alpha_0 = 1.08, \quad \alpha' = 0.25 \ \text{GeV}^{-2} \tag{22}$$

for the Pomeron, and

$$\alpha_0 = 0.45, \quad \alpha' = 0.94 \ \text{GeV}^{-2} \tag{23}$$

for all the other trajectories. The possible divergences occurring in the expression (20) for $t \neq 0$ are avoided by taking $\alpha(t) \approx \alpha_0$ in the denominators. We shall therefore obtain

$$\sin \frac{\pi \alpha_P(t)}{2} \approx 1,$$

$$\sin \frac{\pi \alpha_V(t)}{2} \approx \frac{1}{\sqrt{2}}, \quad V = f, \ A_2, \ K_2^*,$$

$$\cos \frac{\pi \alpha_V(t)}{2} \approx \frac{1}{\sqrt{2}}, \quad V = \rho, \ K^*. \tag{24}$$

As far as the Regge residua $\gamma^V_{P_3 P_4 ; P_1 P_2}(t)$ are concerned, they are supposed to satisfy the factorization relation

$$\gamma^V_{P_3 P_4 ; P_1 P_2}(t) = \gamma_{P_3 P_1 V}(t) \gamma_{P_4 P_2 V}(t), \tag{25}$$

and their values at $t = 0$ will be determined using the optical theorem and the phenomenological Regge-like parametrizations of the total cross sections [15,16] (details will be given in the next section). The t dependence of these functions is, however, poorly known and we assume they are simply constants.

In order to perform the integration (19), the Mandelstam variable t is expressed in terms of the scattering angle θ:

$$t(z) = t_0(z) + 2 k_{12}(z) k_{34} \cos \theta, \tag{26}$$

with

$$t_0(z) = z + m_3^2 - \frac{(m_B^2 + m_3^2 - m_4^2)(m_B^2 + z - m_2^2)}{2 m_B^2},$$

$$k_{12}(z) = \frac{1}{2 m_B} \sqrt{(m_B^2 - z - m_2^2)^2 - 4 z m_2^2},$$

$$k_{34} = \frac{1}{2 m_B} \sqrt{(m_B^2 - m_3^2 - m_4^2)^2 - 4 m_3^2 m_4^2}, \tag{27}$$

where we indicated explicitly only the dependence on the variable z. With these kinematic variables the integration over the momenta $\mathbf{k}_3$ and $\mathbf{k}_4$ in Eq. (19) is straightforward, and the coefficients $C_{P_3 P_4 ; P_1 P_2}$ can be expressed as

$$C_{P_3 P_4 ; P_1 P_2}(z) = \sum_V \gamma^V_{P_3 P_4 ; P_1 P_2}(0) \kappa^V_{P_3 P_4 ; P_1 P_2}, \tag{28}$$

where

$$\kappa^V_{P_3 P_4 ; P_1 P_2}(z) = \xi_V \frac{k_{34}}{16 \pi m_B} \mathcal{R}_V^{-1}(z) [e^{\mathcal{R}_V(z)} - e^{-\mathcal{R}_V(z)}]$$

$$\times \exp \left[(\alpha_{0,V} + \alpha'_V t_0) \left(\ln \frac{m_B^2}{s_0} - i \frac{\pi}{2} \right) \right], \tag{29}$$

$$\mathcal{R}_V(z) = 2 \alpha'_V k_{12}(z) k_{34} \left(\ln \frac{m_B^2}{s_0} - i \frac{\pi}{2} \right), \tag{30}$$

and ξ_V is a numerical factor equal to -1 for the Pomeron, $i\sqrt{2}$ for $C = -1$ trajectories, and $-\sqrt{2}$ for $C = 1$ physical trajectories.

By inserting the expression (18) of the spectral function in the dispersion relation (9) and recalling that the decay amplitudes $A_{B \to P_3 P_4}$ do not depend on z, we obtain (for simplicity we omit now the mass arguments when the amplitudes are on-shell)

$$A_{B \to P_1 P_2} = \sum_{\{P_3 P_4\}} \bar{\Gamma}_{P_3 P_4 ; P_1 P_2} A_{B \to P_3 P_4}, \tag{31}$$

where

$$\bar{\Gamma}_{P_3 P_4 ; P_1 P_2} = \sum_V \gamma^V_{P_3 P_4 ; P_1 P_2}(0) \bar{\eta}^V_{P_3 P_4 ; P_1 P_2} \tag{32}$$

and

$$\bar{\eta}^V_{P_3 P_4 ; P_1 P_2} = \frac{1}{\pi} \int_0^{(m_B - m_2)^2} dz \frac{\kappa^{V*}_{P_3 P_4 ; P_1 P_2}(z)}{z - m_1^2 - i \epsilon}. \tag{33}$$

For further use we also define

$$\Gamma_{P_3 P_4 ; P_1 P_2} = \sum_V \gamma^V_{P_3 P_4 ; P_1 P_2}(0) \eta^V_{P_3 P_4 ; P_1 P_2} \tag{34}$$

and

$$\eta^V_{P_3 P_4 ; P_1 P_2} = \frac{1}{\pi} \int_0^{(m_B - m_2)^2} dz \frac{\kappa^V_{P_3 P_4 ; P_1 P_2}(z)}{z - m_1^2 - i \epsilon}. \tag{35}$$

Before considering applications, let us make a few comments about the approximations adopted when deriving the above formulas. First, we notice that the Regge expression (20) is valid for large s and t close to 0. The value $s = m_B^2$ satisfies this condition, but the values of t appearing in the integral upon the scattering angle in Eq. (19) can be large, outside the range of validity of the Regge theory. However, the hadronic amplitudes decrease at large $|t|$, so the contribution of the large scattering angles in the unitarity integral is expected to be small and not very sensitive to the inaccuracy of the dynamical model. Another difficulty is related to the fact that the particle P_1 is off shell, and its mass $k_1^2 = z$ becomes very large at the upper limit of integration in the dispersion relation (9). Here again, one of the assumptions for the validity of the Regge expression, namely, $\sqrt{s} \gg m_i$ [14] is not met.

However, this part of the integral, which is not correctly evaluated, brings a small contribution in the dispersion integral due to the denominator in Eq. (9) (this statement is true when the masses of the intermediate particles P_3 and P_4 are not too large). We emphasize that the main advantage of the formalism is that it provides, with no approximation, an algebraic relation involving only physical decay amplitudes. Indeed, as we mentioned, all the quantities $A_{B \to P_3 P_4}$ appearing in Eq. (31) are on shell, the dynamical approximations affecting only the coefficients $\bar{\Gamma}_{P_3 P_4; P_1 P_2}$.

IV. CONSTRAINTS ON THE AMPLITUDES OF $B^0 \to \pi^+ \pi^-$ DECAY

Unitarity and the dispersion relations were used in previous works [1–4] in order to estimate the FSI corrections to the decay amplitudes calculated in an approximation which does not include strong rescattering (such as, for instance, factorization). Such an evaluation was made in Ref. [4] for the magnitude of final state interactions in $B^- \to \pi^- \bar{K}^0$. In the notations used in Sec. III, it corresponds to $P_1 P_2 = \pi^- \bar{K}^0$, with the intermediate states $P_3 P_4 = \pi^0 K^-$ and ηK^0 inserted in the unitarity sum. Then only physical trajectories ($V = \rho$, K^*) contribute to Eq. (20), for which the intercept and the slope are given in Eq. (23).

With values of the residua extracted from the phenomenological parametrization of the cross sections the authors of Ref. [4] suggested a modification of the magnitude of the amplitude of the decay $B^- \to \pi^- \bar{K}^0$ by a factor of 10%. However, the coefficients $\bar{\Gamma}_{P_3 P_4; P_1 P_2}$ appearing in a relation of the type (31) were calculated in Ref. [4] by using the rescattering absorbtive function in a dispersion relation with respect to the variable s [see Ref. [4], Eq. (2.17)]. By performing the correct calculations with the same values of the parameters, we find that the coefficient $\bar{\Gamma}_{\pi^0 K^-; \pi^- \bar{K}^0}$, for instance, is larger by a factor of about 2.5 compared to the value reported in Ref. [4]. This shows that the correct treatment can modify the conclusions about the magnitude of FSI in $B \to \pi K$ decays.

In the present paper we consider an application of the dispersive formalism to the decay $B^0 \to \pi^+ \pi^-$. The time dependent CP asymmetry in this decay is considered as one of the ways of extracting the angle α of the unitarity triangle [17]. However, the unknown strong phase difference between the tree and the penguin amplitudes of the process affects the accuracy of this determination. Additional theoretical constraints on these amplitudes would be very helpful for reducing the uncertainty of the method. As we shall show below, the dispersion relations can provide such a constraint. We investigate the problem by combining the relations (18) and (31) derived above with isospin or $SU(3)$ symmetry [18,19]. The idea is that by unitarity and dispersion relations we obtain a set of correlations between exact decay amplitudes, containing both weak and strong phases. By imposing in addition $SU(3)$ flavor symmetry, all the amplitudes can be expressed in terms of a small number of parameters, for which unitarity and the dispersion relations provide non-

trivial constraints. Following Ref. [18] we write most generally the amplitude of the decay $B^0 \to \pi^+ \pi^-$ as a sum of diagram contributions

$$A_{B^0 \to \pi^+ \pi^-} = -(A_T e^{i\gamma} + A_P e^{-i\beta} + A'_P e^{i\gamma} + A_E e^{i\gamma} + A_{PA} e^{-i\beta}), \qquad (36)$$

where A_T, $A_P(A'_P)$, A_E, and A_{PA} denote the amplitudes of the tree, penguin, exchange and penguin annihilation diagrams, respectively. We indicated explicitly the weak phases defined as $\beta = \mathrm{Arg}(-V_{td}^*)$ and $\gamma = \mathrm{Arg}(-V_{ub}^*)$ [16]. As intermediate states $P_3 P_4$ in Eqs. (17) and (31) we keep $\pi^+ \pi^-$ giving the elastic channel, as well as two meson states responsible for the soft inelastic scattering, e.g., $\pi^0 \pi^0$, $K^+ K^-$, $K^0 \bar{K}^0$, $\pi^0 \eta_8$, and $\eta_8 \eta_8$ [η_8 is the η, η' superposition belonging to the $SU(3)$ octet]. We notice that $\pi^0 \eta_8$ will not contribute finally due to isospin conservation in the strong rescattering. Of course, besides these states, other inelastic channels, like for instance multipion states can contribute. The states $D^+ D^-$ and $D^0 \bar{D}^0$ are not included because they contribute to the hard scattering [1].

Assuming $SU(3)$ flavor symmetry we express the decay amplitudes $B \to P_3 P_4$ of interest as [18,19]

$$A_{B^0 \to \pi^0 \pi^0} = \frac{1}{\sqrt{2}}(-A_C e^{i\gamma} + A_P e^{-i\beta} + A'_P e^{i\gamma} + A_E e^{i\gamma}),$$

$$A_{B^0 \to K^+ K^-} = -A_E e^{i\gamma},$$

$$A_{B^0 \to K^0 \bar{K}^0} = A_P e^{-i\beta} + A'_P e^{i\gamma},$$

$$A_{B^0 \to \pi^0 \eta_8} = -\frac{1}{\sqrt{3}}(A_P e^{-i\beta} + A'_P e^{i\gamma} - A_E e^{i\gamma}),$$

$$A_{B^0 \to \eta_8 \eta_8} = \frac{1}{3\sqrt{2}}(A_C e^{i\gamma} + A_P e^{-i\beta} + A'_P e^{i\gamma} + A_E e^{i\gamma}),$$

$$(37)$$

where A_T, A_P, A'_P, and A_E are the same as in Eq. (36) and A_C denotes the amplitude of the tree color suppressed diagrams.

Because of the lack of detailed dynamical calculations, various phenomenological assumptions are made in the literature about the above amplitudes. The conservative bound $|A_P/A_T| < 1$ for the ratio of the penguin and tree amplitudes is mentioned in Ref. [19] (a more specific estimate $|A_P/A_T| \approx 0.2$ is also quoted in this reference). The penguin annihilation amplitude A_{PA} corresponds to Okubo-Zweig-Iizuka- (OZI)-suppressed diagrams [19], while A_C and A_E are color suppressed by a factor of about 0.25 with respect to the corresponding color favored amplitude. Finally, the two penguin amplitudes A_P and A'_P are assumed to satisfy $|A'_P/A_P| \approx 0.4$ [19]. The presence of strong rescattering is expected to modify these qualitative estimates based on quark diagrams [20,21]. In particular, this conclusion was obtained for the $B \to \pi K$ decays in a model based on isospin

symmetry in the quasielastic approximation [22,23]. A similar analysis for $B \to \pi\pi$ was not yet performed, but it is quite possible that the contribution of the ''suppressed'' amplitudes A'_P, A_C, A_E, and A_{PA} is enhanced by the rescattering effects. Our approach, based on unitarity and dispersion relations, aims actually to bring clarifications on the magnitude of various amplitudes. The next step is to introduce the decay amplitudes $A_{B \to P_i P_j}$ discussed above in the dispersion relation (31). We should recall, however, that due to the truncation of the unitarity sum, the imaginary part of the decay amplitude $A_{B \to P_1 P_2}$ obtained from Eq. (31) [or equivalently from the unitarity relation (18) evaluated on-shell] might be not real. In order to avoid this situation we apply the procedure suggested in Ref. [12] which maintains the proper reality condition of the spectral function and simulates the effect of other inelastic channels. As discussed at the end of Sec. II, this method amounts to insert in the unitarity sum the complete set of states $1/2|\text{in}\rangle + 1/2|\text{out}\rangle$. We recall that this method was applied to include inelastic effects through complex phases in the dispersive analysis of the electromagnetic form factors [10,11]. In our case this procedure yields, instead of Eq. (31), the modified dispersion relation

$$A_{B \to P_1 P_2} = \frac{1}{2} \sum_{\{P_3 P_4\}} \Gamma_{P_3 P_4 ; P_1 P_2} A^*_{B \to P_3 P_4}$$
$$+ \frac{1}{2} \sum_{\{P_3 P_4\}} \bar{\Gamma}_{P_3 P_4 ; P_1 P_2} A_{B \to P_3 P_4}, \quad (38)$$

where $\Gamma_{P_3 P_4 ; P_1 P_2}$ and $\bar{\Gamma}_{P_3 P_4 ; P_1 P_2}$ are defined in Eqs. (34) and (32), respectively. We notice that Eq. (38) can be split in two relations, one for the real part and another for the imaginary part of the decay amplitude. In particular, the relation giving the imaginary part is

$$iA^*_{B \to P_1 P_2} - iA_{B \to P_1 P_2} = \sum_{\{P_3 P_4\}} C_{P_3 P_4 ; P_1 P_2}(m_\pi^2) A^*_{B \to P_3 P_4}$$
$$+ \sum_{\{P_3 P_4\}} C^*_{P_3 P_4 ; P_1 P_2}(m_\pi^2) A_{B \to P_3 P_4}, \quad (39)$$

and can be obtained also directly from the unitarity relation (18) evaluated on shell. Concerning the real part, it is obtained by taking the principal value of the dispersion integrals appearing in Eqs. (33) and (35).

We describe now briefly the determination of the Regge residua $\gamma^V_{P_3 P_4 ; P_1 P_2}(0)$ which enters the expressions (32) and (34) of the coefficients of the dispersion relation (38). We use the optical theorem and the Regge parametrization of the total hadronic cross sections [15,16], which gives

$$\frac{s_0}{s} \text{Im} \, \mathcal{M}_{f \to f}(s,0) \approx s_0 \sigma_{\text{tot}} = s_0 X \left(\frac{s}{s_0} \right)^{\alpha_P(0)-1}$$
$$+ s_0 Y \left(\frac{s}{s_0} \right)^{\alpha(0)-1}, \quad (40)$$

TABLE I. Contributing trajectories to various hadronic processes.

Process	Contributing trajectories
$\pi^- p$	$P + f + f' - \rho$
$\pi^+ p$	$P + f + f' + \rho$
$\bar{p}p$	$P + f + f' + \rho + \omega + \phi + A_2$
$\bar{p}n$	$P + f + f' - \rho + \omega + \phi - A_2$
pp	$P + f + f' - \rho - \omega - \phi + A_2$
pn	$P + f + f' + \rho - \omega - \phi - A_2$

where $s_0 \approx 1\,\text{GeV}^2 \approx 1/0.38\ \text{mb}^{-1}$. The first term represents the Pomeron contribution, the second the contributions of all the other trajectories. By comparing Eq. (40) with the Regge parametrization (20) we obtain

$$s_0 X = \gamma^P_{f;f}, \quad s_0 Y = \sum_{V \neq P} \gamma^V_{f;f}.$$

For the Pomeron, which contributes to the elastic $\pi^+ \pi^-$ channel, we assumed that the coupling constant is proportional to the number of quarks, taking as in Ref. [1] $\gamma^P_{\pi^+ \pi^- ; \pi^+ \pi^-} = (\frac{2}{3})^2 s_0 X_{NN}$. The residua of the physical trajectories (which in our case are ρ, f, $f' K^*$, K_2^*, and A_2) were estimated by taking into account the factorization property (25) combined with the experimental data on several hadron-hadron scattering processes. We used the processes given in Table I, for which we wrote the contributions of various trajectories as in Ref. [14]. Using the NN channels we write in particular

$$s_0 Y_{pp} = \gamma^2_{N\bar{N}f} - \gamma^2_{N\bar{N}\rho} - \gamma^2_{N\bar{N}\omega} - \gamma^2_{N\bar{N}\phi} + \gamma^2_{N\bar{N}A_2},$$

$$s_0 Y_{pn} = \gamma^2_{N\bar{N}f} + \gamma^2_{N\bar{N}\rho} - \gamma^2_{N\bar{N}\omega} - \gamma^2_{N\bar{N}\phi} - \gamma^2_{N\bar{N}A_2}. \quad (41)$$

Noticing that experimentally $Y_{pp} \approx Y_{pn}$ [16] we obtain

$$\gamma^2_{N\bar{N}A_2} \approx \gamma^2_{N\bar{N}\rho} \quad (42)$$

and further

$$s_0 (Y_{\bar{p}p} - Y_{\bar{p}n}) = 2 \gamma^2_{N\bar{N}\rho} + 2 \gamma^2_{N\bar{N}A_2} \approx 4 \gamma^2_{N\bar{N}\rho}. \quad (43)$$

Also, by replacing the contributions of f and f' with the octet member f_8, we obtain

$$s_0 (Y_{\pi^- p} - Y_{\pi^+ p}) = 2 \gamma_{\pi^+ \pi^- \rho} \gamma_{N\bar{N}\rho},$$

$$s_0 (Y_{\pi^- p} + Y_{\pi^+ p}) = 2 \gamma_{\pi^+ \pi^- f_8} \gamma_{N\bar{N}f_8},$$

$$s_0 (Y_{pn} + Y_{pp} + Y_{\bar{p}n} + Y_{\bar{p}p}) = 4 \gamma^2_{N\bar{N}f_8}. \quad (44)$$

The coupling constants $\gamma^2_{\pi^+ \pi^- \rho^0}$ and $\gamma^2_{\pi^+ \pi^- f_8}$ can be easily calculated from these equations using the experimental values of the parameters X and Y for πN and NN scattering

TABLE II. The values of the Regge residua and of the coefficients entering the dispersion relation.

P_3P_4	V	$\gamma^V_{P_3P_4}$	$\kappa^V_{P_3P_4;P_1P_2}$	$\eta^V_{P_3P_4;P_1P_2}$	$\bar{\eta}^V_{P_3P_4;P_1P_2}$
$\pi^+\pi^-$	P	25.6	$-0.0089+0.0270i$	$-0.0547+0.0768i$	$-0.0007-0.0946i$
$\pi^+\pi^-$	ρ	31.4	$-0.0005-0.0015i$	$0.0001-0.0051i$	$-0.0029+0.0040i$
$\pi^+\pi^-$	f_8	35.3	$-0.0015+0.0005i$	$-0.0051-0.0001i$	$-0.0040-0.0029i$
$\pi^0\pi^0$	ρ	31.4	$-0.0005-0.0015i$	$0.0001-0.0051i$	$-0.0029+0.0040i$
$\bar{K}^0K^0$	K^*	15.7	$-0.0003-0.0007i$	$-0.0004-0.0030i$	$-0.0019+0.0024i$
$\bar{K}^0K^0$	K_2^*	52.9	$-0.0007+0.0003i$	$-0.0030+0.0004i$	$0.0024-0.0019i$
$\eta_8\eta_8$	A_2	35.3	$-0.0015+0.0005i$	$-0.0051-0.0001i$	$-0.0040-0.0029i$

[16]. Other coupling constants we need are obtained from the previous ones by using $SU(3)$ symmetry, namely,

$$\gamma^2_{\pi^0\pi^-\rho^-}=\gamma^2_{\pi^+\pi^-\rho^0},$$

$$\gamma^2_{\bar{K}^0\pi^-K^{*-}}=\frac{1}{2}\gamma^2_{\pi^+\pi^-\rho^0},$$

$$\gamma^2_{\bar{K}^0\pi^-K_2^{*-}}=\frac{3}{2}\gamma^2_{\pi^+\pi^-f_8},$$

$$\gamma^2_{\pi\eta_8A_2}=\gamma^2_{\pi^+\pi^-f_8}. \tag{45}$$

In Table II we give the values of the Regge residua calculated with the aid of the above relations using the parameters of the total cross sections quoted in Ref. [16], the coefficients $\kappa^V_{P_3P_4;P_1P_2}$ are defined in Eq. (29), while the coefficients $\bar{\eta}^V_{P_3P_4;P_1P_2}$, $\eta^V_{P_3P_4;P_1P_2}$ are given by Eqs. (33) and (35), respectively. We notice that the dominant contribution is brought by the elastic channel, and in particular by the Pomeron exchange.

With the numbers given in Table II, the real and imaginary parts of the dispersion relation (38) can be completely evaluated, yielding two algebraic relations for the decay amplitudes written in Eqs. (36) and (37). Of course, the predictive power of the method is increased if it is supplemented by some correlations among the amplitudes in the right-hand side of these equations, or if some amplitudes are very small and can be neglected. As discussed above, it is most probable that the naive estimates for these amplitudes based on the diagrammatic method are no longer valid when the

strong interactions are taken into account. However, in order to illustrate the method, we shall make first a crude approximation, keeping in Eqs. (36) and (37) only the contribution of the tree and the penguin diagrams A_T and A_P, whose strong phases are denoted as δ_T and δ_P, respectively. A more complete treatment exploiting both dispersion relations and estimates based on isospin symmetry as in Refs. [22,23] will be performed in a future work. Let us write

$$\frac{A_P}{A_T}=Re^{i\delta}, \tag{46}$$

where $R=|A_P/A_T|$ and $\delta=\delta_P-\delta_T$. For simplicity, let us make also the additional assumption that the tree amplitude A_T is real. Then this amplitude can be factored out from the dispersion relation (38), which gives the following two constraints for the ratio R and the phase difference δ of the penguin and tree amplitudes describing the $B^0\to\pi^+\pi^-$ decay:

$$1.1189\sin(\gamma+0.721)+R\sin(\delta-\beta+0.686)=0,$$

$$-1.239\sin(\gamma-0.0403)+R\sin(\delta-\beta+1.452)=0. \tag{47}$$

These equations can be explicitly solved as

$$\delta=\beta+\epsilon(\gamma),$$

$$R=-1.1189\frac{\sin(\gamma+0.721)}{\sin[\epsilon(\gamma)+0.686]}, \tag{48}$$

where

$$\epsilon(\gamma)=-\arctan\left[\frac{\sin(\gamma-0.0403)\sin0.686+0.9027\sin(\gamma+0.721)\sin1.452}{\sin(\gamma-0.0403)\cos0.686+0.9027\sin(\gamma+0.721)\cos1.452}\right]. \tag{49}$$

We recall that in these relations β and γ are the angles of the unitarity triangle which are expected [24,25] to be in the ranges $0.17\leq\beta\leq0.52$ and $0.349\leq\gamma\leq2.79$.

In Figs. 1 and 2 we represent the strong phase difference δ as a function of γ for two values of β at the limits of the

allowed intervals mentioned above, and the ratio R as a function of γ, according to Eq. (48). One can see that despite the crude approximations we made, the results are qualitatively reasonable. We notice that the equation for δ is not restrictive for the weak angles, while the expected condition R
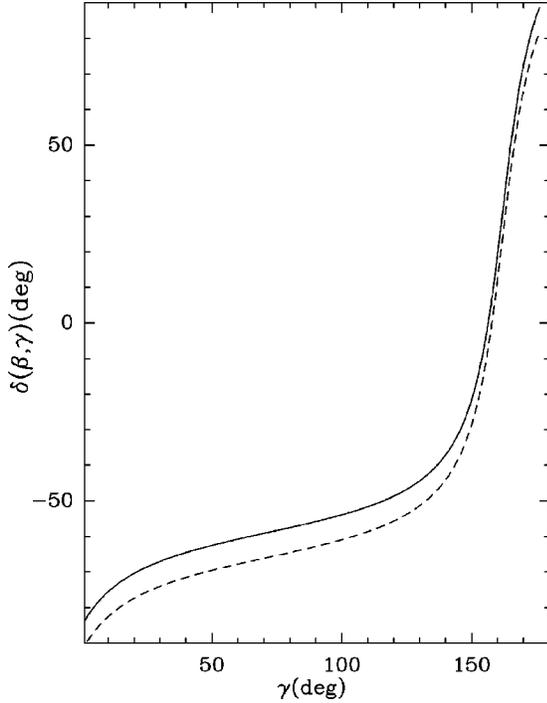
FIG. 1. The strong phase difference as function of γ, solid curve $\beta = 1°$, dashed curve $\beta = 174°$.

FIG. 2. The ratio R as a function of γ.

<1 is satisfied only for a small range above $\gamma = 2.4$. However, this somewhat intriguing limitation disappears if we relax the last approximation made above, namely that the tree amplitude A_T is real. It can be easily seen that by allowing a nonzero δ_T in the dispersion relation we get two equations of the form (47), with γ replaced by $\gamma + \delta_T$ and δ replaced by δ_P. The two constraints similar to Eq. (47) now involve three parameters R, δ_T, and δ_P.

V. CONCLUSIONS

In the present paper we investigated a recent treatment of the final state interactions in the nonleptonic B decays, based on unitarity and dispersion relations [1–4]. By considering the analytic continuation in the external mass variable in the frame of LSZ formalism, we established the connection between the dispersion variable and the part of the unitarity sum defining the spectral function. The strong rescattering part is shown to appear as a discontinuity in a dispersion relation in terms of the mass of one final particle. Our results prove that the dispersion relations written in Refs. [1–4], based on the analytic continuation in the mass of B, are not consistent. We derived the correct dispersion relation, and showed that it modifies the conclusions of Ref. [4] on the magnitude of FSI effects in $B \rightarrow \pi K$ decay by a factor of approximately 2.5. We also applied the formalism to derive a theoretical constraint for the amplitudes of the $B^0 \rightarrow \pi^+ \pi^-$ decay. We included in the unitarity sum a few channels, connecting them by the $SU(3)$ symmetry [17,18] and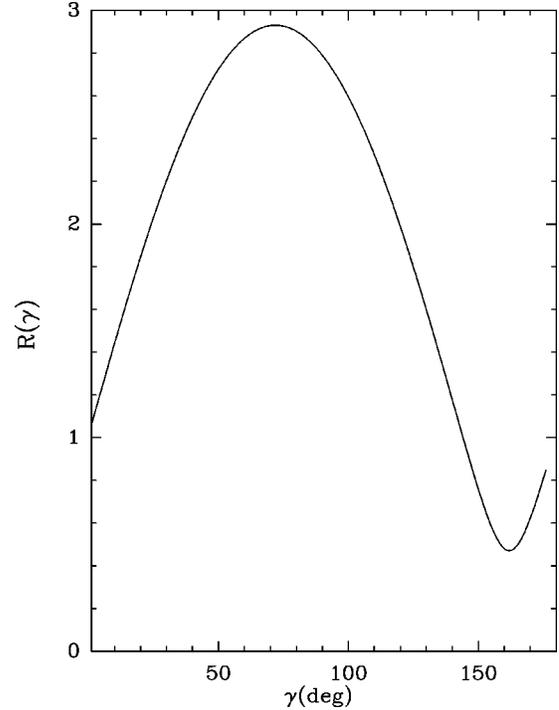 took into account qualitatively the effect of higher inelastic channels by a procedure applied in the study of the electromagnetic form factors [13,10]. In spite of the various dynamical assumptions mentioned in the text our results (48) for the ratio R and the strong relative phase δ of the penguin and tree amplitudes in terms of the weak angles (represented in Figs. 1 and 2) are qualitatively reasonable. As we mentioned, these results can be immediately modified to incorporate a nonzero strong phase for the tree amplitude. A more complete analysis must include also the amplitudes neglected in the present analysis, which are suppressed in the diagrammatic approach but may be enhanced by the strong interactions. Such an investigation is in progress. The results might be useful as additional constraints in the extraction of the angles of the unitarity triangle from the time dependent CP asymmetry in the decay $B^0 \rightarrow \pi^+ \pi^-$.

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