Phenomenological predictions of the properties of the B_c system

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We present a comprehensive calculation of the energies, splittings and electromagnetic decays of the low-lying levels of the bottom charmed meson system. In order to incorporate running coupling constant effects, we choose Richardson's potential for the central potential and take the spin-dependent potentials from the radiative one-loop calculation of Pantaleone, Tye and Ng. The effects of a nonperturbative spin-orbit potential are also included. Our parameters are determined from the low-lying levels of charmonium (average deviation of 19.9 MeV) and of the upsilon system (average deviation of 4.3 MeV). We carry out a detailed comparison with the calculations of Eichten and Quigg and the lattice calculations of the NRQCD Collaboration. Our predicted result for the ground state energy is 6286^{+15}_{-6} MeV. Our results are generally in agreement with earlier calculations. However, we find the two lowest 1^+ states to be very close to the j-j limit, in contrast with some of the earlier calculations. The implications of this finding for the photon spectra of the $1P$ and $2S$ states are discussed in detail. Some strategies for the observation of these states are discussed, and a table of their cascades to the ground state is presented. Our calculated value for the ground state lifetime is 0.38 ± 0.03 ps, in good agreement with the recent CDF measurement. [S0556-2821(99)03517-1]

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I. INTRODUCTION

The Collider Detector at Fermilab (CDF) Collaboration [1] has reported the discovery of the B_c system in 1.8 TeV $p-\bar{p}$ collisions at Fermilab. They have observed about 20 decays in the J/ψ-lepton channel, which are interpreted as decays of the ground state. For the mass of the ground state, the CDF collaboration quotes $M_{B_c} = 6.40 \pm 0.39 \pm 0.13$ GeV. Their value for the lifetime is $\tau_{B_c} = 0.46^{+0.18}_{-0.16} \pm 0.03$ ps. This state should be one of a number of states lying below the threshold for emission of B and D mesons. Because these states cannot decay by gluon annihilation, they should be very stable in comparison with their counterparts in charmonium and the upsilon system.

The purpose of this paper is to give a detailed account of the energies, splittings and electromagnetic decay rates for the B_c states below the continuum threshold and to propose strategies for detecting some of these states. We will use a potential model that includes running coupling constant effects in both the central potential and the spin-dependent potentials to give a simultaneous account of the properties of charmonium, the upsilon system and the B_c system. We choose Richardson's potential [2] to represent the central potential and insist upon strict flavor-independence of its parameters. Since one would expect the average values of the momentum transfer in the various quark-antiquark states to be different, some variation in the values of the strong coupling constant and the renormalization scale in the spin-dependent potentials should be expected. In order to minimize the role of flavor-dependence, we use the same values for the coupling constant and the renormalization scale for each of the levels in a given system and require that these values be consistent with a universal QCD scale. Since one can calculate the central potential and the spin-dependent potentials from first principles on the lattice [3], it should be possible at some point in future calculations to connect subsets of these potential parameters in more fundamental ways. Such a program would be an important step towards finding a more rigorous way to formulate a potential model calculation.

In 1991 Kwong and Rosner [4] predicted the masses of the lowest vector and pseudoscalar states of the B_s and the B_c systems using an empirical mass formula and a logarithmic potential. Eichten and Quigg [5] gave a more comprehensive account of the energies and decays of the of the B_c system that was based on the QCD-motivated potential of Buchmüller and Tye [6]. Gershtein *et al.* [7] also published a detailed account of the energies and decays of the B_c system and established contact with QCD sum-rule calculations. Both of these latter calculations included running coupling constant effects in the central potential, but used spin-dependent potentials that were restricted to the tree level.

One of the most important goals of the present calculation is to extend the treatment of the spin-dependent potentials to the full radiative one-loop level and thus include effects of the running coupling constant in these potentials. In a previous calculation [8] we have shown that such effects offer a substantial improvement in the calculation of the spectra of charmonium and the upsilon system. In particular, it becomes possible to offer a good account of both the fine structure splittings and the hyperfine structure. Our previous calculation used the renormalization scheme developed by Gupta and Radford [9,10]. However for this calculation, we have chosen the modified-minimal subtraction scheme used by Pantaleone, Tye and Ng (PTN) [11] to extend their perturbative QCD calculation to the one-loop level. We supplement the PTN formalism with a long-range spin-orbit potential to be consistent with the Gromes consistency condition

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[12] and recent lattice calculations [3]. The flavor dependence allowed for the string constant in the 1991 calculation is not permitted in the present calculation. Thus this calculation should also be viewed as part of the effort to formulate potential model calculations under a more restrictive set of assumptions. We note in passing a number of additional important calculations of the properties of the B_c system [13–17].

In Sec. II we present preliminary calculations of the lowest vector and pseudoscalar states of the B_c system with several potential models. These exercises are similar to some carried out by Eichten and Quigg [5] and represent one way of estimating the uncertaintities in the predicted energies of the new system. In Sec. III we introduce Richardson's potential and the expressions necessary to include the one-loop corrections in the spin-dependent potentials and discuss the determination of our parameters from charmonium and the upsilon system. The formalism required for the mixing of the P states in the B_c system is presented in Sec. IV. The last section contains our results for the energies and decays of the low-lying B_c states and a discussion of their implications for the development of strategies to detect these states.

II. SOME POTENTIAL MODELS

The three systems that we consider in this paper, upsilon, charmonium and B_c, are often considered as nonrelativistic systems, and thus our treatment is based upon the Schrödinger equation with a Hamiltonian of the form,

$$H = p^2/2\mu + V(r) + V_{\rm SD}, \tag{1}$$

where $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass. No spin-independent relativistic corrections are included. For the purpose of making some preliminary estimates of the energies of the lowest two states of the B_c system, it is necessary to consider only the spin-spin part of the spin-dependent potential since these are S states. Thus,

$$V_{SD} \rightarrow V_{SS} = \frac{32\pi\alpha_S}{9m_1m_2}\delta^3(\mathbf{r})\mathbf{S}_1\cdot S_2. \tag{2}$$

Our solutions to the Schrödinger equation are generated numerically with the given form for the central potential [18] and the effects of the spin-dependent parts are added as a perturbation improvement. Thus, the $1S$-state hyperfine splitting is given by

$$\Delta E_{\rm hfs} = \frac{8\alpha_S}{9m_1m_2}|R_{1S}(0)|^2. \tag{3}$$

We address the question of the validity of first-order perturbation theory for the contact potential of Eq. (2) in the Appendix.

All of the potential parameters in this section are strictly flavor-independent. This includes any additive constants. The central potential parameters and the constituent mass parameters are fit to the low-lying energy levels of charmonium and the upsilon system. The strong coupling constant α_S is fit to the observed charmonium hyperfine splitting of 117 MeV. In Table I we present results with five different central potentials. These include the Cornell potential [19]

TABLE I. Ground state energies of the heavy-quark systems (MeV).

State	Cornell	Martín	logarithm	Song-Lin	Turin
α_S	0.313	0.437	0.372	0.396	0.373
m_b (GeV)	5.232	5.174	4.905	5.199	5.171
m_c (GeV)	1.840	1.800	1.500	1.820	1.790
$1^3S_1(c\bar{c})$	3097	3097	3097	3097	3097
1^1S_0	2980	2980	2980	2980	2980
ΔE_{1S}	117	117	117	117	117
$1^3S_1(b\bar{b})$	9461	9461	9460	9460	9460
1^1S_0	9316	9397	9395	9380	9365
ΔE_{1S}	145	64	65	80	95
$1^3S_1(c\bar{b})$	6337	6319	6333	6324	6327
1^1S_0	6246	6247	6266	6247	6247
ΔE_{1S}	91	72	67	77	80

$$V(r) = Ar - \kappa/r + C, \tag{4}$$

where $A = 0.1756$ GeV^2, $\kappa = 0.52$ and $C = -0.8578$ GeV; Martín's power-law potential [20],

$$V(r) = -8.093 + 6.898(r \times 1\ \ \text{GeV})^{0.1} \tag{5}$$

(units of the potential are GeV), and the logarithmic potential [21],

$$V(r) = -0.6631 + 0.733\ln(r \times 1\ \ \text{GeV}) \tag{6}$$

(potential units are also GeV). Each of these forms was used by Eichten and Quigg. However, our values for the potential parameters are slightly different. We also have done calculations with the potential of Song and Lin [22],

$$V(r) = Ar^{1/2} - Br^{-1/2}, \tag{7}$$

where $A = 0.511$ GeV$^{3/2}$ and $B = 0.923$ GeV$^{1/2}$, and the Turin [23] potential,

$$V(r) = -ar^{-3/4} + br^{3/4} + C, \tag{8}$$

where $a = 0.620$ GeV$^{1/4}$, $b = 0.304$ GeV$^{7/4}$ and $C = -0.823$ GeV.

Our results in the first three columns of Table I are very similar to those presented in Table I of Eichten and Quigg [5]. The results obtained with the Song-Lin and the Turin potentials in all cases fall between the extremes defined by the first three columns. Averaging over all five cases presented in Table I yields

$$M_{B_c} = 6251^{+15}_{-5}\ \text{MeV}, \quad M_{B_c^*} = 6328^{+9}_{-8}\ \text{MeV}, \tag{9}$$

as the predicted energies of the two lowest states. If we treat Eichten and Quigg's results in the same way, then we get 6258^{+8}_{-10} MeV for B_c and 6333^{+10}_{-14} MeV for B_c^*, in good

agreement with our preliminary analysis. All of these results fall comfortably within the ranges found by Kwong and Rosner [4], that is,

$$6194\ \text{MeV} \leqslant M_{B_c} \leqslant 6292\ \text{MeV}, \tag{10a}$$

$$6284\ \text{MeV} \leqslant M_{B_c^*} \leqslant 6357\ \text{MeV}. \tag{10b}$$

III. RICHARDSON'S POTENTIAL AND THE SPIN-DEPENDENT POTENTIALS

The starting point for the derivation [24] of Richardson's potential is the one-loop expression for the running coupling constant [25], namely,

$$\alpha_S(|\mathbf{q}^2|) = \frac{12\pi}{(33-2n_f)\ln(|\mathbf{q}^2|/\Lambda^2)}, \tag{11}$$

where n_f denotes the number of quark degrees of freedom accessible to the propagating gluon [26] and Λ is the QCD scale. Richardson [2] realized that one could tame the infrared singularity of Eq. (11) and produce a linear confining potential by making the replacement,

$$|\mathbf{q}^2|/\Lambda^2 \rightarrow |\mathbf{q}^2|/\Lambda^2 + 1, \tag{12}$$

in the argument of the logarithm. Our 1991 calculation [8] was based on a modification of Richardson's potential suggested by Moxhay and Rosner [27], that is, the string constant is treated as a free parameter instead of being connected to the scale parameter Λ. Then the potential takes the form

$$V(r) = Ar - \frac{8\pi f(\Lambda r)}{(33-2n_f)r}, \tag{13}$$

where

$$f(t) = \frac{4}{\pi}\int_0^\infty \frac{\sin tx}{x}\left[\frac{1}{\ln(1+x^2)} - \frac{1}{x^2}\right]dx. \tag{14}$$

To specify the spin-dependent potentials, we use a generalization of the Eichten-Feinberg formalism [13] derived by PTN [11]. Our notation is closely akin to that of Eichten and Quigg [5]. Thus,

$$V_{SD} = \frac{\mathbf{L}\cdot\mathbf{S}_1}{2m_1^2}T_a + \frac{\mathbf{L}\cdot\mathbf{S}_2}{2m_2^2}T_a' + \frac{\mathbf{L}\cdot(\mathbf{S}_1+\mathbf{S}_2)}{m_1 m_2}T_b + \frac{\mathbf{L}\cdot(\mathbf{S}_1-\mathbf{S}_2)}{m_1 m_2}T_b' + \frac{S_{12}}{m_1 m_2}T_c + \frac{\mathbf{S}_1\cdot\mathbf{S}_2}{m_1 m_2}T_d, \tag{15}$$

where S_{12} denotes the tensor operator, that is, $4[3\mathbf{S}_1\cdot\hat{\mathbf{r}}\mathbf{S}_2\cdot\hat{\mathbf{r}} - \mathbf{S}_1\cdot\mathbf{S}_2]$ and m_1 denotes the mass of the lighter (charmed) quark, if the two constituent masses are not equal. The quantities T are connected with the PTN potentials as follows:

$$T_a = \frac{1}{r}\frac{d}{dr}(E-Ar) + \frac{2}{r}\frac{dV_1}{dr} + 2V_5, \quad T_a' = T_a - 4V_5, \tag{16a}$$

$$T_b = \frac{1}{r}\frac{dV_2}{dr}, \quad T_b' = V_5, \quad T_c = V_3/12, \quad T_d = V_4/3, \tag{16b}$$

where we have supplemented the PTN potentials with a long range spin-orbit contribution to include nonperturbative effects. Such a contribution is clearly indicated by the lattice calculations of Bali, Schilling and Wachter [3] and is required to generalize the consistency condition,

$$\frac{d}{dr}[E(r) + V_1(r) - V_2(r)] = 0, \tag{17}$$

as it applies to the one-loop potentials, to include the effects of the confining potential, as shown by Gromes [12].

The PTN potentials are expressed in terms of the strong coupling constant α_S^- and the renormalization scale μ. The function $E(r)$ includes the leading correction to the central potential,

$$E(r) = -\frac{4\alpha_S^-}{3r}\left[1 + \frac{\alpha_S^-}{6\pi}\left[(33-2n_f)(\ln\mu r + \gamma_E) + \frac{31}{2} - \frac{5n_f}{3}\right]\right], \tag{18}$$

where $\gamma_E = 0.5772\cdots$ is Euler's constant. This function is of course a part of the contribution to the short-distance behavior of the running coupling constant in Eq. (11), although its contribution to Richardson's potential is not made explicit. The first two spin-orbit potentials are given by

$$V_1(r) = -\frac{\alpha_S^2}{2\pi r}\left[\frac{16}{9} - 4(\ln\sqrt{m_1 m_2}\,r + \gamma_E)\right], \tag{19}$$

$$V_2(r) = -\frac{4\alpha_S^-}{3r}\left[1 + \frac{\alpha_S^-}{6\pi}\left[(33-2n_f)(\ln\mu r + \gamma_e) + \frac{39}{2} - \frac{5n_f}{3} - 9(\ln\sqrt{m_1 m_2}\,r + \gamma_E)\right]\right]. \tag{20}$$

The tensor potential takes the form

$$V_3(r) = \frac{4\alpha_S^-}{r^3}\left[1 + \frac{\alpha_S^-}{6\pi}\left[(33-2n_f)\left(\ln\mu r + \gamma_E - \frac{4}{3}\right) + \frac{65}{2} - \frac{5n_f}{3} - 18\left(\ln\sqrt{m_1 m_2}\,r + \gamma_E - \frac{4}{3}\right)\right]\right]. \tag{21}$$

The lengthy expression for the spin-spin potential,

$$V_4(r)=\frac{32\pi\alpha_S^-}{3}\Bigg\{1+\frac{\alpha_S}{6\pi}\Bigg[\frac{11}{2}-\frac{5n_f}{3}-\frac{3}{4}\left(\frac{9m_1^2+9m_2^2-14m_1m_2}{m_1^2-m_2^2}\right)\ln(m_2/m_1)\Bigg]\Bigg\}\delta^3(\mathbf{r})-\frac{4\alpha_S^{2-}}{9\pi}\Bigg[(33-2n_f)\nabla^2\left(\frac{\ln\mu r+\gamma_E}{r}\right)-\frac{63}{2}\nabla^2\left(\frac{\ln\sqrt{m_1m_2}r+\gamma_E}{r}\right)\Bigg], \tag{22}$$

reduces to an especially simple form for states with nonzero angular momentum, that is,

$$V_4\rightarrow\frac{\alpha_S^{2-}}{\pi r^3}\left[\frac{2}{3}-\frac{8n_f}{9}\right], \tag{23}$$

since one can take the derivatives present in Eq. (22) in this case, and the delta function there does not contribute. If the two constituent quarks have equal masses, then it is necessary to add a second-order contribution from the annihilation graphs,

$$\delta V_4=8\alpha_S^{2-}(1-\ln 2)\delta^3(\mathbf{r}), \tag{24}$$

to the expression of Eq. (22). The spin-orbit potential discovered by PTN is given by

$$V_5(r)=\frac{\alpha_S^{2-}}{\pi r^3}\ln(m_1/m_2). \tag{25}$$

The equations describing the PTN potentials use the modified minimal subtraction scheme to define the strong coupling constant α_S^-. This coupling constant can be related to that defined in a scheme developed by Gupta *et al.* [9,10], which was used in the 1991 calculation [8], that is,

$$\alpha_{GRR}(\mu^2)=\alpha_S^-(\mu^2)\left[1+\frac{\alpha_S^-(\mu^2)}{12\pi}\left(49-\frac{10n_f}{3}\right)\right]+\cdots, \tag{26}$$

to the second order in α_S. It is straightforward to verify that PTN's expressions for the potentials are the same as those of Gupta *et al.* to the second order in α_S by using Eq. (26) to connect the two renormalization schemes. The signs of the terms in Eq. (26) indicate that we should expect smaller coupling constants in the present calculation than in our 1991 calculation.

In determining the matrix elements for the perturbation improvement of the central potential energies, we must exercise some care since the total spin, $\mathbf{S}=\mathbf{S}_1+\mathbf{S}_2$, is not a good quantum number in the most general case. This is a consequence of the fact that its magnitude does not commute with the spin-dependent potential because, for example,

$$[\mathbf{L}\cdot\mathbf{S}_1,S^2]=2i\mathbf{L}\cdot(\mathbf{S_2}\times\mathbf{S_1}). \tag{27}$$

TABLE II. Charmonium energies (MeV).

State	EQ94	FUI98	FUII98	Expt.
1^3S_1	3097	3098	3098	3097 ± 0.04
1^1S_0	2980	2981	2980	2980 ± 2
2^3S_1	3686	3692	3693	3686 ± 0.1
2^1S_0	3608	3617	3615	
1^3P_2	3507	3515	3530	3556 ± 0.1
1^3P_1	3486	3492	3482	3511 ± 0.1
1^3P_0	3436	3443	3391	3415 ± 1.0
1^1P_1	3493	3499	3501	3526 ± 0.1
$\sqrt{\delta^2}$	25.5	22.7	19.9	

We will deal with the more complicated case of the mass-mixing matrix in the next section. For the remainder of this section, we will consider only the diagonal matrix elements. This will suffice to determine the requisite expectation values for the case of equal masses, since then the total spin is a good quantum number.

The diagonal matrix elements can all be evaluated in terms of the expectation values [5] of $\langle\mathbf{L}\cdot\mathbf{S}\rangle$ since

$$\langle\mathbf{L}\cdot\mathbf{S}_1\rangle=\langle\mathbf{L}\cdot\mathbf{S}_2\rangle=\frac{1}{2}\langle\mathbf{L}\cdot\mathbf{S}\rangle, \tag{28}$$

and

$$\langle S_{12}\rangle=\frac{4\langle L^2\rangle\langle S^2\rangle-6\langle\mathbf{L}\cdot\mathbf{S}\rangle-12\langle\mathbf{L}\cdot\mathbf{S}\rangle^2}{(2l+3)(2l-1)}, \tag{29}$$

where l is the orbital angular momentum quantum number, as we have verified [28–30]. Hence, the diagonal matrix elements of the spin-dependent potentials may be expressed as

$$\langle V_{SD}\rangle=\langle\mathbf{L}\cdot\mathbf{S}\rangle\left[\frac{\langle T_a\rangle}{4m_1^2}+\frac{\langle T_a'\rangle}{4m_2^2}+\frac{\langle T_b\rangle}{m_1m_2}\right]+\frac{1}{m_1m_2}[\langle S_{12}\rangle\langle T_c\rangle+\langle\mathbf{S}_1\cdot\mathbf{S}_2\rangle\langle T_d\rangle]. \tag{30}$$

Our results for the charmonium and upsilon energies obtained with Richardson's potential and the one-loop expressions for the spin-dependent potentials are presented in Tables II and III (listed in column 4 as FUII98), where they are compared with the results of Eichten and Quigg [5]. The flavor-independent potential parameters and constituent masses used to obtain these results are

$$A=0.152\ \text{GeV}^2,\quad \Lambda=0.431\ \text{GeV}, \tag{31a}$$

$$m_b=4.889\ \text{GeV},\quad m_c=1.476\ \text{GeV}. \tag{31b}$$

For the upsilon system the value of the coupling constant $\alpha_S^-=0.30$, and the value of the renormalization scale $\mu=1.95$ GeV. For charmonium these values are $\alpha_S^-=0.486$ and $\mu=0.80$ GeV. In both cases the value of the universal QCD scale determined from

TABLE III. Upsilon energies (MeV).

State	EQ94	FUI98	FUII98	Expt.
1^3S_1	9464	9461	9461	9460 ± 0.2
1^1S_0	9377	9368	9406	
2^3S_1	10007	10022	10027	10023 ± 0.3
2^1S_0	9963	9978	10001	
3^1S_1	10339	10358	10364	10353 ± 0.5
3^1S_0	10298	10325	10344	
1^3P_2	9886	9902	9910	9913 ± 0.6
1^3P_1	9864	9881	9891	9892 ± 0.7
1^3P_0	9834	9852	9863	9860 ± 1
1^1P_1	9873	9889	9899	
2^3P_2	10242	10261	10269	10269 ± 0.4
2^3P_1	10224	10244	10255	10255 ± 0.5
2^3P_0	10199	10222	10234	10232 ± 0.6
2^1P_1	10231	10251	10261	
$\sqrt{\delta^2}$	24.9	8.3	4.3	

$$\Lambda_{QCD}=\mu e^{-6\pi/(33-2n_f)\alpha_S^-} \tag{32}$$

is 0.190 GeV, a value consistent with other determinations [31]. Agreement between the calculated results and the measured values [31] in Tables II and III is extremely good, the average deviation being only 4.3 MeV for the upsilon system. Most of the 19.9 MeV deviation in the charmonium case arises from a 25 MeV discrepancy in the center-of-gravity of the $1P$ state. Much of this difference could probably be removed by a proper account of the spin-independent relativistic corrections [32].

Our results for the upsilon and charmonium leptonic widths are presented in Table IV, where they are compared with those calculated by Eichten and Quigg [5]. Our decay rates were obtained with the formula [33]

$$\Gamma_{ee}=\frac{4\alpha^2e_Q^2}{M^2(Q\overline{Q})}|R(0)|^2\left[1-\frac{16\alpha_S}{3\pi}\right], \tag{33}$$

where α is the fine structure constant, e_Q denotes the quark charge, and M denotes the mass of the quark-antiquark state. Since the average values of the momentum transferred in the annihilation graphs underlying Eq. (33) should be much larger than that associated with the scattering processes of the spin-dependent potentials above, we should expect the appropriate value for α_S in Eq. (33) to be smaller than that used above for the fine structure splittings. Thus we choose

TABLE IV. Leptonic widths (keV).

State	EQ94	FU98	Expt.
$\Upsilon(1S)$	1.71(1.19)	1.34	1.32 ± 0.05
$\Upsilon(2S)$	0.76(0.53)	0.57	0.52 ± 0.03
$\Upsilon(3S)$	0.55(0.38)	0.40	0.48 ± 0.06
$\psi(1S)$	8.00(5.55)	5.81	5.26 ± 0.37
$\psi(2S)$	3.67(2.55)	2.61	2.14 ± 0.21

$\alpha_S=0.18$, a value obtained from heavy quarkonium decays [8,34]. Our leptonic widths also agree with experiment very well. In order to carry out a fair comparison with the widths of Eichten and Quigg (EQ), it is necessary to correct their published results for radiative QCD corrections. Doing this as described above gives the EQ results shown in parentheses and moves their results much closer to the measured values. The differences between the values listed in columns 2 and 3 of Table IV give a measure of the differences in wave functions calculated with Richardson's potential and the Buchmüller-Tye potential.

In order to have a good basis for comparison, we have carried out a Richardson's potential calculation with tree-level expressions for all the spin-dependent potentials. These may be readily obtained by omitting all terms of $O(\alpha_S^2)$ in the spin-dependent potentials above, and thus there is no reference to the renormalization scale. Then the expressions for the functions T are especially simple, that is,

$$T_a=T_a'=\frac{4\alpha_S^-}{3r^3}-\frac{A}{r}, \tag{34a}$$

$$T_b=\frac{4\alpha_S^-}{3r^3},\quad T_b'=0,\quad T_c=\frac{\alpha_S^-}{3r^3},\quad T_d=\frac{32\pi\alpha_S^-}{9}\delta^3(\mathbf{r}). \tag{34b}$$

The central potential parameters in this calculation are the same as those listed in Eqs. (31). The constituent mass for the charmed quark is 1.476 GeV, the same as in Eqs. (31), and $m_b=4.884$ GeV, slightly smaller. The coupling constant $\alpha_S=0.339$. Results of this calculation are also presented in Tables II and III (designated as FUI98 in column 3). For both charmonium and the upsilon system, the average deviation is smaller for the one-loop calculation, but this does not tell the whole story. The fit to the fine structure splittings of charmonium with the one-loop calculation is much better than either of the tree-level calculations presented in Table II. This is our basis for the expectation that a one-loop calculation should give a more accurate rendering of the fine structure and the hyperfine structure of the B_c system. Since the central potential parameters are the same, the leptonic widths for the FUI and FUII calculations are the same.

One of the most dramatic differences in the one-loop and tree level results is in the prediction for upsilon hyperfine splittings in Table III. For example, the one-loop prediction for the $1S$ state is 55 MeV, which is consistent with earlier predictions [8,10] and the average of the tree-level predictions is 90 MeV. It is interesting to note that the one-loop level prediction is much closer to the lattice results (44–50 MeV) of Bali, Schilling and Wachter [3]. Thus, measurement of the energies of the singlet S energies would serve to clarify the proper input for a good phenomenological calculation of the properties of heavy quark systems. Some experimental group should give these measurements a high priority.

IV. MASS MIXING MATRIX FOR *P* STATES

In determining the eigenvalues and eigenfunctions for the B_c system, we have a choice of using basis functions from either the L-S coupling scheme or the j-j coupling scheme. We follow the lead of Eichten and Quigg [5] and choose the j-j basis. First we form the total angular momentum of the charmed quark, $\mathbf{J}_1=\mathbf{L}+\mathbf{S}_1$, and then we form the total angular momentum of the system, $\mathbf{J}=\mathbf{J}_1+\mathbf{S}_2$. For P states, the $J=2$ states and the $J=0$ state are the same in either basis, that is,

$$\psi_{2m}\left(\frac{3}{2}\,\frac{1}{2}\right)=\psi_{2m}(11), \quad \psi_{00}\left(\frac{1}{2}\,\frac{1}{2}\right)=\psi_{00}(11), \tag{35}$$

and the determination of the eigenvalues of the spin-dependent potentials requires only the diagonal elements of Eq. (30). On the other hand, the $J=1$ states are a combination,

$$\psi_{1m}=a_1\psi_{1m}\left(\frac{3}{2}\,\frac{1}{2}\right)+a_2\psi_{1m}\left(\frac{1}{2}\,\frac{1}{2}\right), \tag{36}$$

and one must diagonalize the mass mixing matrix in order to determine the eigenvalues and eigenfunctions. In the basis described above, we have derived the following forms for the spin-dependent operators of Eq. (15), that is,

$$(\mathbf{L}\cdot\mathbf{S}_1)=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -1\end{pmatrix}, \quad (\mathbf{L}\cdot\mathbf{S}_2)=\begin{pmatrix}-\frac{5}{6} & -\frac{\sqrt{2}}{3}\\ -\frac{\sqrt{2}}{3} & \frac{1}{3}\end{pmatrix}, \tag{37a}$$

and

$$(S_{12})=\begin{pmatrix}\frac{2}{3} & \frac{2\sqrt{2}}{3}\\ \frac{2\sqrt{2}}{3} & \frac{4}{3}\end{pmatrix}, \quad (\mathbf{S}_1\cdot\mathbf{S}_2)=\begin{pmatrix}-\frac{5}{12} & \frac{\sqrt{2}}{3}\\ \frac{\sqrt{2}}{3} & -\frac{1}{12}\end{pmatrix}. \tag{37b}$$

To obtain the matrices for the operators $\mathbf{L}\cdot\mathbf{S}$ and $\mathbf{L}\cdot(\mathbf{S}_1-\mathbf{S}_2)$, one simply takes the appropriate linear combination of the matrices in Eqs. (37a). By inspection one can verify that our first three matrices in Eqs. (37) are consistent with Eqs. (2.21) of Eichten and Quigg [5].

It is straightforward to take the equal-mass, or L-S, limit. Then one can find a unitary transformation that simultaneously diagonalizes the operators, $\mathbf{L}\cdot\mathbf{S}$, S_{12} and $\mathbf{S}_1\cdot\mathbf{S}_2$. The eigenvectors are given by

$$\psi_{1m}(^3P_1)=\sqrt{\frac{1}{3}}\psi_{1m}\left(\frac{3}{2}\,\frac{1}{2}\right)+\sqrt{\frac{2}{3}}\psi_{1m}\left(\frac{1}{2}\,\frac{1}{2}\right), \tag{38a}$$

TABLE V. Energies of the B_c system (MeV).

State	EQ94	GKLT95	CK92	FUI98	FUII98	LAT96
1^3S_1	6337	6317	6355	6341	6341	6321 ± 30
1^1S_0	6264	6253	6310	6267	6286	$6280\pm30\pm190$
2^3S_1	6899	6902	6917	6911	6914	6990 ± 80
2^1S_0	6856	6867	6890	6869	6882	6960 ± 80
$1P_2$	6747	6743	6773	6761	6772	6783 ± 30
$1P1^{+\prime}$	6736	6729		6750	6760	6765 ± 30
$1P1^{+}$	6730	6717		6742	6737	6743 ± 30
$1P_0$	6700	6683	6728	6713	6701	6727 ± 30
1^3D_3	7005	7007		7022	7032	
1^3D_2	7012			7025	7028	
1^3D_1	7012			7024	7019	
1^1D_2	7009	7008		7023	7028	

$$\psi_{1m}(^1P_1)=\sqrt{\frac{2}{3}}\psi_{1m}\left(\frac{3}{2}\,\frac{1}{2}\right)-\sqrt{\frac{1}{3}}\psi_{1m}\left(\frac{1}{2}\,\frac{1}{2}\right). \tag{38b}$$

The first eigenvector of Eqs. (38) has the lower eigenvalue since it is determined by choosing the solution to the quadratic eigenvalue equation with the minus sign. Our choice of overall sign for the second eigenvector of Eqs. (38) is opposite that of Eichten and Quigg, but the same as that of Gershtein *et al.* [7].

V. RESULTS AND DISCUSSION

Since the central potential parameters are strictly flavor independent and the constituent masses are not allowed to vary, the only decision that one has to make to address the B_c system is to decide on the value of $\alpha_{\bar{S}}$. The simplest choice is the average of the values in the charmonium and the upsilon system, that is, $\alpha_{\bar{S}}=0.393$. This choice requires a value of $\mu=1.12$ GeV, to preserve the value of 0.190 GeV for the QCD scale. Our results for the low-lying S, P and D states of the B_c system are presented in column 6 of Table V, where they are compared with the predictions of Eichten and Quigg [5], Gershtein *et al.* [7], Chen and Kuang [16], the nonrelativistic QCD (NRQCD) lattice calculation [35] and our tree level calculation. The energy of each of these states lies below the B-D meson threshold at 7143 MeV. In determining the D-state splittings, which are much smaller than the P-state splittings, we followed the example of Eichten and Quigg and did not use the mass-mixing matrix approach. Several running coupling constant effects are noticeable in Table V, although our results are in general agreement with the earlier calculations. Our S-state hyperfine splittings are smaller than those of Eichten and Quigg, and our results for the P-state fine structure splittings are larger. We do not see the inverted order of the D states predicted by both Eichten and Quigg and Gershtein *et al.* The order of the fine-structure levels [37,38] is a consequence of the relative sizes of the perturbative and the nonperturbative contributions to the spin-orbit potentials of Eq. (15).

Comparison with the lattice NRQCD results [35] is of special interest. Their result for the B_c mass is based on a

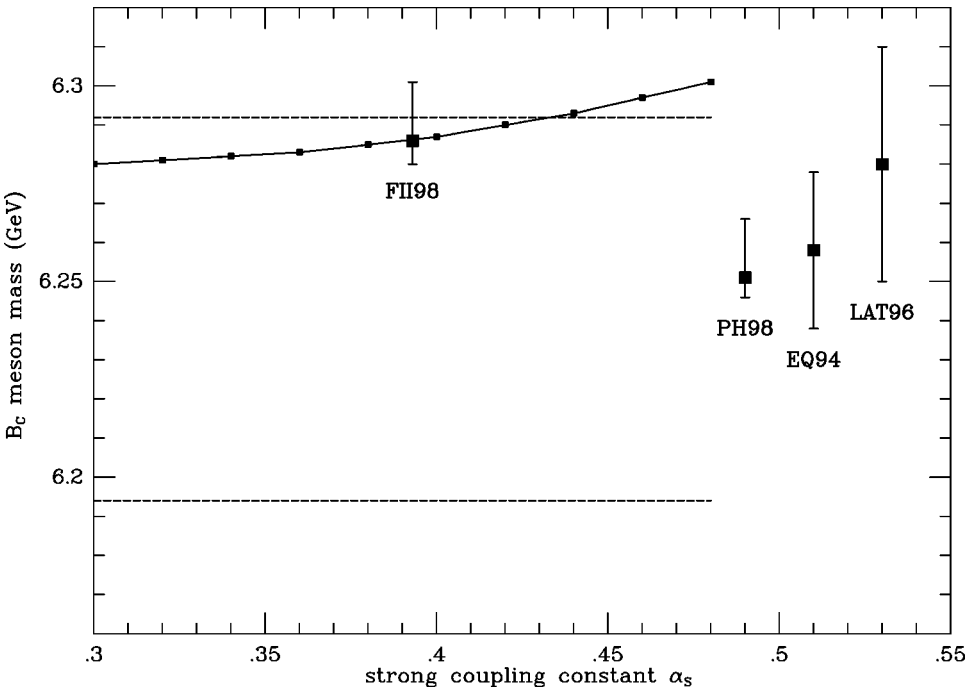

FIG. 1. Ground state energy of the B_c system as a function of the running coupling constant. The PH98 result is based on the calculations presented in Table I.

lattice calculation of the kinetic mass, 4.76 ± 0.02, and a determination of the scale factor by comparison with charmonium and the upsilon system, which gives $a^{-1} = 1.32 \pm 0.04$ GeV. Thus, we list two errors for the lattice calculation of the B_c mass in Table V. The smaller error is a consequence of the error in the kinetic mass calculation and the larger error is a consequence of the uncertainty in the overall scale. In determining the other lattice errors, we simply list the largest possible source of error without including the error due to the overall scale. The lattice result for the $1S$ hyperfine splitting is 41 ± 3 MeV, which is independent of the larger errors discussed above, and somewhat smaller than our 55 MeV result. The lattice result for the $1P^2 - 1P^0$ splitting is 59 ± 5 MeV, in reasonable agreement with our result of 71 MeV. Although the number quoted in Ref. [35] for the B_c mass is very close to our result, its importance as a confirmation of our work is undermined by the large error in the overall scale. Another important lattice result is now available from the UKQCD Collaboration [36]. Their result for the ground state energy is $6386 \pm 9 \pm 98 \pm 15$ MeV, which is consistent with our result because of the large error bars. It will be interesting to see whether further refinements of the lattice calculations will support our results, or offer the experimenters alternative predictions.

In order to get some idea of an error estimate for our predictions, we have calculated the ground state mass of the B_c system as a function of α_S^- in a range bounded by the values determined in charmonium and the upsilon system. Our results are shown in Fig. 1, where they are compared with the result of Table I, the Eichten-Quigg quote and the NRQCD lattice result. The error shown for the lattice result simply ignores the large error associated with the overall scale. The horizontal lines there describe the limits determined by Kwong and Rosner [4]. Using the largest and smallest values of α_S^- in Fig. 1 to determine the errors, we have

$$M_{B_c} = 6286^{+15}_{-6} \text{ MeV}, \quad M_{B_c^*} = 6341^{+2}_{-5} \text{ MeV}, \tag{39}$$

as our predicted value for the $1S$ energies. It is interesting to note how close our results are to the earlier predictions of Godfrey and Isgur [14] (6270 and 6340 MeV) and the predictions of Baker, Ball and Zachariasen [15] (6287 and 6372 MeV), although our prediction for the hyperfine splitting is smaller than both. Clearly, the precision of the experiments [1] requires a very substantial improvement to be sensitive to the energy differences between the various calculations listed in Table V and Fig. 1.

The pseudoscalar decay constant is given by the Van Royen-Weisskopf formula modified for color [39], that is,

$$f_{B_c}^2 = \frac{3|R_{1S}(0)|^2}{\pi M_{B_c}}, \tag{40}$$

and we find that

$$f_{B_c} = 517 \text{ MeV}, \tag{41}$$

in excellent agreement with the result of Eichten and Quigg, $f_{B_c} = 500$ MeV, and in reasonable agreement with the lattice result [35], $f_{B_c} = 440 \pm 20$ MeV.

The empirical result obtained by Collins *et al.* [40] for potential model wave functions at the origin, that is,

$$|R_{B_c}(0)|^2 \cong |R_{J/\psi}(0)|^{1.3} |R_\Upsilon(0)|^{0.7}, \tag{42}$$

provides another touchstone for our numerical work. Using this relationship and input from our charmonium and upsilon calculations, we get $|R_{B_c}(0)|^2 \cong 1.81$ GeV^3, about 3% higher than our numerical result. Using their empirical relationship for the ground state hyperfine splittings, that is,

TABLE VI. Electric dipole matrix elements and transition rates.

Transition	Photon energy (MeV)	$\langle f\|r\|i\rangle$ (GeV)$^{-1}$			Γ(keV)	
		GKLT95	EQ94	FU98	EQ94	FU98
$1P_2 \to 1^3S_1$	417				113	126
$1P1^{+\prime} \to 1^3S_1$	406				0.1	26.2
$1P1^{+} \to 1^3S_1$	384				99.5	75.8
$1P_0 \to 1^3S_1$	350	1.568	1.714	1.683	79.2	74.2
$1P1^{+\prime} \to 1^1S_0$	457				56.4	128
$1P1^{+} \to 1^1S_0$	436				0.0	32.5
$2^3S_1 \to 1P_2$	141				17.7	14.5
$2^3S_1 \to 1P1^{+\prime}$	152				0.0	2.5
$2^3S_1 \to 1P1^{+}$	175				14.5	13.3
$2^3S_1 \to 1P_0$	210	−2.019	−2.247	−2.253	7.8	9.6
$2^1S_0 \to 1P1^{+\prime}$	121				5.2	13.1
$2^1S_0 \to 1P1^{+}$	143				0.0	6.4

$$M_{B_c^*} - M_{B_c} \simeq 0.7(M_{J/\psi} - M_{\eta_c})^{0.65}(M_\Upsilon - M_{\eta_b})^{0.35}, \quad (43)$$

yields a splitting of 63 MeV, about 14% larger than our result listed in Table V. Both of these results are reasonable in view of the spread of the results Collins *et al.* obtained with different forms for the central potential.

Diagonalizing the *P*-state mixing matrix, we obtain the following combinations for the two lowest $J=1$ P states:

$$\psi_{1m}(1^+) = 0.118\psi_{1m}\left(\frac{3}{2}\,\frac{1}{2}\right) + 0.993\psi_{1m}\left(\frac{1}{2}\,\frac{1}{2}\right), \quad (44a)$$

$$\psi_{1m}(1^{+\prime}) = 0.993\psi_{1m}\left(\frac{3}{2}\,\frac{1}{2}\right) - 0.118\psi_{1m}\left(\frac{1}{2}\,\frac{1}{2}\right), \quad (44b)$$

which is very close to the j-j coupling limit, that one would expect to be valid in the heavy-quark limit. Using the inverse of Eqs. (38), we can determine the probability of observing spin 1 in the lowest 1^+ state, $P_{1^+}(S=1)=0.773$. Our result is consistent with the lattice calculation [35], where the mixing angle in the L-S basis was found to be close to that of the j-j limit. From this mixing angle ($\theta = 33.4° \pm 1.5°$) we obtain $P_{1^+}(S=1)=0.697\pm 0.020$, in reasonable agreement with our result. Our results for the mixing angles of the 1^+ and $1^{+\prime}$ states is very different from that of Eichten and Quigg, whose results were much closer to the L-S limit. Below we show that this difference has important implications for the spectrum of photons emitted in electric dipole transitions.

The electric dipole rate for the emission of a photon [5] of energy k is given by

$$\Gamma_{E1}(i \to f + \gamma) = \frac{4\alpha\langle e_Q\rangle^2}{27} k^3 (2J_f+1)|\langle f|r|i\rangle|^2 S_{if}, \quad (45)$$

where the statistical factor $S_{if}=1$ for transitions between triplet S and triplet P states [41], and $S_{if}=3$ for transitions between spin-singlet states. The mean charge in Eq. (45) is

$$\langle e_Q\rangle = \frac{m_2 e_1 - m_1 e_2}{m_1 + m_2}, \quad (46)$$

where e_1 denotes the charge of the charmed quark (in units of the proton's charge) and e_2 denotes the charge of the bottom antiquark. Our results for the $1P \to 1S$ and the $2S \to 1P$ transition rates are shown in Table VI, where they are compared with the results of Eichten and Quigg. Our rates for transitions involving the $1P^2$ and $1P^0$ states are rather close to their counterparts calculated by Eichten and Quigg. However for the $J=1$ states important differences arise. In particular, we predict a larger number of nonzero transition probabilities for each of the $J=1$ states since each state has substantial overlaps with both triplet and singlet spin states. Thus, each of our simulated photon spectra presented in Fig. 2 and Fig. 3 has six lines instead of four. These additional lines are a consequence of the fact that our mixture of states in Eqs. (44) is not close to the L-S limit.

The magnetic dipole transition rate between S states is given by

$$\Gamma_{M1}(i \to f + \gamma) = \frac{16\alpha}{3}\mu_{maq}^2 k^3 (2J_f+1)|\langle f|j_0(kr/2)|i\rangle|^2, \quad (47)$$

where the mean magnetic dipole moment is

$$\mu_{mag} = \frac{m_2 e_1 - m_1 e_2}{4 m_1 m_2}. \quad (48)$$

Our results for the magnetic dipole transition rates are presented in Table VII, where they are compared with those of Eichten and Quigg [5] and Gershtein *et al.* [7]. Most of the

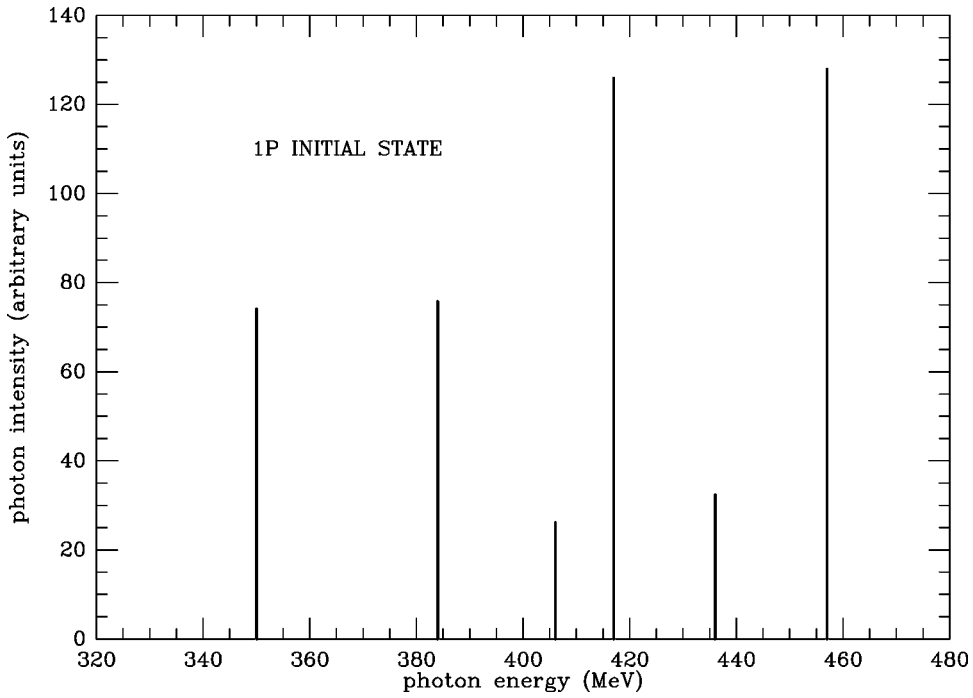

FIG. 2. Simulated photon spectrum for $1P \rightarrow 1S$ transitions. The probability of populating each of the initial states is assumed to be equal.

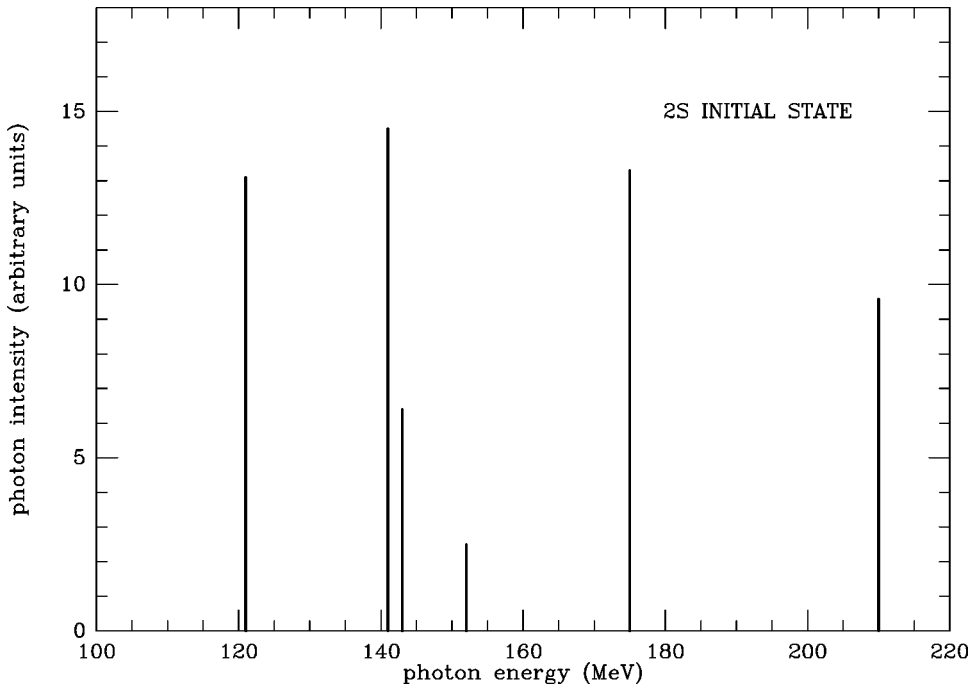

FIG. 3. Simulated photon spectrum for $2S \rightarrow 1P$ transitions.

TABLE VII. Magnetic dipole matrix elements and transition rates.

Transition	Photon energy (MeV)	$\langle f\|j_0(kr/2)\|i\rangle$			Γ (keV)	
		EQ94	FU98	GKLT95	EQ94	FU98
$2^3S_1 \rightarrow 2^1S_0$	32	0.9990	0.9995	0.010	0.029	0.012
$2^3S_1 \rightarrow 1^1S_0$	599	0.0395	0.0399	0.098	0.123	0.122
$2^1S_0 \rightarrow 1^3S_1$	520	0.0265	0.0305	0.096	0.093	0.139
$1^3S_1 \rightarrow 1^1S_0$	55 09.9993	0.9996	0.060	0.135	0.059	

TABLE VIII. Decays and branching ratios of the B_c system.

State	Total width (keV)	Decay mode	Branching ratio (percent)
1^3S_1	0.059	$1^1S_0+\gamma$	100
$1P_2$	126	$1^3S_1+\gamma$	100
$1P1^{+\prime}$	154	$1^3S_1+\gamma$	17
		$1^1S_0+\gamma$	83
$1P1^+$	108	$1^3S_1+\gamma$	70
		$1^1S_0+\gamma$	30
$1P_0$	74.2	$1^3S_1+\gamma$	100
2^3S_1	89.9 ± 7	$1^3S_1+\pi\pi$	56 ± 8
		$1P_2+\gamma$	16 ± 1
		$1P1^{+\prime}+\gamma$	3 ± 0.2
		$1P1^++\gamma$	15 ± 1
		$1P_0+\gamma$	11 ± 1
2^1S_0	69.5 ± 7	$1^1S_0+\pi\pi$	72 ± 10
		$1P1^{+\prime}+\gamma$	19 ± 2
		$1P1^++\gamma$	9 ± 1

differences are a consequence of different energies for the hyperfine splittings, since the results for the matrix elements are rather close.

The photon energies and transition rates in Tables VI and VII suggest at least two good strategies for experimental searches for the $1P$ and $1S$ states. One could detect one or both of the high energy photons (457, 436 MeV) associated with the decay of the lowest two 1^+ states to the ground state and then search for a leptonic decay of the ground state. An alternative would be to look for some of the high-energy photons (417, 406, 384, 350 MeV) in the $1P\rightarrow1^3S_1$ transitions and then seek a coincidence with the 55 MeV photon associated with the decay to the ground state.

Since the charmed quark and the bottom antiquark cannot annihilate into gluons, the only additional complication that arises in the decays of these low-lying states is the $\pi\pi$ channel. If we take the rates for the $\pi\pi$ decays from Eichten and Quigg, then we can work out complete decay schemes and branching ratios for the $2S$ and $1P$ states as well as the 1^3S_1 state. These are shown in Table VIII. Our table of decay rates and branching ratios differs from that of Eichten and Quigg in two important respects. More photon channels are available to the 1^+ states and the $2S$ states. Our decay width for the 1^3S_1 is 59 eV, more than a factor of 2 smaller than theirs.

Our final calculation is that of the lifetime of the ground state of the B_c system. We follow the approach used by several researchers [42–44] where the decay of the B_c meson is taken to be the sum of three distinct contributions, namely weak decay of the $\bar{b}$ antiquark while the c quark behaves as a spectator, weak decay of the c quark while the $\bar{b}$ antiquark behaves as a spectator and an annihilation of the $\bar{b}$ antiquark and the c quark into an intermediate vector boson that subsequently decays into a lepton-neutrino pair or a quark-antiquark pair. Thus the total decay rate is the sum,

$$\Gamma(B_c\rightarrow X)=\Gamma(\bar{b}\rightarrow X)+\Gamma(c\rightarrow X)+\Gamma(annih). \quad (49)$$

If one neglects quark binding effects, then the first two terms are given by

$$\Gamma(\bar{b}\rightarrow X)=\frac{9G_F^2|V_{cb}|^2m_b^5}{192\pi^3}, \quad \Gamma(c\rightarrow X)=\frac{5G_F^2|V_{cs}|^2m_c^5}{192\pi^3}, \quad (50)$$

where the subscripted quantities V denote the appropriate Cabibbo-Kobayashi-Maskawa matrix elements [45] and G_F denotes the Fermi coupling constant. Using the constituent masses listed in Eq. (31), we obtain

$$\Gamma(\bar{b}\rightarrow X)=(8.73\pm1.34)\times10^{-10}\ \text{MeV},$$
$$\Gamma(c\rightarrow X)=(7.59\pm0.02)\times10^{-10}\ \text{MeV}. \quad (51)$$

The annihilation width is given by

$$\Gamma(annih)=\frac{G_F^2}{8\pi}|V_{bc}|^2f_{B_c}^2M_{B_c}\sum_i\ m_i^2\left(1-\frac{m_i^2}{M_{B_c}^2}\right)^2C_i, \quad (52)$$

where m_i denotes the mass of the heavier Fermion in the given decay channel. The most important channels in the sum are the $\tau\nu$ and the $\bar{c}s$ channels. For the former $C_i=1$, and for the latter $C_i=3|V_{cs}|^2$. Using Eq. (41) for f_{B_c} and Eq. (39) for M_{B_c}, we have

$$\Gamma(annih)=(1.13\pm0.17)\times10^{-10}\ \text{MeV}. \quad (53)$$

Adding these three widths yields a lifetime

$$\tau_{B_c}=0.38\pm0.03\ \text{ps}, \quad (54)$$

in good agreement with the measured CDF result [1]. Our result is also in reasonable agreement with the recent calculation of El-Hady, Lodhi and Vary [44], who obtained $\tau_{B_c}=0.46$ ps, since the spectator widths of Eqs. (50) are very sensitive to small differences in the constituent masses. Although different authors may wish to interpret relatively small differences between $\Gamma(\bar{b}\rightarrow X)$ and $\Gamma(c\rightarrow X)$ as the domination of one process over the other, we feel that the safest characterization of our results is that the two spectator processes are almost equally important and that the annihilation processes are less important, consistent with the earlier conclusion of Gershtein *et al.* [42].

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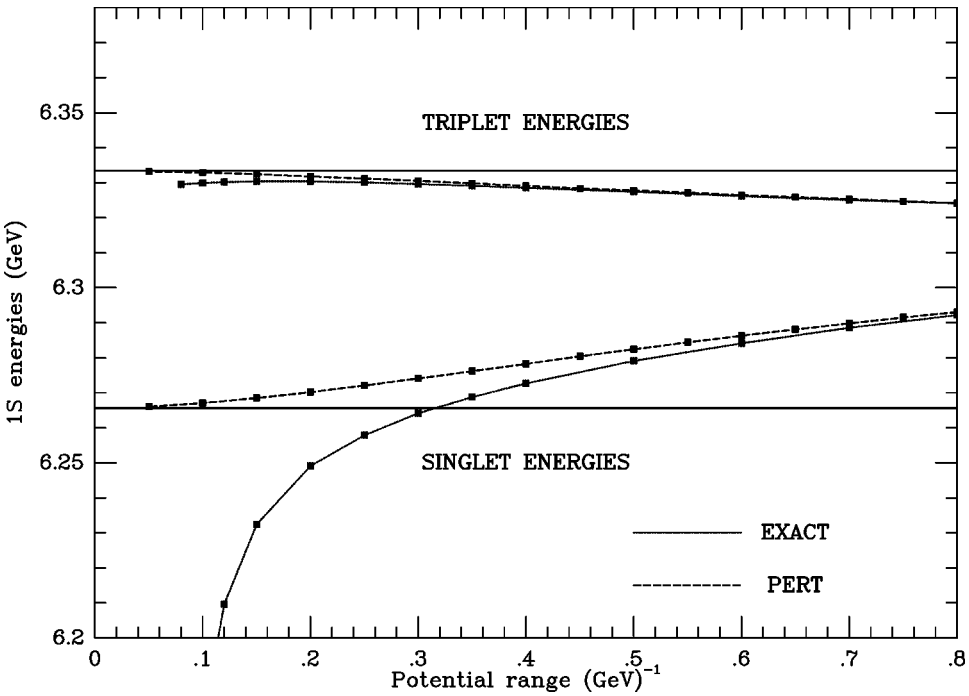

FIG. 4. Hyperfine splittings of the 1*S* state for a finite range spin-spin potential.

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APPENDIX: HYPERFINE SPLITTING AND CONTACT POTENTIALS

Determining the energy shifts from the hyperfine splitting can be problematic because of the delta function that is often present in the spin-spin potential. As Lucha, Schöberl and Gromes point out in their review [24], the energy of the singlet state is not actually bounded from below, in marked contrast to the first-order perturbation theory result of Eq. (3). Although the use of Eq. (3) in the literature is fairly common, seldom does one see any discussion of its validity. It is straightforward to create a context for addressing this question, by considering a more general form for the spin-spin potential, which allows for a finite range, that is,

$$V_{SS}=\frac{32\alpha_S}{9m_1m_2}\frac{e^{-r^2/b^2}}{\pi^{1/2}b^3}\mathbf{S}_1\cdot\mathbf{S}_2. \tag{A1}$$

The advantage of such a form is that one can calculate the singlet and triplet *S*-state eigenvalues of the Hamiltonian of Eq. (1) exactly and examine the limit as the range parameter *b* becomes smaller and smaller. We have done such a calculation with the logarithmic potential of Eq. (6) and the parameters listed in Table I. Our results for the exact singlet and triplet as a function of the range *b* are shown in Fig. 4, where they are compared with the results of a first-order perturbation calculation of the singlet and triplet energies produced by the potential of Eq. (A1). The contact potential results of Eq. (3) are presented as two horizontal lines. It is gratifying to see the first-order perturbation result from Eq. (A1) above approach the contact potential result in the limit $b\rightarrow 0$. Extrapolating the exact result for the triplet energy to the $b=0$ limit gives 6330 MeV, about 3 MeV lower than the result listed in column 4 of Table I, which was obtained with the contact potential. Thus, it is clear that first-order perturbation theory and the contact potential give a good account of the 1S triplet energy.

Figure 4 gives a clear signal of the instability of the singlet energies as $b\rightarrow 0$. However the difficulty begins to appear only as *b* decreases below 0.3 GeV^{-1}. The short range required for the effects of the instability to manifest itself provides a means of resolving this dilemma. As Lucha, Schöberl and Gromes point out, the contact potential expression is not really valid for such short ranges since one must take the nonrelativistic limit in order to obtain it. Such a limit requires that nonlocal effects associated with the normalization of Dirac wave functions be ignored. Thus it might be reasonable to choose $b=0.36$ GeV, the geometric mean of the Compton wavelengths of the two constituent quarks. Such a choice would lead to a value of the singlet energy very close to that found in column 4 of Table I.

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