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Topological solitons in a vacuumless system

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We investigate a model for a real scalar field in bidimensional space-time, described in terms of a positive semidefinite potential that presents no vacuum state. The system presents topological solutions of the BPS type, with an energy density that follows a Lorentzian law. These BPS solutions differ from the standard tanh-type kink, but they also support bosonic and fermionic zero modes. [S0556-2821(99)05016-X]

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Topological defects such as domain walls, strings, and monopoles, can appear in models of symmetry-breaking phase transitions in the early universe [1]. These standard defects appear in models where the potential presents at least two degenerate vacuum states. For instance, domain walls require a countable set of vacuum states, in models where one breaks some discrete symmetry, and cosmic strings and monopoles require an uncountable set of vacuum states in models where one breaks some continuum symmetry.

Despite this standard situation, however, there are other models such as the ones considered for instance in Refs. [2,3], which engender interesting features. For instance, in Ref. [2] one considered the Liouville theory to offer a way of breaking spontaneously the translation invariance of the spatial portion of space-time. More recently, in Ref. [3] one considered other forms of systems defined by vacuumless potentials. In the present work we shall be mainly concerned with the model defined by

$$\mathcal{L}=\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi-U(\phi) \tag{1}$$

with the potential

$$U(\phi)=\frac{1}{2}\frac{\mu^2}{\lambda^2}\operatorname{sech}^2(\lambda\phi). \tag{2}$$

We consider μ real and positive, and λ real. The potential is positive semidefinite, presents a maximum at $\phi=0$ and has no vacuum state. Despite the absence of vacuum states, the system is still able to support topological defects. This model was recently investigated in Ref. [3], and also in Ref. [4], and there attention was given mainly on issues concerning gravitational aspects of the new topological defect. In the present work, however, we shall focus attention on exposing other features of the system, considering the $(1+1)$-dimensional Minkowski space-time: $x^0=x_0=t$ and $x^1=-x_1=x$. Here λ is dimensionless, and μ has dimension inverse of distance, $\dim(\mu)=1/\dim(x)$. The field ϕ is dimensionless, and the system behaves standardly in bidimensional space-time.

The equation of motion is

$$\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}-\frac{1}{\lambda}\mu^2\operatorname{sech}^2(\lambda\phi)\tanh(\lambda\phi)=0 \tag{3}$$

and for static $\phi=\phi(x)$ we get

$$\frac{d^2\phi}{dx^2}=-\frac{1}{\lambda}\mu^2\operatorname{sech}^2(\lambda\phi)\tanh(\lambda\phi). \tag{4}$$

As shown in Ref. [3], this equation is solved by

$$\phi(x)=\pm\frac{1}{\lambda}\operatorname{arcsinh}(\mu x) \tag{5}$$

These solutions diverge asymptotically, and their specific forms depend on μ and λ. Although they are somehow similar to the usual tanh-type kink and antikink that appear is the ϕ^4 model, they are much more diffuse than the standard tanh-type defect, and have divergent amplitude.

To expose key features of this classical solution, let us consider the energy of static configurations. Here we have

$$E=\frac{1}{2}\int_{-\infty}^{\infty}dx\left[\left(\frac{d\phi}{dx}\right)^2+\frac{\mu^2}{\lambda^2}\operatorname{sech}^2(\lambda\phi)\right]. \tag{6}$$

We can write

$$E=E_B+\frac{1}{2}\int_{-\infty}^{\infty}dx\left(\frac{d\phi}{dx}-\frac{\mu}{\lambda}\operatorname{sech}(\lambda\phi)\right)^2. \tag{7}$$

E_B is the value that minimizes the energy. It can be written as $E_B=\Delta W$, where

$$\Delta W=W[\phi(x\to\infty)]-W[\phi(x\to-\infty)]. \tag{8}$$

It only depends on the asymptotic values of the function

$$W(\phi)=\frac{\mu}{\lambda^2}\arctan[\sinh(\lambda\phi)]. \tag{9}$$

The procedure here is similar to the case of coupled fields, as considered in Refs. [5,6] and in applications to condensed matter [7] and field theory itself [8].

The function $W(\phi)$ obeys

$$\frac{dW}{d\phi}=\frac{\mu}{\lambda}\,\mathrm{sech}(\lambda\phi) \tag{10}$$

and the potential in Eq. (2) can be written as

$$U(\phi)=\frac{1}{2}\left(\frac{dW}{d\phi}\right)^2. \tag{11}$$

For static fields the energy is bounded to $E=E_B$ for field configurations that solve the first-order equation

$$\frac{d\phi}{dx}=\frac{\mu}{\lambda}\,\mathrm{sech}(\lambda\phi). \tag{12}$$

We see that solutions of this first-order equation also solve the equation of motion. Yet, we can check explicitly that

$$\bar{\phi}(x)=\frac{1}{\lambda}\,\mathrm{arcsinh}(\mu x) \tag{13}$$

solves the first-order equation (12). This means that this solution is stable [6] and of the Bogomol'nyi-Prasad-Sommerfield (BPS) [9] type [8,10]. We return to the issue of stability below, making the argument explicit.

The BPS solutions present the interesting feature of allowing the energy to be written as

$$E_B=\int_{-\infty}^{\infty}dx\left(\frac{d\bar{\phi}}{dx}\right)^2=\frac{\mu^2}{\lambda^2}\int_{-\infty}^{\infty}dx\,\mathrm{sech}^2(\lambda\bar{\phi}). \tag{14}$$

The energy density has the form

$$\varepsilon(x)=\frac{1/\lambda^2}{x^2+1/\mu^2}. \tag{15}$$

Interestingly, the energy density of the BPS solution obeys a Lorentzian law, and accordingly is more diffuse than the standard tanh-type kink. Explictly, the tanh-type kink appears when the potential is

$$U_s(\phi)=\frac{1}{2}\lambda^2(\phi^2-a^2)^2. \tag{16}$$

Here a is real and positive, and the kink solutions are $\phi(x)=\pm a\tanh(\lambda a x)$. The corresponding energy density reads

$$\varepsilon_s(x)=\frac{1}{2}\lambda^2a^2\,\mathrm{sech}^4(\lambda a x). \tag{17}$$

In spite of this, however, the energy density $\varepsilon(x)$ in Eq. (15) is still integrable and gives the finite value $E_B=\mu\pi/\lambda^2$. Alternatively, although the classical static solution $\bar{\phi}(x)$ diverges asymptotically, the asymptotic values of $W(\phi)$ are finite and give a well-defined ΔW. This fact can be used to expose the topological aspects of the solution. In $1+1$ dimensions we can introduce the topological current

$$J^\mu=\epsilon^{\mu\nu}\partial_\nu W(\phi). \tag{18}$$

Here we are following the first work in Ref. [7], using $W(\phi)$ to define the current; it makes the conserved (topological) charge identical to the energy. This procedure of using $W(\phi)$ is in general more appropriate than considering the field ϕ itself, as it is usually done in the case of the standard tanh-type solution. This is evident here, since the classical solution diverges asymptotically and would make the charge (artificially) ill defined if one uses $\epsilon^{\mu\nu}\partial_\nu\phi$ to define the topological current.

To investigate classical or linear stability, or to compute the first quantum corrections [11,12] introduced by the bosonic field (13) we consider

$$\phi(x,t)=\bar{\phi}(x)+\sum_n\ \eta_n(x)\cos(w_n t). \tag{19}$$

We substitute this into the time-dependent equation of motion (3), and consider the case of small fluctuations $\eta_n(x)$ about the classical field $\bar{\phi}(x)$ to get

$$\left(-\frac{d^2}{dx^2}+V(x)\right)\eta_n(x)=w_n^2\eta_n(x), \tag{20}$$

where

$$V(x)=\frac{-1/\mu^2+2x^2}{(x^2+1/\mu^2)^2} \tag{21}$$

is the potential of this Schrödinger-like equation.

Stability of the classical solution (13) implies that w_n in Eq. (19) should be real. This makes $w_n^2\geqslant 0$, and so the Schrödinger-like Hamiltonian that appears in Eq. (20) has to be positive semidefinite. But this is indeed the case, since we can factorize the Hamiltonian in Eq. (20) in a very specific way. To show this explicitly we use Eqs. (10) and (12) to write

$$\frac{d\phi}{dx}=\frac{dW}{d\phi}. \tag{22}$$

This equation can be used to introduce the operators

$$a_\pm=\pm\frac{d}{dx}+W_{\phi\phi}=\pm\frac{d}{dx}-\mu\ \ \mathrm{sech}(\lambda\phi)\tanh(\lambda\phi), \tag{23}$$

where $W_{\phi\phi}$ stands for $d^2W/d\phi^2$. These operators obey $a_\pm^\dagger=a_\mp$, and can be used to introduce $H_+=a_+^\dagger a_+$ and $H_-=a_-^\dagger a_-$ as the Hamiltonians

$$H_\pm=-\frac{d^2}{dx^2}+V_\pm, \tag{24}$$

where

$$V_\pm=W_{\phi\phi}^2\mp W_\phi W_{\phi\phi\phi}. \tag{25}$$

We use the classical solution (13) to write these potentials in the explicit forms

$$V_+(x) = \frac{1/\mu^2}{(x^2+1/\mu^2)^2}, \tag{26}$$

$$V_-(x) = \frac{-1/\mu^2+2x^2}{(x^2+1/\mu^2)^2}. \tag{27}$$

The potential $V_-(x)$ is exactly $V(x)$, the potential that appears in the Schrödinger-like equation (20). We use this result and $|n\rangle = \eta_n(x)$ and $\langle n|m\rangle = \delta_{nm}$ to write

$$\begin{aligned} w_n^2 &= \left\langle n \left| \left(-\frac{d^2}{dx^2} + \frac{-1/\mu^2+2x^2}{(x^2+1/\mu^2)^2} \right) \right| n \right\rangle \\ &= \left\langle n \left| \left(\frac{d}{dx} - \frac{x}{(x^2+1/\mu^2)} \right) \left(-\frac{d}{dx} - \frac{x}{(x^2+1/\mu^2)} \right) \right| n \right\rangle \\ &= \int_{-\infty}^{\infty} dx \left| \zeta_n(x) \right|^2, \end{aligned} \tag{28}$$

where

$$\zeta_n(x) = \left(-\frac{d}{dx} - \frac{x}{(x^2+1/\mu^2)} \right) \eta_n(x). \tag{29}$$

This result shows that w_n^2 cannot be negative. This proof follows the work in Ref. [6], where it was done in general, for systems of coupled real scalar fields.

The Schrödinger-like equation (20) has at least one bound state, the bosonic zero mode that is present due to translational invariance. The normalized eigenfunction $\eta_0(x)$ is given by

$$\eta_0(x) = \frac{\sqrt{(1/\mu\pi)}}{\sqrt{(x^2+1/\mu^2)}}. \tag{30}$$

It is not hard to see that there is no other bound state. In this case no meson can be binded to the soliton, and only scatering states may appear [11,12].

The Lagrangian density (1) can be seen as the bosonic portion of a supersymmetric theory, and in the extended supersymmetric model the function $W = W(\phi)$ plays the role of the superpotential—see, for instance, Ref. [10]. We can introduce fermions with the usual Yukawa coupling $f(\phi)\bar{\psi}\psi$. We may consider $f(\phi) = g\phi$ or

$$f(\phi) = g\frac{d^2W}{d\phi^2} = -g\mu\, \mathrm{sech}(\lambda\phi)\tanh(\lambda\phi). \tag{31}$$

The possibility of introducing fermions in a supersymmetric way requires the coupling (31), with $g = \pm 1$ and Majorana spinors. In the model with fermions we search for fermionic zero modes. We consider the more general case where the spinors are Dirac spinors. Here the relevant Dirac equation is

$$i\gamma^1 \frac{d\psi}{dx} + f(\phi)\psi = 0. \tag{32}$$

We use $\psi_\pm$ as the upper $(+)$ and lower $(-)$ components of the Dirac spinor ψ, and the representation $i\gamma^1 \to \sigma_3$ to get

$$\pm \frac{d\psi_\pm}{dx} + f(\phi)\psi_\pm = 0. \tag{33}$$

For $f(\phi)$ given by Eq. (31), for instance, the fermionic zero mode depends on μ and g, and may not exist for specific values of g, as for instance for $|g| = 1/2$. There are fermionic zero modes for $|g| > 1/2$. For $g > 1/2$ we get

$$\psi_0 = C(\mu,g) \sqrt{\left(\frac{1}{(x^2+1/\mu^2)} \right)^g} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{34}$$

and for $g < 1/2$ we get

$$\psi_0 = C(\mu,g) \sqrt{\left(\frac{1}{(x^2+1/\mu^2)} \right)^{-g}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{35}$$

where $C(\mu,g)$ is the normalization constant, which depends on both μ and g. For instance, $C(\mu, g = \pm 1) = \sqrt{1/\mu\pi}$, and in this case the fermionic zero mode very much resembles the bosonic zero mode (30): this reflects the fact that $g = \pm 1$ are the only two values of g that allow the supersymmetric extension.

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