Field dynamics on the light cone: Compact versus continuum quantization

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Compact canonical quantization on the light cone is examined in the limit of infinite periodicity length L. Pauli-Jordan commutators are found to approach continuum expressions with marginal noncausal terms of order $L^{-3/4}$ traced back to the handling of the IR divergence through the elimination of zero modes. In contrast direct quantization in the continuum in terms of field operator valued distributions is shown to provide the standard causal result while at the same time ensuring consistent IR and UV renormalization.
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Light front quantization has emerged as an important tool in the study of nonperturbative aspects of field theories [1]. However, a major problem in this approach resides in the infrared behavior of the continuum theory. Recently this issue was clarified on the basis of a mathematically well-defined procedure [2]. In the early attempts to deal with these infrared problems, discretized light front quantization (DLCQ) [3] played an important role. The popularity of DLCQ resides in the easy and conceptually simple treatment of the infrared regularization: zero modes in the expansion of the fields were simply eliminated and later on understood as the LC counterpart of the nontrivial ground state of equal-time (ET) quantization. The study of critical phenomena in the framework of effective theories requires using a continuum version of the quantum field theory on the light front. Indeed critical points, critical exponents, etc., are accessible only from a complete knowledge of the cutoff dependence of the critical mass, which can only be given by the continuum theory. In DLCQ the limit of infinite periodicity length L cannot be achieved in a straightforward manner without further insight both on the handling of zero modes and restoration of covariance and causality in the limiting process [4]. Our approach [2] was to propose a genuine continuum treatment (CLCQ) in which fields are treated as operator valued distributions, thereby leading to a well-defined handling of ultraviolet and light cone induced infrared divergences and of their renormalization. We focused in Ref. [2] on the comparison of the critical coupling in the LC and ET framework, showing that the continuum nonperturbative LC approach is no more complex than usual perturbation theory in lowest order. The LC-critical coupling is in essential agreement with the renormalization-group- (RG-) improved perturbative result at fourth order. Here we want to report on a detailed comparison between DLCQ and CLCQ treatments of important quantities such as Pauli-Jordan commutator functions, which, due to necessary concision and lack of space, could not be treated therein.

Within the particle sector in DLCQ, periodic boundary conditions are imposed, L being the periodicity length, leading to the usual Fock expansion. Restricting to $1+1$ dimension[1] the particle sector field is written as

$$\phi(x)=\sum_{n=1}^{\infty}(1/\sqrt{4\pi n})\left[a_n e^{-ik_n x}+a_n^+ e^{ik_n x}\right] \tag{1}$$

with $[a_n,a_m^+]=\delta_{n,m}$, $n,m\geq 1$ and $k_n=n\pi/L$, $n\in\mathbb{Z}$. The CLCQ approach relies on the introduction of field operator-valued distributions defined with respect to C^∞-test functions with compact support [5]. Apart from formal considerations there exists a fundamental physical argument which demonstrates that it is compelling to treat the field amplitudes in the distributional sense in order to guarantee that the LC quantization procedure by itself is correct. Due to the hyperbolic form of the LC Laplacian, initial field values have to the prescribed on characteristics, i.e., on $x^+=0$ and $x^-=0$. In order to be able to transform this characteristic value problem into a problem with periodic boundary conditions, test functions $f(p^+,p^-)$ have to be introduced with the property [6]

$$\lim_{p^+\to 0}\frac{1}{p^+}f\left(p^+,m^2/p^+\right)=0 \tag{2}$$

[see Eq. (3.20) of Ref. [6]].

This is exactly what happens automatically with the test functions defined below. Condition (2) ensures, as discussed in detail in Ref. [6], that the field values on the characteristic $x^-=0$ become dependent quantities and, as a consequence, the quantization can be performed prescribing boundary values for $x^+=0$ at $x^-=-L$ and $x^-=L$, where $L\to\infty$. The field can be expressed in a chart independent way as a surface integral over a manifold, thereby showing that the ultraviolet (UV) behavior on the Minkowski manifold dictates the UV and IR behavior on the LC manifold. This is due to the regularization properties of the test function which are automatically transferred from the first to the second case.

[1]For massive field the IR problematics can be discussed independently of higher dimensionalities.

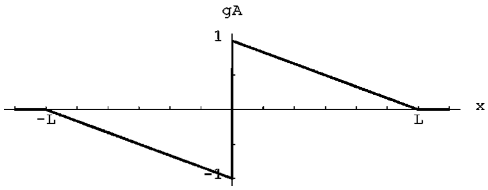

FIG. 1. The DLCQ function $g_A(x^-)$.

In this context the field is written as

$$\phi_{\rm LC}(x)=\int_0^\infty (dp^+/4\pi p^+)[a(p^+)e^{-ipx} + a^+(p^+)e^{ipx}]f_{\rm LC}[p^+,\hat{p}^-(p^+)] \quad (3)$$

with $[a(p^+),a^+(p'^+)]=4\pi p^+\delta(p^+-p'^+)$. In Eq. (3) $\hat{p}^-(p^+)$ stands for the on-shell condition m^2/p^+ and $f_{\rm LC}$ is the test function in momentum space which falls off with all its derivatives sufficiently fast as a function of the Minkowski arguments p_0,p_z $[p^+=\frac{1}{2}(p^0+p^3),\ p^-=\frac{1}{2}(p^0-p^3)]$. Its behavior as a function of p^+ is discussed in Ref. [2]: the singular behavior of $1/p^+$ in Eq. (2) is completely damped out by the behavior of $f_{\rm LC}$ for $p^+\to 0$, eliminating $p^+=0$ as an accumulation point. The ensuing renormalization is independent of the particular choice of $f_{\rm LC}$.

We examine first the Pauli-Jordan commutator $\Delta(x)=[\phi(x),\phi(0)]$ evaluated at $x^+=0$. In the DLCQ case one finds

$$\Delta_{\rm DLCQ}(x^+=0,x^-)=\sum_{n=-\infty,\neq 0}^{\infty}(1/4\pi n)\,e^{-i(n\pi x^-/L)} = -(i/4)[{\rm sgn}(x^-)-(x^-/L)], \quad (4)$$

where ${\rm sgn}(x)=\pm 1$ if $x \gtrless 0$, ${\rm sgn}(0)=0$.

Within CLCQ, with $\hat{f}(p^+)\equiv f_{\rm LC}[p^+,\hat{p}^-(p^+)]$, the corresponding expression is

$$\Delta_{\rm CLCQ}(x^+=0,x^-)=-\frac{i}{2\pi}\int_0^\infty \frac{dp^+}{p^+}\hat{f}^2(p^+)\sin(p^+x^-). \quad (5)$$

The test function $\hat{f}$ is strictly one in the interval $[1/\Lambda,\Lambda-1/\Lambda]$, varies between 0 and 1 in the intervals $[0,1/\Lambda]$ and $[\Lambda-1/\Lambda,\Lambda]$, and is zero outside.

The behavior of $g_A(x^-)=4i\Delta_{\rm DLCQ}(x^+=0,x^-)$ is sketched in Fig. 1. To evaluate $g_B(x^-)=4i\Delta_{\rm CLCQ}(x^+=0,x^-)$, we choose

$$f(p)=\begin{cases} 1-\exp\left[\dfrac{1}{\Lambda^2p^2-1}+1\right], & 0\leqslant p<\dfrac{1}{\Lambda},\\ 1, & \dfrac{1}{\Lambda}\leqslant p\leqslant \Lambda-\dfrac{1}{\Lambda},\\ 1-\exp\left[\dfrac{1}{\Lambda^2(p-\Lambda)^2-1}+1\right], & \Lambda-\dfrac{1}{\Lambda}<p\leqslant\Lambda,\\ 0, & p>\Lambda \end{cases} \quad (6)$$

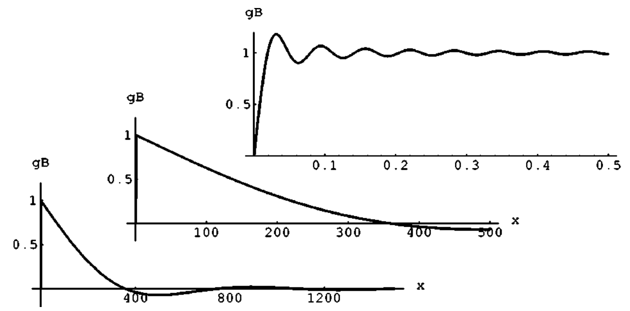

FIG. 2. The CLCQ function $g_B(x^-)$ at different spatial scales.

with $\Lambda=100$, and calculate $g_B(x^-)$ numerically. The results are plotted in Fig. 2 at three different spatial scales. Near the origin $g_B(x^-)$ rises to 1 over distances shorter with increasing Λ. It is followed by an oscillatory fall-off with an average slope in $1/\Lambda$, corresponding to the straight line of $g_A(x^-)$ in DLCQ. Finally for large values of x^- $(\geqslant 10\Lambda)$ $g_B(x^-)$ remains oscillating around zero.

Hence in both cases the decay zone and the asymptotic region where $g(x^-)$ is null or quasinull, reflect the elimination of the zero mode $n=0$ for DLCQ and a halo around $p^+=0$ for CLCQ. However, it is the presence of the UV regularization in CLCQ which is responsible for the smeared out rise near $x^-=0$ and small short wave length oscillations for small x^-, at variance with DLCQ where no such regularization is present. Clearly the n summation can be arbitrarily cut off to deal with the UV divergence but the approach to the continuum is not under control since the limiting procedure of infinite cut off and infinite periodicity length compatible with causality is not known. To discuss these points we examine now the commutator for space or timelike separation.

For DLCQ we have

$$\Delta_{\rm DLCQ}(x^+,x^-)=-\frac{i}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left[\frac{n\pi x^-}{L}+\frac{1}{4}\frac{m^2Lx^+}{\pi n}\right] \quad (7)$$

and for CLCQ the corresponding expression is

$$\Delta_{\rm CLCQ}(x^+,x^-)=-\frac{i}{2\pi}\int_0^\infty\frac{dp^+}{p^+}\sin\left[\frac{1}{4}\frac{m^2x^+}{p^+}+p^+x^-\right]\hat{f}^2(p^+). \quad (8)$$

The integral in Eq. (8) is convergent even if $\hat{f}=1$ everywhere and a straightforward change of the integration variable shows that $\Delta_{\rm CLCQ}$ depends only on the product x^+x^-. The limit $\Lambda\to\infty$ can be taken safely with the result

$$\Delta_{\rm CLCQ}(x^+,x^-)=-\frac{i}{4}[{\rm sgn}(x^+)+{\rm sgn}(x^-)]J_0(m\sqrt{x^+x^-}), \quad (9)$$

which is the correct causal covariant expression, with $J_0(x)$ the Bessel function of order zero.

Clearly for $\Delta_{\rm DLCQ}$ the limit $L\to\infty$ cannot be taken before the sum is carried out, as the sinus becomes ill defined. As shown in the Appendix this limit requires some care. Using Eq. (A14) one finds

$$\Delta_{\rm DLCQ}(x^+,x^-)|_{L\to\infty}=-(i/4)[\mathrm{sgn}(x^+)+\mathrm{sgn}(x^-)]\times J_0(m\sqrt{x^+x^-})+\frac{i}{2mLx^+}\sqrt{x^+x^-+2Lx^+\mathrm{sgn}(x^+)}\times J_1[m\sqrt{x^+x^-+2Lx^+\mathrm{sgn}(x^+)}]+O(L^{-5/4}). \tag{10}$$

Hence the causal covariant expression is retrieved in the limit $L\to\infty$. However, the marginal noncausal term in J_1 in Eq. (10) originates from the elimination of the zero mode in the infinite sum of Eq. (7) [for $x^+=0$ it is just $(i/4)(x^-/L)$, cf., Eq. (4)]. Its disappearance as $L\to\infty$ indicates that in the continuum the infrared problems would remain at variance with CLCQ. Thus in DLCQ, L has to be kept finite to achieve IR regularization, at the expense of the appearance of a causality violating term of order $(L^{-3/4})$. Unfortunately this does not mean that the two versions coincide in the limit $L\to\infty$ since in this limit the infrared regularization of DLCQ is lost. Due to the regularization properties of the test functions, the situation in CLCQ is far more satisfactory since the approach provides a well defined handling of UV and IR divergences and of their renormalization.

To conclude we want to add a remark concerning the LC-lattice method introduced by Destri and de Vega [7] and elaborated by Faddeev and co-workers [8]. This approach works on a LC-space-time lattice. The basic building blocks of field dynamics being causal transfer matrices between neighboring points along lightlike directions, problems with causality are avoided by construction in this discretization scheme. However, the main argument in favor of this approach lies in the integrability properties in closest connection to those of the continuum.

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APPENDIX

In this appendix we derive the expression of the periodic Pauli-Jordan function in the limit of infinite periodicity length L. Consider the periodic distribution with period $\lambda=2\pi/K$

$$f(x)=\sum_{n=-\infty}^{\infty}C_n e^{inKx} \tag{A1}$$

and the class of C^∞-test function $\varphi(x)$ with the properties

$$\left\{ x\in[0,1];\quad \varphi(x)+\varphi(x-1)=1;\quad \varphi(0)=1,\varphi(1)=0,\left.\frac{d^P\varphi(x)}{dx^P}\right|_{x=1}=0\quad \forall p\geq 1\right\}. \tag{A2}$$

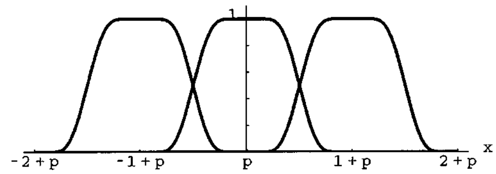

FIG. 3. A function $\varphi(x)$ decomposing unity.

This constitutes a decomposition of unity since by construction

$$\sum_{p=-\infty}^{\infty}\varphi(x+p)=1,\quad \forall x. \tag{A3}$$

The Fourier transform $\phi(k)$ of $\varphi(x)$ has the property

$$\phi(0)=1,\quad \phi(2p\pi)=0,\quad \forall p \text{ integer } \neq 0. \tag{A4}$$

The coefficient C_n in the expansion of $f(x)$ is then given by

$$C_n=\frac{1}{\lambda}\int_{-\infty}^{\infty}f(x)\varphi\left(\frac{x}{\lambda}\right)e^{inKx}dx. \tag{A5}$$

If $f(x)$ is a standard integrable function of period λ, C_n is just the usual Fourier coefficient since $\varphi(x/\lambda)+\varphi(x/\lambda-1)=1$.

We consider now, for $(a,b)\in\mathbb{R}$, the distribution

$$T_{ab}(x)=\frac{1}{2i}\sum_{p=-\infty}^{\infty}\frac{e^{i(ax+b/x)}}{x}\left(1-\frac{\sin(\pi x)}{\pi x}\right)\delta(x-p). \tag{A6}$$

With the C^∞-test function $\Omega(x)$ which decomposes unity we have

$$\begin{aligned}T_{ab}(x)(\Omega)&=\frac{1}{2i}\sum_{p=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i(ax+b/x)}}{x}\left(1-\frac{\sin(\pi x)}{\pi x}\right)\times\delta(x-p)\Omega(x)dx\\&=\frac{1}{2i}\sum_{p=-\infty}^{\infty}\frac{e^{i(ap+b/p)}}{p}\left(1-\frac{\sin\pi p}{\pi p}\right)\Omega(p)\\&=\sum_{p=1}^{\infty}\frac{1}{p}\sin\left(ap+\frac{b}{p}\right)\end{aligned} \tag{A7}$$

since $\Omega(p)=1$, $\forall p$ integer or zero (see Fig. 3).

On the other hand the periodic distribution $f(x)=\Sigma_{p=-\infty}^{\infty}\delta(x-p)$ also admits the Fourier expansion (A1) with $K=2\pi$ and $C_n=1$, directly from Eq. (A5). Hence we have the well-known representation

$$\sum_{p=-\infty}^{\infty} e^{2ip\pi x} = \sum_{p=-\infty}^{\infty} \delta(x-p). \tag{A8}$$

$T_{ab}(x)(\Omega)$ is then also given by

$$T_{ab}(x)(\Omega) = \sum_{p=-\infty}^{\infty} \int_0^{\infty} (dx/x)\, \Omega(x) \sin[(a+2p\pi)x + (b/x)]\left(1-\sin\pi x/\pi x\right). \tag{A9}$$

$\Omega(x)$ is a decomposition of unity and since the integral is well defined with $\Omega(x)=1$ on the whole integration domain, we have

$$\int_0^{\infty} \frac{dx}{x} \sin\left[(a+2p\pi)x + \frac{b}{x}\right] = \frac{\pi}{2}[\text{sgn}(a+2p\pi)+\text{sgn}(b)] \times J_0[2\sqrt{(a+2p\pi)b}], \tag{A10}$$

$$\int_0^{\infty} \frac{dx}{x} \sin\left[(a+2p\pi)x + \frac{b}{x}\right] \frac{\sin\pi x}{\pi x} = \frac{1}{4b}\Big(\{\text{sgn}[a+(2p+1)\pi]+\text{sgn}(b)\}\sqrt{[a+(2p+1)\pi]b} \times J_1\{2\sqrt{[a+(2p+1)\pi]b}\} - \{\text{sgn}[a+(2p-1)\pi]+\text{sgn}(b)\}\sqrt{[a+(2p-1)\pi]b} \times J_1\{2\sqrt{[a+(2p-1)\pi]b}\}\Big). \tag{A11}$$

Here $\text{sgn}(x)=\pm 1 \quad x \gtrless 0, \quad \text{sgn}(0)=0$, and $J_n(x)$ is the ordinary Bessel function of order n.

Specializing to the discretized light-cone variables $a=\pi x^-/L, \quad b=(m^2/4)(Lx^+/\pi)$, we have

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left[\frac{n\pi x^-}{L}+\frac{m^2}{4}\frac{Lx^+}{n\pi}\right] = [\text{sgn}(x^+)+\text{sgn}(x^-)]J_0(m\sqrt{x^+x^-}) + \lim_{N\to\infty}\Bigg\{2\,\text{sgn}(x^+)\sum_{p=1}^{N} J_0[m\sqrt{x^+x^-+2pLx^+\text{sgn}(x^+)}] - \frac{2}{mLx^+}\sqrt{x^+x^-+(2N+1)Lx^+\text{sgn}(x^+)}\,J_1[m\sqrt{x^+x^-+(2N+1)Lx^+\text{sgn}(x^+)}]\Bigg\}. \tag{A12}$$

The limit $N\to\infty$ in Eq. (A12) is still elusive because the compensation between the two diverging terms in N is not explicit. However, the remaining sum in Eq. (A12) can be given in an integral form using a contour integral representation of $J_0(z)$. Then the sum over p becomes geometric and can be performed. We have the result

$$2\,\text{sgn}(x^+)\sum_{p=1}^{N} J_0[m\sqrt{x^+x^-+2pLx^+\text{sgn}(x^+)}] = -\frac{2}{mLx^+}\sqrt{x^+x^-+Lx^+\text{sgn}(x^+)}\,J_1[m\sqrt{x^+x^-+Lx^+\text{sgn}(x^+)}] + \frac{2}{mLx^+}\sqrt{x^+x^-+(2N+1)Lx^+\text{sgn}(x^+)}\,J_1[m\sqrt{x^+x^-+(2N+1)Lx^+\text{sgn}(x^+)}] + O(L^{-5/4}). \tag{A13}$$

Now the limit $N\to\infty$ can be taken in Eq. (A12) as the diverging term in N in Eq. (A12) is cancelled exactly by the one in Eq. (A13), leaving the result

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x^-}{L}+\frac{m^2}{4}\frac{Lx^+}{n\pi}\right) = [\text{sgn}(x^+)+\text{sgn}(x^-)]J_0(m\sqrt{x^+x^-}) - \frac{2}{mLx^+}\sqrt{x^+x^-+2Lx^+\text{sgn}(x^+)} \times J_1[m\sqrt{x^+x^-+2Lx^+\text{sgn}(x^+)}] + O(L^{-5/4}). \tag{A14}$$

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