Dual symmetry and the vacuum energy

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A hidden symmetry for the scale factor of the $k=0$ Friedmann-Robertson-Walker model which is different from the known ones is presented herein. This exact symmetry implies a zero cosmological constant and can be interpreted as a string-type dual symmetry. [S0556-2821(99)04016-3]

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Astronomical observations of the universe indicate that the cosmological constant, if it is nonzero, is very small. The vacuum energy density ρ_v multiplied by $8\pi G_N=\kappa^2$ (where G_N is the Newton constant) is usually called the cosmological constant Λ [1]. The cosmological data imply that in the present-day the vacuum energy density is not much greater than the critical density $\rho_c\sim 10^{-48}$ GeV^4.

Vacuum energy in quantum field theory is not zero, but it has the value of m^4, where "m" is a characteristic particle physical mass parameter [2]. For instance, the masses of intermediate bosons of weak interactions are $m_z\sim 10^2$ GeV, which gives $\rho\sim 10^8$ GeV^4 and the observable vacuum energy density ρ_v is fantastically small $\rho_v<10^{-56}\rho$. This fact could be due to the accidental compensation of different contributions to ρ_v, but there is a small chance for a compensation with an accuracy of one part in 10^{60}. It may be that the compensation of different contributions to ρ_v is secured by a symmetry principle.

The natural candidate is supersymmetry. Experiment shows, however, that this symmetry is broken in the observed universe, since the boson and fermion masses are different. Therefore, vacuum energy is not exactly canceled. In the best case the contributions to vacuum energy are proportional to $m_{3/2}^4$, where $m_{3/2}\sim 10^2-10^3$ GeV (gravitino mass parameter) describes the scale of supersymmetry breaking [3].

Ideally, we would like to explain the vanishing of the cosmological constant in the observed universe in terms of an exact symmetry principle. In the case of early stages of the universe, the symmetry is broken and the cosmological constant is nonvanishing. The symmetry must include a nontrivial new transformation on the metric $g_{\mu\nu}(x^\lambda)$, since general coordinate transformations on the metric $g_{\mu\nu}$ must not constrain the cosmological term $\sqrt{-g}\Lambda$.

In this work we show that the minisuperspace formulation allows such a symmetry in any theory of gravitation, including Einstein theory. We consider a simple model of the universe described by a homogeneous and isotropic Friedmann-Robertson-Walker (FRW) metric

$$ds^2=-N^2(t)dt^2+R^2(t)d^3\Omega, \tag{1}$$

where $d^3\Omega$ is the interval on the spatial sector with constant curvature $k=0,\mp 1$, corresponding to plane, hyperbolic or spherical three-space, respectively.

The metric is described by a single scale factor $R(t)$ and as the matter source we shall consider a homogeneous scalar field $\varphi(t)$ which induce the potential $V(\varphi)$.

The action for FRW interacting with the scalar field $\varphi(t)$ is described by

$$S=\int\left[-\frac{3}{\kappa^2}\frac{R\dot{R}^2}{N}+\frac{R^3}{2N}\dot{\varphi}^2+\frac{3}{\kappa^2}kNR-NR^3V(\varphi)\right]dt, \tag{2}$$

where $\dot{R}=dR/dt$, $\dot{\varphi}=d\varphi/dt$ and $\kappa=(8\pi G_N)^{1/2}$ have length dimensions ℓ. The dimension of the scalar field $\varphi(t)$ is of the form ℓ^{-1}, while the potential $V(\varphi)$ has dimension ℓ^{-4}. We assume units in which $c=\hbar=1$.

It turns out that the action S is invariant under the time reparametrization $t\rightarrow t'=t+a(t)$, if the variables $N(t)$, $R(t)$ and $\varphi(t)$ are transformed as

$$\delta N=(aN)^{\dot{}},\quad \delta R=a\dot{R},\quad \delta\varphi=a\dot{\varphi}. \tag{3}$$

In fact, under the transformation (3) the action (2) becomes

$$\delta S=\int (aL)^{\dot{}}dt, \tag{4}$$

where L is the corresponding Lagrangian. So, up to a total derivative, the action S is invariant under the transformation (3).

We can see that the first and the second terms in the action (2), which are the kinetic terms for the scale factor

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$R(t)$ and the scalar field $\varphi(t)$, respectively, are invariant under the following transformations, and in what follows we call dual transformation:[1]

$$R(t)\to R'(t)=\frac{\kappa^2}{R(t)},\quad N(t)\to N'(t)=\frac{\kappa^6 N(t)}{R^6(t)},$$

$$\varphi(t)\to\varphi(t)=\varphi(t). \tag{5}$$

Using Eq. (3) we find that by a time reparametrization the transformation law for R' and N' has the form

$$\delta R'=a\dot{R}' \quad \text{and} \quad \delta N'=(aN')^{\dot{}}. \tag{6}$$

The term $(3/\kappa^2)kNR$ is invariant under the transformation (5) only for the plane three space, $k=0$. Finally, the last term in action (2), $-NR^3V(\varphi)$, is the effective cosmological constant term and defines the contributions to cosmological constant from the potential $V(\varphi)$. If $V(\varphi)$ vanishes at $\varphi=\langle\varphi\rangle=\varphi_0$, for example for a potential of the form $V=\lambda(\varphi^2-a^2)^2$, the action (2) is invariant under the dual transformation (5), while for the vacuum energy $V(\varphi=0)=a^4\neq 0$ the dual symmetry is broken and the cosmological constant is nonvanishing.

Let us find the canonical Hamiltonian of the model. The momenta conjugate to R and φ are defined in the usual way

$$\Pi_R=-\frac{6}{\kappa^2}\frac{R\dot{R}}{N},\quad \Pi_\varphi=\frac{R^3\dot{\varphi}}{N}, \tag{7}$$

and under the dual transformations (5) we have the following relations between the old and the transformed momenta:

$$\Pi_R\to\Pi'_R=-\frac{1}{\kappa^2}R^2\Pi_R,\quad \Pi_\varphi\to\Pi'_\varphi=\Pi_\varphi. \tag{8}$$

With these relations we find, that the canonical Poisson brackets are invariant under dual transformations (5):

$$\{R,\Pi_R\}=\{R',\Pi'_R\}=1,\quad \{\varphi,\Pi_\varphi\}=\{\varphi',\Pi'_\varphi\}=1. \tag{9}$$

Thus, the Hamiltonian can be calculated in the usual way. We have the classical canonical Hamiltonian

$$H_c=NH=N\left[-\frac{\kappa^2}{12R}\Pi_R^2+\frac{1}{2R^3}\Pi_\varphi^2+R^3V(\varphi)\right], \tag{10}$$

where H is the Hamiltonian of the system. This form of the canonical Hamiltonian explains the fact, that the lapse function N is a Lagrange multiplier, which enforces the only first-class constraint $H=0$. This constraint, of course expresses the invariance of the action (2) under reparametrization transformations.

We note that for the case $k=0$ and for the effective cosmological term $R^3V(\varphi_0)=0$ the canonical Hamiltonian is invariant under dual transformations (5). In fact, since under Eq. (3) the classical constraint H transform as $H\to H'=HR^6/\kappa^6$, we have $N'H'=NH$.

According to the Dirac's constraints Hamiltonian quantization procedure, the wave function is annihilated by operator version of the classical constraint. In the usual fashion, the canonical momenta are replaced by operators $\hat{\Pi}_R=-i(\partial/\partial R),\hat{\Pi}_\varphi=-i(\partial/\partial\varphi)$.

It turns out that the commutators of the quantum operators $[R,\hat{\Pi}_R]=i$, $[\varphi,\hat{\Pi}_\varphi]=i$ are also invariants under the dual transformations.

In order to find a correct quantum expression for the Hamiltonian we must always consider factor ordering ambiguities. This is true in our case because the operator Hamiltonian contains the product noncommutating operator R and $\hat{\Pi}_R$. Thus, of the first term in the classical Hamiltonian H we consider the following operator form: [6]

$$\frac{\kappa^2}{12}R^{-p-1}\frac{\partial}{\partial R}R^p\frac{\partial}{\partial R}=\frac{\kappa^2}{12}\left(\frac{1}{R}\frac{\partial^2}{\partial R^2}+\frac{p}{R^2}\frac{\partial}{\partial R}\right), \tag{11}$$

where p is a real parameter that measures the ambiguity in the factor ordering [7] in the first term of Eq. (10).

The quantum Hamiltonian has the form

$$\hat{H}=\frac{\kappa^2}{12}R^{-p-1}\frac{\partial}{\partial R}R^p\frac{\partial}{\partial R}-\frac{1}{2R^3}\frac{\partial^2}{\partial\varphi^2}+R^3V(\varphi),$$

$$=\frac{\kappa^2}{12R}\frac{\partial^2}{\partial R^2}+\frac{\kappa^2 p}{12R^2}\frac{\partial}{\partial R}-\frac{1}{2R^3}\frac{\partial^2}{\partial\varphi^2}+R^3V(\varphi). \tag{12}$$

On the other hand, under duality transformations the Hamiltonian H becomes

$$\hat{H}'=\frac{R^6}{\kappa^6}\hat{H}, \tag{13}$$

Thus, we may assume the final form

$$\hat{H}=\frac{\kappa^2}{12}R^{p-3}\frac{\partial}{\partial R}R^{-p+2}\frac{\partial}{\partial R}-\frac{1}{2R^3}\frac{\partial^2}{\partial\varphi^2}+\frac{\kappa^{12}}{R^9}V(\varphi), \tag{14}$$

It is straightforward to show that $\hat{H}$ is dual invariant only if the parameter $p=1$ and $V(\varphi_0)=0$. If $V(\varphi_0)\neq 0$ the last term in Eq. (12) is broken under the dual symmetry.

The physical states $|\Psi\rangle$ are those that are annihilated by $\hat{H}$:

[1]The target space in the modular transformations of the string contains the well-known dual transformations $r\to 1/r$, where r is the ''radius'' of the internal six-dimensional space of the string [4,5].

$$\hat{H}|\Psi\rangle=\left[\tilde{R}^2\frac{\partial^2}{\partial\tilde{R}^2}+p\tilde{R}\frac{\partial}{\partial\tilde{R}}-6\frac{\partial^2}{\partial\tilde{\varphi}^2}-36k\tilde{R}^4+12\lambda\tilde{R}^6(\tilde{\varphi}^2-\tilde{a}^2)^2\right]|\Psi\rangle=0. \tag{15}$$

This equation can be identified with the Wheeler-DeWitt equation for minisuperspace models. Clearly, Eq. (15) has many different solutions, and one of the most fundamental questions facing us is which of these solutions actually describes our universe.

Let us consider the following ansatz:

$$\text{(i)}\quad \Psi(\tilde{R},\varphi_0)=\Psi\left(\frac{1}{\tilde{R}},\tilde{\varphi}_0\right),\quad V(\tilde{\varphi}_0=\tilde{a})=0,$$

$$\text{(ii)},\quad V(\tilde{\varphi}=0)=\tilde{a}^4,\ \text{dual breaking invariance,}$$

$$\Psi(\tilde{R})\neq\Psi\left(\frac{1}{\tilde{R}},\tilde{\varphi}=0\right). \tag{16}$$

Here, we consider the scale factor and the scalar field dimensionless; $\tilde{R}=R/\kappa$, $\tilde{\varphi}=\kappa\varphi$, $\tilde{a}=\kappa a$. Considering these two cases we shall find the exact solution to the quantum equation (15) for any p.

Case (i) means that we need to ensure the invariance of the Hamiltonian (12) under dual transformation

$$\tilde{R}(t)\rightarrow\frac{1}{\tilde{R}(t)},\quad N(t)\rightarrow\frac{N(t)}{\tilde{R}^6(t)},$$

$$\tilde{\varphi}(t)\rightarrow\tilde{\varphi}(t)\quad k:=0,\quad\text{and}\quad p:=1. \tag{17}$$

Note, that in this case the values of the parameter k, the factor ordering p, and the potential for the scalar field are fixed. Thus, under the duality transformations (17) we have

$$\hat{H}|\Psi\rangle=\left[\tilde{R}^2\frac{\partial^2}{\partial\tilde{R}^2}+\tilde{R}\frac{\partial}{\partial\tilde{R}}-6\frac{\partial^2}{\partial\tilde{\varphi}^2}\right]|\Psi\rangle=0, \tag{18}$$

$$\Psi(\tilde{R},\tilde{\varphi})=\begin{cases}A(\tilde{R}^{\nu}+\tilde{R}^{-\nu})e^{\pm(\nu/\sqrt{6})\tilde{\varphi}}, & m=-\nu^2<0,\\ B\cos(\nu\ln\tilde{R})e^{\pm i(\nu/\sqrt{6})\tilde{\varphi}}, & m=\nu^2>0,\end{cases} \tag{19}$$

where A, B, and ν are integration constants.

Now, for case (ii) the factor ordering is not fixed and we obtain the solution

$$\Psi(\tilde{R},\tilde{\varphi})=\tilde{R}^{(1-p)/2}Z_{\nu}\left(\frac{\sqrt{2}\tilde{a}^2}{3}\tilde{R}^3\right)e^{\pm im\tilde{\varphi}},$$

$$\nu=\frac{1}{6}\sqrt{(1-p)^2-(2m)^2}, \tag{20}$$

where $Z_\nu[(\sqrt{2}\tilde{a}^2/3)\tilde{R}^3]$ is the Bessel function defined as $AJ_\nu+BY_\nu$, where ν is a real (or imaginary [8]) number with a real argument $(\sqrt{2}\tilde{a}^2/3)\tilde{R}^3$.

It is straightforward to generalize our procedure to all Bianchi-type cosmological models. In the proposed framework it is also possible to include the supersymmetric minisuperspace models [9]. Thus, as further research, it may be interesting to consider the close minisuperspace models.

Furthermore, these simple dual transformations may be applied to the metric $g_{\mu\nu}(x^\lambda)$ in the Arnowitt-Deser-Misner (ADM) formalism [10]. According to the ADM prescription, of general relativity, they are considered as a slicing of the space time by a family of spacelike hypersurfaces labeled by a parameter t. The space-time metric $g_{\mu\nu}(x^\lambda)$ is decomposed into lapse $N(t,x^k)$, shift $N_i(t,x^k)$ and the three-metric of the slice $h_{ij}(t,x^k)$. Thus, these quantities under dual transformations have the following form:

$$N\rightarrow N'=\frac{N}{h},\quad N_i\rightarrow N_i'=\frac{N_i}{h^{2/3}},\quad h_{ij}\rightarrow h_{ij}'=\frac{h_{ij}}{h^{2/3}}, \tag{21}$$

where $h=\det(h_{ij})$.

A gravitational vacuum polarization correction to effective action [11] must induce new terms which will not be invariant under the dual transformations (21). In fact, even if $\sqrt{-g}V(\varphi)$ vanishes at $\langle\varphi\rangle=\varphi_0$, the quantum correction to $V(\varphi)$ would generate a nonzero contribution $\sqrt{-g}V_{\text{eff}}(\varphi_0)$ and the effective action is not invariant under dual transformations (21). Therefore, the condition for such an exact dual symmetry leads to the cancellation of these two contributions of the vacuum energy.

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