Thermodynamic curvature of the BTZ black hole

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In this paper we apply the concept of thermodynamic geometry to the Bañados-Teitelboim-Zanelli (BTZ) black hole. We find the thermodynamic curvature diverges at the extremal limit of the black hole, which means the extremal black hole is the critical point with the temperature zero. We also study the effective dimensionality of the underlying statistical model. Near the critical point, the picture is clear; the spatial dimension of the underlying statistical model is just one, which agrees with other results. However, far from the critical point, the dimension becomes less than one and even negative. In order to interpret this result, we resort to a qualitative analogy with the Takahashi gas model. [S0556-2821(99)04516-6]

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Over the decades the statistical interpretation of black-hole entropy has been one of the most fascinating subjects. There have been many approaches to the problem, although nothing was completely successful. One curiosity about black-hole thermodynamics is that it looks different from an ordinary thermodynamical system, due to the negative heat capacity. This makes it hard to compose its thermodynamic ensemble and to make the underlying statistical model. One way to study the statistical aspects is to assume the microcanonical ensemble for an isolated black hole. Actually, in this way one can understand the critical behavior for the near extremal black hole. The critical exponents satisfying the scaling law even tells us the dimensionality of the underlying statistical model.

In this paper, we suggest another tool useful for studying the statistical properties including the fluctuation, the critical behavior, and so on. This is the thermodynamic geometry. It defines a metric on the space of thermodynamic variables. (Of course, it has nothing to do with the geometry of space and time.) Here, the thermodynamic potential becomes the geometrical potential generating the metric components. The thermodynamic variables constitute the coordinates for the geometry. Details will be given below through the example of the Bañados-Teitelboim-Zanelli (BTZ) black hole [2]. As is known, the BTZ black hole could play an important role in understanding entropy and some dynamical properties of certain five- and four-dimensional black holes in supergravity theories, because of the U duality between the BTZ black hole and those high-dimensional black holes [3]. For a review see [4].

In general, it is technically difficult to define the zero-temperature critical point, as is the case with the black hole. Conventional definition for the zero-temperature critical point is the point where at least one of the second derivatives of some thermodynamic potential diverges [1]. However, in the language of thermodynamic geometry one can define it unambiguously as the point where the thermodynamic curvature (the curvature with respect to the thermodynamic metric) diverges. This is based on the fact that thermodynamic curvature is proportional to the correlation volume [5]:

$$\Re = \kappa_2 \xi^{\bar{d}}, \tag{1}$$

where κ_2 is a constant of order of unity, ξ is the correlation length, and $\bar{d}$ is the spatial dimension of the statistical system.

Another strong point of the thermodynamic curvature is that the above relation is valid even far from the critical point. This allows one to get the effective dimension $\bar{d}$ of the underlying statistical model for the general nonextremal black hole. (The analysis in [6], for the effective dimension, is valid only near the critical point.) In this paper, we apply the concept of thermodynamic geometry to the BTZ black hole. We find the thermodynamic curvature diverges, which means the extremal black hole is the critical point with the temperature zero. We also study the effective dimensionality of the underlying statistical model. Near the critical point, the picture is clear; the spatial dimension of the underlying statistical model is just one, which agrees with other results. However, far from the critical point, the dimension becomes less than one and even negative. In order to interpret this result, we resort to a qualitative analogy with the Takahashi gas model. Interpretation is summarized as follows: Near the critical point, the model is defined in (1+1)-dimensional spacetime. Far from the critical point, it becomes the model of the world sheet.

The BTZ black hole is a solution of the (2+1)-dimensional Einstein gravity with a negative cosmological constant l^{-2}. Its metric is

$$ds^2 = -N(r)dt^2 + N^{-1}(r)dr^2 + r^2(N^{\phi}(r)dt + d\phi)^2, \tag{2}$$

where $N(r) = -M + r^2/l^2 + J^2/(4r^2)$, $N^{\phi}(r) = -J/(2r^2)$, M and J are the mass and angular momentum of the black hole. The black hole has two horizons

$$r_{\pm}^2 = \frac{1}{2}Ml^2(1\pm\Delta), \quad \Delta = [1-(J/Ml)^2]^{1/2}. \tag{3}$$

The Hawking temperature T of the hole is

$$T = \frac{r_+^2 - r_-^2}{2\pi r_+ l^2} = \frac{M\Delta}{2\pi r_+}. \tag{4}$$

and the entropy is $S = 2\pi r_+$. These thermodynamic quantities obey the first law of thermodynamics

$$dS=\beta dM-\mu dJ, \tag{5}$$

where $\beta=1/T$, $\mu=\beta\Omega_H$, and $\Omega_H=J/2r_+^2$ is the angular velocity of the hole. Another feature of the BTZ black hole is that its heat capacity, $C_J=(\partial M/\partial T)_J$, is always positive definite [7]

$$C_J=\frac{4\pi r_+\Delta}{2-\Delta}, \tag{6}$$

due to $0\leqslant\Delta\leqslant 1$. This means the temperature increases with the mass. Therefore, the BTZ black hole can be stable in thermal equilibrium with an arbitrary volume of heat bath. When $\Delta=1$, i.e., $J=0$, we have $C_J=4\pi l\sqrt{M}$. And when $\Delta=0$, i.e., $J=Ml$, we have $C_J=0$, corresponding to extremal BTZ black holes. In that case, the two horizons of the hole coincide, and the Hawking temperature becomes zero.

Recently, it was reported for some near extremal black holes including the BTZ black hole, that the relevant degrees of freedom concerned with the entropy can be described by a one-dimensional ideal-gas model [8,9]. This means that their corresponding statistical model must be $(1+1)$-dimensional. Indeed, the statistical entropy of the BTZ black-hole can be calculated by using a two-dimensional boundary conformal field theory [10]. In order to understand better the statistical origin of the black-hole thermodynamics, it is desirable to study the fluctuations of thermodynamic quantities of black holes, from which one (possibly) obtains some information on the corresponding statistical model. Although we do not know the details of the microscopic theory, this is possible because thermodynamic fluctuation theory can be put completely on a thermodynamic basis.

The thermodynamic geometry is described by the so-called Ruppeiner metric [5]. It is defined as the second derivative of the entropy with respect to the internal energy and other extensive variables of a thermodynamic system (these variables are considered as the coordinates). This geometry comes from the thermodynamic fluctuation theory. The details of the geometry can be found in a review paper [5]. Now we consider the BTZ black hole as a thermodynamic system. One can write down its Ruppeiner metric in the $a=(M,J)$ coordinates as (the Ruppeiner metric in other coordinates may be obtained through the canonical Legendre transformations):

$$\begin{aligned} ds_{\mathrm{R}}^2 &\equiv -\left(\frac{\partial^2 S}{\partial M^2}\right)_J dM^2-\left(\frac{\partial^2 S}{\partial J^2}\right)_M dJ^2 \\ &=\frac{1}{T^2C_J}dM^2+\frac{1}{TI_M}dJ^2, \end{aligned} \tag{7}$$

where I_M is defined as

$$I_M^{-1}=\frac{1}{2r_+^2}+\frac{J^2}{8Mr_+^4\Delta}+\frac{J^2}{2M^2l^2r_+^2\Delta^2}.$$

Because C_J and I_M are always positive, the line element ds_{R}^2 is positive definite.

One can easily see from Eq. (7) that at the extremal limit, the Ruppeiner metric becomes singular. In fact, the extremal limit is a critical point, i.e., at least one of the second derivatives of certain thermodynamic potential diverges there [1]. Indeed if one chooses the Helmholtz free energy $f=M-TS$ as the thermodynamic potential, the appropriate coordinates become (T,J) and the Ruppeiner metric is

$$ds_{\mathrm{R}}^2=\frac{1}{T}\left(-\frac{\partial^2 f}{\partial T^2}dT^2+\frac{\partial^2 f}{\partial J^2}dJ^2\right)=C_JT^{-2}dT^2+(TI_T)^{-1}dJ^2, \tag{8}$$

where I_T is

$$I_T^{-1}=\frac{2Mr_+^2\Delta+J^2}{4Mr_+^4\Delta}+\frac{J^3+2MJr_+^2\Delta}{4M^2r_+^4\Delta}\mathcal{C}$$

with

$$\mathcal{C}=-\frac{J(2+\Delta)}{4r_+^2\Delta}\left[\frac{\Delta}{2}+\frac{(2+\Delta)J}{4Mr_+^2\Delta}\right]^{-1}.$$

I_T vanishes at the extreme limit, accordingly the second derivative of the Helmholtz free energy with respect to the angular momentum J diverges. This tells us that the extremal black hole is a critical point and its Hawking temperature $T_H=0$ is the corresponding critical temperature. The components of the Ruppeiner metric have the meaning of the second moments for fluctuations [5]. Thus we can immediately read off the following second moments from Eq. (8):

$$\langle\delta T\delta T\rangle=T^2/C_J,\quad \langle\delta J\delta J\rangle=TI_T. \tag{9}$$

These are quite different results from the ordinary critical point at nonvanishing critical temperature. There, the fluctuations diverge for the extensive quantities while they vanish for the intensive quantities at critical points.

As is mentioned already the thermodynamic curvature gives an unambiguous definition for the zero critical temperature. For the BTZ black hole, we can easily calculate the curvature scalar in the coordinates (M,J). The curvature scalar of the Ruppeiner metric (7) is[1]

$$\mathfrak{R}=\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial M}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{22}}{\partial M}\right)+\frac{\partial}{\partial J}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{11}}{\partial J}\right)\right], \tag{10}$$

where $g=1/(T^3C_JI_M)$, $g_{11}=1/(T^2C_J)$, $g_{22}=1/(TI_M)$. The full expression for the thermodynamic curvature (10) in terms of the thermodynamic variables is somewhat complicated. Here we do not present it explicitly. Instead we show the numerical behavior of the thermodynamic curvature.

Figure 1 shows the behavior of the thermodynamic curvature with respect to the angular momentum when mass $M=1$ and cosmological constant $l=100$. The thermody-

[1]Our conventions are $\mathfrak{R}^{\lambda}_{\mu\nu\sigma}=\Gamma^{\lambda}_{\mu\nu,\sigma}-\Gamma^{\lambda}_{\mu\sigma,\nu}+\Gamma^{\lambda}_{\sigma\eta}\Gamma^{\eta}_{\mu\nu}-\Gamma^{\lambda}_{\nu\eta}\Gamma^{\eta}_{\mu\sigma}$, and $\mathfrak{R}_{\mu\nu}=\mathfrak{R}^{\lambda}_{\mu\lambda\nu}$, $\mathfrak{R}=g^{\mu\nu}\mathfrak{R}_{\mu\nu}$.

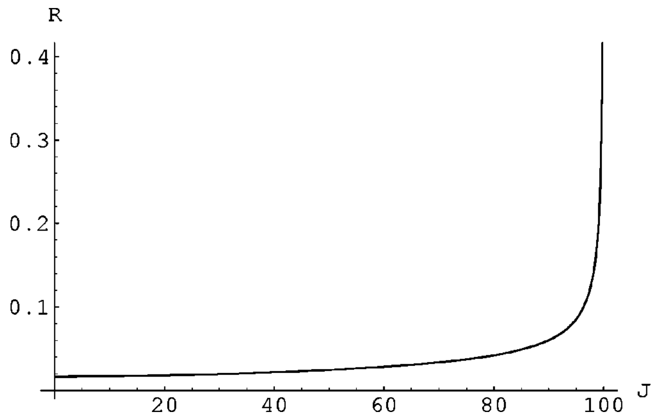

FIG. 1. Thermodynamic curvature $\mathfrak{R}$ of the BTZ black hole versus the angular momentum J with $M=1$ and $l=100$.

namic curvature becomes small as $J\to 0$ but does not vanish. It reflects the fact that the Ruppeiner geometry describes the fluctuations of the thermodynamic system; although the thermodynamic variable J approaches zero, its fluctuation does not vanish. In this sense, the fluctuation property is well incorporated into the thermodynamic curvature. When the extremal limit is approached, the curvature diverges strongly. This is consistent with the result from thermodynamics: the extremal limit is a critical point. Since the divergence of the thermodynamic curvature is manifest, there is no ambiguity in defining the critical point even if the temperature vanishes. The differences between the extremal and nonextremal BTZ black holes also suggest the critical nature of the point. In particular, it is found that the extremal BTZ black hole preserves some of the full supersymmetries of the supergravity, while the supersymmetry is absent for the nonextremal black hole [10]. Note that the change of symmetry of thermodynamic systems is a characteristic feature of critical phenomena in the ordinary thermodynamics [1].

Further inspection of Eq. (10) shows that the divergence near the extremal limit behaves like

$$\mathfrak{R}\sim\Delta^{-1}. \tag{11}$$

In order to study the system further near the critical point, we need to know the correlation function and correlation length of the statistical system. Unfortunately, we do not have knowledge about these. In Ref. [11], Traschen investigated the dynamical behavior of a scalar field propagating in the Reissner-Nordström (RN) black-hole background and obtained the correlation function of the scalar field. It was found that for the extremal black hole, the scalar field has scaling symmetry and long-range correlation, i.e., the effect of the source falls off like y^{-1}, while for the nonextremal black hole, there is no scaling symmetry and the influence of the source falls off exponentially fast, like $e^{2\kappa y}$, where κ is the surface gravity of the black hole and y is the usual tortoise coordinate. Therefore, the inverse surface gravity plays the role of the correlation length. Considering the similarity between the geometries of the near extremal RN black hole and BTZ black hole [7], as an assumption, we suppose that this is valid for the BTZ black hole as well. Then we have

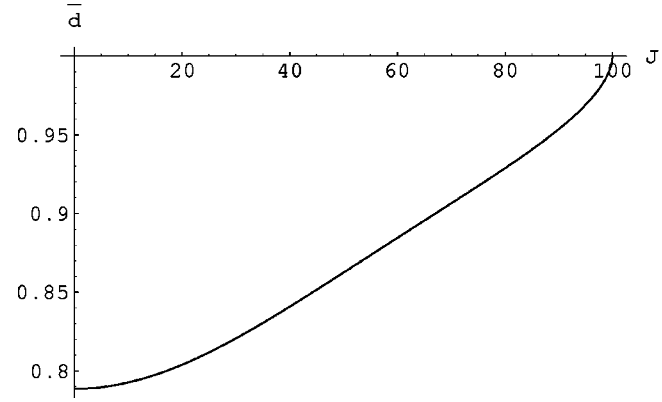

FIG. 2. The effective dimension $\bar{d}$ versus the angular momentum J with $M=1$ and $l=100$. The coefficient c is determined to be $3/(1250\pi)$.

$$\xi=1/(2\kappa), \tag{12}$$

where $\kappa=2\pi T$. We see from Eq. (4) that $\kappa\sim\Delta$ near the extremal point. Combining Eqs. (11) and (1), we obtain the effective spatial dimension of the statistical model underlying the near-extremal BTZ black hole as

$$\bar{d}=1. \tag{13}$$

Reference [6] used the ansatz $\xi\sim 1/\kappa$ to discuss the scaling laws occurring near the extremal limit of the black p branes. It was found from the scaling laws that the effective spatial dimension is just p for the nondilaton black p branes, which explains very well why the entropy may be given a simple world volume interpretation only for the nondilaton black p branes.

We now turn to the relation (1) between the thermodynamic curvature and the correlation volume of the corresponding statistical system. Near the extremal limit, we have already found that the effective spatial dimension is one. Note that, according to the interpretation of the thermodynamic curvature, the relation (1) still holds even far from the critical point. We extrapolate this relation to the nonextremal BTZ black hole. The spatial dimension $\bar{d}$ is not a constant. With Eqs. (1) and (12), we have

$$\mathfrak{R}=c\left(\frac{l\sqrt{1+\Delta}}{2\sqrt{2M}\Delta}\right)^{\bar{d}}, \tag{14}$$

from which, we get

$$\bar{d}=\ln\frac{\mathfrak{R}}{c}/\ln\frac{l\sqrt{1+\Delta}}{2\sqrt{2M}\Delta}. \tag{15}$$

The coefficient c is determined so that $\bar{d}=1$ at the critical point, which is clear from Eq. (11).

Figure 2 shows the effective spatial dimension versus the angular momentum when mass $M=1$ and cosmological constant $l=100$. c is determined to be $3/(1250\pi)$. In this parameter region of Fig. 2 we see that the spatial dimension varies from 1 to 0.79. The numerical results show that the spatial dimension becomes less than one and even negative as we go off the extremal point. For instance, when the mass $M=1$ and the cosmological constant $l=3$, the effective spatial dimension decreases to the negative value -1.05 near the spinless black hole. The results are sensitive to the values of the parameters M and l but it keeps the value 1 near the extremal limit according to Eq. (13). On the other hand, the variation of thermodynamic curvature with respect to the angular momentum in Fig. 1 is insensitive to the value of M and l.

It becomes important to understand these results through the corresponding statistical model. Before doing so, let us first recall the Takahashi gas, which is a one-dimensional system composed of rigid rods. An attractive square potential is assumed between adjacent rods. In this model, the thermodynamic curvature shows a sharp peak at a point (considered as the pseudophase transition point in the sense that it is still finite; see Fig. 11 of [5]). This pseudophase transition indicates the transition from the liquidlike phase to the gaslike phase. Note that it shows the Gaussian curvature versus the inverse density and the former is defined as $\xi_G \equiv -\Re/2$. We are interested in this model because, in the liquid phase, the dependency of the thermodynamic curvature on the inverse density looks very similar in shape to Fig. 1 of BTZ black-hole case. Taking into account the fact that the shape of the graph in Fig. 1 is insensitive to the values M and l, one can infer that the statistical model must share some common features with the Takahashi gas model.

The first feature is the dimensionality of the model. Indeed, there is a one-dimensional conformal field theory, with which the entropy of the BTZ black hole can be reproduced [10]. In particular, Ghosh [9] recently produced the correct expression of the entropy for the extremal and near-extremal BTZ black holes using a one-dimensional ideal gas model. But this model fails for the BTZ black hole far from the extremal limit.

We focus on the second feature of this. The gas is composed of extended objects. We may say, employing the terminology of Takahashi gas model, that the corresponding statistical model go into the gaslike phase from the liquidlike phase as the BTZ black hole approaches its extremal limit, and thereby the corresponding statistical model can be described by the one-dimensional ideal-gas model. The appearance of the negative spatial dimension is not surprising. For instance, the string gas [12] is just of the negative spatial dimension in some cases as scaled by the ideal-gas thermodynamics. The thermodynamics of the intersecting M-brane configurations [13] also has the effective negative spatial dimension, which can be explained by the dynamical p branes.

One can understand the whole results in the spirit of the correspondence principle [14]. From five- and four-dimensional black holes U dual to the BTZ black holes, we have learned that the entropy of the near-extremal black hole can be statistically understood by the open string gas model living on D branes. For the black holes far from the extremal limit, their entropy can be matched by one (a few) long string(s). Of course, the statistical behavior of long strings is quite different from that of the ideal gas, i.e., of the open strings. In this regard, it would be very interesting to see further the behavior of the Takahashi gas. At the inverse density $\rho^{-1}=3/2$, the thermodynamic curvature shows a high and narrow peak (see Fig. 11 of [5]), which is a critical point from the liquidlike phase to the gaslike phase. If one grows the size of the rigid rod, the density becomes large and the system then enters the liquidlike phase. A further increase of the rod length terminates with the zero scalar curvature at the point where the whole system is composed of one long rigid rod. Of course, the Takahashi gas is not the statistical model of the BTZ black hole. However, from this analogy we note that the entropy for the near-extremal BTZ black hole can be described in terms of the one-dimensional ideal gas, while the entropy for the BTZ black hole far from the extremal limit can be matched by a single highly excited long string.

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