Vacuum amplification of the high-frequency electromagnetic radiation

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When an electrically charged source is capable of both emitting the electromagnetic waves and creating charged particles from the vacuum, its radiation gets so amplified that only the back reaction of the vacuum makes it finite. The released energy and charge are calculated in the high-frequency approximation. The technique of expectation values is advanced and employed. [S0556-2821(99)01016-4]

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I. INTRODUCTION AND SUMMARY

A. Introduction

The reaction of the vacuum on rapidly moving sources, or strongly variable fields, is important for the evolution of black holes and the early universe but is also interesting in electrodynamics. We know that in electrodynamics the vacuum attenuates an external charge. Suppose now that the external source is not a monopole but, say, a dipole, and let this dipole be capable of emitting the electromagnetic waves so that the information about it reaches infinity. Then what will the vacuum effect be on such a dipole?

The answer obtained below is that the effect is the opposite: the radiation of the dipole gets amplified. This effect becomes noticeable as soon as the typical frequency of the dipole exceeds the threshold of pair creation. A flux of charged particles that appears in this case is accompanied by an *increase* of the electromagnetic radiation. Generally, there is a nonlocal tail of radiation caused by the vacuum stress but, at high frequency, the effect boils down to a multiplication of the classical radiation rate by a renormalization constant. Since the dipole is a nonlocal object, its renormalization[1] is finite and observable.

The vacuum amplification of the electromagnetic waves emitted by a source is analogous to the effect of the vacuum gravitational waves [1]. The difference is only in the theoretical mechanisms and in the dimensions of the coupling constants.[2] The dimension of the coupling constant causes the gravitational effect to never boil down to a mere renormalization.

In the case of electromagnetic waves, the mechanism by which this effect emerges in theory is as follows. If one calculates the energy of charged particles created from the vacuum by a given nonstationary electromagnetic field [2], one finds that the result can be obtained only in the case where the electromagnetic field contains no outgoing waves. In the general case this energy is infrared divergent with the divergent term proportional to the energy of the outgoing waves. The appearing divergence is a signal that the calculation is not complete because *the energy of the vacuum of charged particles goes partially into the coherent electromagnetic radiation.* Indeed, the missing contribution comes from the back reaction of the vacuum on the electromagnetic field. If one calculates the effective electromagnetic field that solves the expectation-value equations, one finds that the quantum correction to the energy of the outgoing waves is also divergent, and the two divergences cancel each other. As a result, the total released energy is finite but is no more the energy of created charged particles alone. Rather it is a sum of the energy that goes with charged particles and the energy that goes with the enhanced electromagnetic waves. The two contributions can then be separated by calculating the released charge.

It is worth noting that the vacuum reactions on the low-frequency and high-frequency external fields are very different. Effects such as the anomalous magnetic moment in QED refer to the low-frequency electromagnetic fields and are not related to the effect considered here. On the other hand, in the mechanism described above one easily recognizes the physics that stands behind the so-called infrared disaster. Here this physics actually works and, of course, there is no disaster if one does not consider the transitions between concocted states but considers the evolution of expectation values.

The terms ''high-frequency approximation'' and ''strongly variable field'' are used here as synonyms. Let l be the typical spatial size of the source of an external field and ν be its typical frequency. On the other hand, let m be the mass of the lightest particles interacting with this field. In the problem of the vacuum particle creation, the external field is considered as strongly variable if the energy $\hbar\nu$ dominates both the rest energy of the vacuum particle and its static energy in this field. The first of these conditions is discussed in Sec. IV below, and its more accurate form is

$$\hbar\nu \gg mc^2\left(\frac{mc}{\hbar}l\right). \tag{1.1}$$

The second is exemplified in Ref. [2]. Under condition (1.1) the vacuum particles may be regarded as massless in the calculation of their fluxes. However, the mass m cannot be neglected in the calculation of the static polarization and charge renormalization.

It makes sense to begin with quoting the result for the energy of particles created from the vacuum by strongly variable fields of arbitrary configurations. The respective calcu-

[1] Residual after an infinite renormalization of the monopole.

[2] The main difference is, of course, in the fact that for a creation of the gravitational charge, there is no threshold.

lation was carried out in Ref. [2] for *the standard loop*, i.e., for the vacuum action of the form

$$S_{\rm vac}=\frac{i}{2}\log\det\hat{H}, \tag{1.2}$$

$$\hat{H}=g^{\mu\nu}\nabla_\mu\nabla_\nu\hat{1}+\left(\hat{P}-\frac{1}{6}R\hat{1}\right)-m^2\hat{1}, \tag{1.3}$$

where the operator $\hat{H}$ is defined as acting on an arbitrary set of quantum fields. The hat over a symbol means that this symbol is a matrix in the space of field components, $\hat{1}$ is the unit matrix, and the matrix trace will be denoted tr. The external fields in Eq. (1.3) are the metric $g_{\mu\nu}$, the matrix potential $\hat{P}$, and an arbitrary connection defining the commutator curvature:

$$[\nabla_\mu,\nabla_\nu]=\hat{\mathcal{R}}_{\mu\nu}. \tag{1.4}$$

The sign convention for the Ricci scalar R in Eq. (1.3) is such that when acting on a scalar field, the operator $\hat{H}$ with $\hat{P}=0$ and $m=0$ is conformal invariant.

In the present paper, only the effect of the commutator curvature $\hat{\mathcal{R}}_{\mu\nu}$ is considered, and *nevertheless* the action (1.2) is needed with all the three types of external fields present. The dependence of $S_{\rm vac}$ on the metric is needed because the vacuum energy-momentum tensor is obtained by varying the action (1.2) with respect to the metric

$$T^{\mu\nu}_{\rm vac}=\frac{2}{g^{1/2}}\frac{\delta S_{\rm vac}}{\delta g_{\mu\nu}} \tag{1.5}$$

(and next using the retarded resolvent for the nonlocal form factors[3]). The dependence of $S_{\rm vac}$ on the potential is needed because the results for various quantum field models, e.g., for the spinor QED, are obtained by combining the standard loops with $\hat{P}$ generally depending on $\hat{\mathcal{R}}_{\mu\nu}$ (see Ref. [3] and Sec. VIII below).

For the classical action of the commutator curvature one may take the expression

$$S_{\rm cl}=\frac{1}{16\pi\kappa^2}\int dx\, g^{1/2}{\rm tr}\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu} \tag{1.6}$$

with some coupling constant $\kappa^2>0$. In the case of the electromagnetic connection, the κ^2 is to be chosen so that the expression (1.6) be the Maxwell action

$$S_{\rm cl}=-\frac{1}{16\pi}\int dx\, g^{1/2}F_{\mu\nu}F^{\mu\nu} \tag{1.7}$$

[the matrix trace in Eq. (1.6) is always negative [2]]. Denote

$$T^{\mu\nu}_{\rm cl}=\frac{2}{g^{1/2}}\frac{\delta S_{\rm cl}}{\delta g_{\mu\nu}}. \tag{1.8}$$

The energy of the classical electromagnetic radiation and, in the high-frequency approximation, also the outgoing flux of the vacuum energy can be calculated at the future null infinity $\mathcal{I}^+$ [4,2]. The limit $\mathcal{I}^+$ is defined as the limit of infinite luminosity distance r along the null geodesic that, when traced to the future, comes at the instant u of retarded time to the point ϕ of the celestial two-sphere $\mathcal{S}$. One has

$$\tfrac{1}{4}\nabla_\mu v\nabla_\nu v T^{\mu\nu}_{\rm cl}\big|_{\mathcal{I}^+}=-\frac{1}{r^2}\frac{\partial}{\partial u}\mathcal{E}_{\rm cl}(u,\phi)+O\left(\frac{1}{r^3}\right), \tag{1.9}$$

$$\tfrac{1}{4}\nabla_\mu v\nabla_\nu v T^{\mu\nu}_{\rm vac}\big|_{\mathcal{I}^+}=-\frac{1}{r^2}\frac{\partial}{\partial u}\mathcal{E}_{\rm vac}(u,\phi)+O\left(\frac{1}{r^3}\right), \tag{1.10}$$

where

$$\nabla v\big|_{\mathcal{I}^+}=\nabla u+2\nabla r,\quad (\nabla u,\nabla r)\big|_{\mathcal{I}^+}=-1. \tag{1.11}$$

The notation $\partial\mathcal{E}/\partial u$ is introduced to represent the energy loss. Taken with the minus sign, each $\partial\mathcal{E}/\partial u$ is the density of the respective outgoing flux of energy so that the total emitted energy is obtained by integrating $(-\partial\mathcal{E}/\partial u)$ over the two-sphere $\mathcal{S}$ (normalized to have the area 4π) and the time u. Specifically, the total released vacuum energy equals

$$\int_{-\infty}^{\infty}du\int d^2\mathcal{S}(\phi)\left(-\frac{\partial\mathcal{E}_{\rm vac}}{\partial u}\right)=\sum_A \varepsilon_A\langle \text{in vac}|a^{+A}_{\rm out}a^A_{\rm out}|\text{in vac}\rangle \tag{1.12}$$

and equals the total energy of particles created from the in-vacuum by external fields (see, e.g., Ref. [5]). Here $a^{+A}_{\rm out}$, $a^A_{\rm out}$ are the creation and annihilation operators for the out-vacuum, and ε_A is the energy in the out-mode A. Since $T^{\mu\nu}_{\rm cl}$ is energy-dominant [4], the flux $(-\partial\mathcal{E}_{\rm cl}/\partial u)$ is manifestly positive. The flux $(-\partial\mathcal{E}_{\rm vac}/\partial u)$ is sign indefinite because of the quantum uncertainty but the integrated flux (1.12) is positive [2].

Only the external fields generated by sources are considered in Ref. [2] and the present paper. The sources of external fields in Eq. (1.3) are

$$\hat{J}=\hat{P},\quad \hat{J}^\mu=\nabla_\nu\hat{\mathcal{R}}^{\mu\nu},\quad J^{\mu\nu}=R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R, \tag{1.13}$$

where $R^{\mu\nu}$ is the Ricci tensor of the external metric, and the potential $\hat{P}$ is identified with its own source. These classical sources will be referred to as *bare* sources. The bare sources are assumed to have their supports in a spacetime tube with compact spatial sections and a timelike boundary. Their domain of nonstationarity is assumed compact in both space and time [2].

At a large distance from a source, all its manifestations at both classical and quantum levels are governed by a single quantity, its *radiation moment* [2] defined as an integral of

[3] See Ref. [2], and references therein.

the source over a spacelike *hyperplane*. The hyperplane itself is defined as follows. One considers all timelike geodesics that, when traced to the future, reach infinity with one and the same value of energy per unit rest mass ($E>1$) and at one and the same point of the celestial sphere ($\phi\in\mathcal{S}$). These geodesics make a three-parameter congruence which is hypersurface-orthogonal, and the hyperplanes are the hypersurfaces orthogonal to this congruence [2]. Let

$$T_{\gamma\phi}(x)=\text{const} \tag{1.14}$$

be the equation of these hypersurfaces. The parameter γ that, along with ϕ, labels the function $T_{\gamma\phi}(x)$ is a redefined E:

$$\gamma=\frac{\sqrt{E^2-1}}{E},\quad 0<\gamma<1 \tag{1.15}$$

and the function $T_{\gamma\phi}(x)$ itself is normalized by the condition

$$(\nabla T_{\gamma\phi}(x))^2=-(1-\gamma^2). \tag{1.16}$$

The radiation moments of the sources in Eq. (1.13) are the following integrals [2]:

$$\hat{D}=\frac{1}{4\pi}\int dx g^{1/2}\delta(T_{\gamma\phi}(x)-u)\hat{J}(x), \tag{1.17}$$

$$\hat{D}^{\alpha}=\frac{1}{4\pi}\int dx g^{1/2}\delta(T_{\gamma\phi}(x)-u)g^{\alpha}_{\bar\alpha}\hat{J}^{\bar\alpha}(x), \tag{1.18}$$

$$D^{\alpha\beta}=\frac{1}{4\pi}\int dx g^{1/2}\delta(T_{\gamma\phi}(x)-u)g^{\alpha}_{\bar\alpha}g^{\beta}_{\bar\beta}J^{\bar\alpha\bar\beta}(x), \tag{1.19}$$

where $g^{\alpha}_{\bar\alpha}$ is the propagator of the geodetic parallel transport [6] connecting the integration point with the future end point of the geodesic having the parameters γ,ϕ. The moments are tensors at this end point depending parametrically on time u.

At the limit $\gamma=1$ the hyperplane (1.14) becomes null. The vector and tensor moments taken at $\gamma=1$ govern the classical electromagnetic and gravitational radiation. Specifically, for the energy of the electromagnetic waves one has [2]

$$-\frac{\partial}{\partial u}\mathcal{E}_{\rm cl}(u,\phi)=-\frac{1}{4\pi\kappa^2}\operatorname{tr} g_{\alpha\beta}\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}\right)\left(\frac{\partial}{\partial u}\hat{D}^{\beta}\right)\bigg|_{\gamma=1}. \tag{1.20}$$

The expansion of the vector and tensor moments at $\gamma=0$ gives rise to the usual multipole moments [2]. The radiation moments integrated over γ govern the energy of the vacuum particle production. One has [2]

$$\begin{aligned}-\frac{\partial}{\partial u}\mathcal{E}_{\rm vac}(u,\phi)=\frac{1}{(4\pi)^2}\int_0^1 d\gamma\gamma^2\operatorname{tr}\Bigg[&\left(\frac{\partial^2}{\partial u^2}\hat{D}\right)\left(\frac{\partial^2}{\partial u^2}\hat{D}\right)\\ &-\frac{1}{3}\frac{1}{(1-\gamma^2)}g_{\alpha\beta}\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}\right)\left(\frac{\partial}{\partial u}\hat{D}^{\beta}\right)\\ &+\frac{\hat{1}}{30}\left(g_{\mu\alpha}g_{\nu\beta}-\frac{1}{3}g_{\mu\nu}g_{\alpha\beta}\right)\left(\frac{\partial^2}{\partial u^2}D^{\mu\nu}\right)\\ &\times\left(\frac{\partial^2}{\partial u^2}D^{\alpha\beta}\right)\Bigg]+\text{Q.N.},\end{aligned} \tag{1.21}$$

where the abbreviation Q.N. means quantum noise and denotes the sign-indefinite contribution that is present in the vacuum energy flux because of the quantum uncertainty but sums to zero for the whole history [2,5]:

$$\int_{-\infty}^{\infty}du\int d^2\mathcal{S}(\phi)(\text{Q.N.})=0. \tag{1.22}$$

In the equations below the term Q.N. will often be omitted but its presence will be tacitly assumed in all expressions for the vacuum energy.

The significance of the result (1.21) is that it brings the quantum problem of particle creation to the level of the classical problem of radiation of waves. This is seen from a comparison of Eq. (1.21) with (1.20). The integral over γ in Eq. (1.21) is none other than the integral over the energies of the outgoing particles. The integrand gives, therefore, the energy spectrum of the vacuum radiation.

Expression (1.21) is the starting point of the present work. It is seen that in the case of the vector moment (and only in this case) the validity of this expression is limited by the condition

$$\operatorname{tr} g_{\alpha\beta}\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}\right)\left(\frac{\partial}{\partial u}\hat{D}^{\beta}\right)\bigg|_{\gamma=1}=0 \tag{1.23}$$

which in view of Eq. (1.20) is a condition that the vector source does not radiate classically:

$$\frac{\partial}{\partial u}\mathcal{E}_{\rm cl}(u,\phi)\equiv 0. \tag{1.24}$$

If it does, the integral in Eq. (1.21) has a pole at $\gamma=1$. The appearance of this pole is a manifestation of the infrared disaster that occurs when the classical and quantum radiations overlap, and the back reaction of the vacuum is neglected.

As pointed out in Ref. [2], the calculation in this reference is insufficient for a removal of the limitation (1.24). If this limitation does not hold, the calculation in Ref. [2] needs to be revised. The revised calculation with all the needed amendments is carried out in the present paper.

B. Summary of the results

Below I shall consider only the contribution of the vector source assuming that the other contributions in Eq. (1.21) are absent:

$$-\frac{\partial\mathcal{E}_{\rm vac}}{\partial u}=-\frac{1}{(4\pi)^2}\frac{1}{3}\int_0^1 d\gamma\frac{\gamma^2}{(1-\gamma^2)}{\rm tr}\, g_{\alpha\beta}\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}\right)\left(\frac{\partial}{\partial u}\hat{D}^{\beta}\right)+{\rm Q.N.} \tag{1.25}$$

It is useful to decompose the vector moment over the vector basis at infinity:

$$g_{\alpha\beta}|_{r\to\infty}=-\nabla_\alpha u\nabla_\beta u-(\nabla_\alpha u\nabla_\beta r+\nabla_\alpha r\nabla_\beta u)+\frac{1}{2}(m_\alpha m_\beta^*+m_\alpha^* m_\beta), \tag{1.26}$$

where m is the complex null vector tangent to the two-sphere $\mathcal{S}$, and m^* is its complex conjugate. The projection

$$[\nabla_\mu u+(1-\gamma)\nabla_\mu r]\hat{D}^\mu\equiv\hat{e}={\rm const} \tag{1.27}$$

is the full conserved charge of the bare source [2]. Hence

$$\begin{aligned}{\rm tr}\, g_{\alpha\beta}\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}\right)\left(\frac{\partial}{\partial u}\hat{D}^{\beta}\right)&={\rm tr}\left[\left(m_\alpha\frac{\partial}{\partial u}\hat{D}^{\alpha}\right)\left(m_\beta\frac{\partial}{\partial u}\hat{D}^{\beta}\right)^*+(1-\gamma^2)\right.\\&\left.\times\left(\nabla_\alpha r\frac{\partial}{\partial u}\hat{D}^{\alpha}\right)^2\right].\end{aligned} \tag{1.28}$$

The transverse projections of the moment taken at $\gamma=1$ define the complex *news function* of the electromagnetic waves[4]

$$\frac{\partial}{\partial u}m_\alpha\hat{D}^{\alpha}|_{\gamma=1}\equiv\frac{\partial}{\partial u}\hat{C}(u,\phi) \tag{1.29}$$

so that, by Eqs. (1.20) and (1.28),

$$-\frac{\partial\mathcal{E}_{\rm cl}}{\partial u}=-\frac{1}{4\pi\kappa^2}{\rm tr}\left(\frac{\partial}{\partial u}\hat{C}\right)\left(\frac{\partial}{\partial u}\hat{C}^*\right). \tag{1.30}$$

Finally, the longitudinal projection of the moment

$$\nabla_\alpha r\hat{D}^{\alpha}\equiv\hat{D}_{||}(u,\phi) \tag{1.31}$$

plays no role in classical theory but, as shown in the present paper, it is responsible for the vacuum creation of charge. Upon the insertion of the decomposition (1.28) in Eq. (1.25) it is seen that the contribution of the longitudinal projection of the moment is finite. It turns out that this contribution alone is the correct result for the energy of created charged particles [Eq. (1.38) below]. The contribution of the transverse projections of the moment is divergent but, if the waves are emitted, the calculation of Eq. (1.25) should be revised [2].

The result of the revised calculation is that the quantity (1.25) *indeed* diverges. The pole at $\gamma=1$ goes but its place is taken up by an infrared divergence. Only the total energy flux

$$-\frac{\partial\mathcal{E}_{\rm tot}}{\partial u}=-\frac{\partial\mathcal{E}_{\rm cl}}{\partial u}-\frac{\partial\mathcal{E}_{\rm vac}}{\partial u} \tag{1.32}$$

is finite. The point here is that the electromagnetic field to be inserted in Eq. (1.32) should solve the expectation-value equations. To an appropriate order in the coupling constant $\partial\mathcal{E}_{\rm vac}/\partial u$ can be calculated with the bare source but $\partial\mathcal{E}_{\rm cl}/\partial u$ should already be quantum corrected, and this correction does not boil down to a renormalization of the coupling constant. This correction is infrared divergent, and it cancels the divergence in $\partial\mathcal{E}_{\rm vac}/\partial u$. The final result for the total energy flux (1.32) is

$$\begin{aligned}-\frac{\partial\mathcal{E}_{\rm tot}}{\partial u}&=-\frac{1}{4\pi\kappa^2}{\rm tr}\left(\frac{\partial}{\partial u}\hat{C}_{\rm eff}\right)\left(\frac{\partial}{\partial u}\hat{C}^*_{\rm eff}\right)-\frac{1}{(4\pi)^2}\\&\times\frac{1}{3}\int_0^1 d\gamma\frac{\gamma^2}{(1-\gamma^2)}{\rm tr}\, g_{\alpha\beta}\left[\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}\right)\left(\frac{\partial}{\partial u}\hat{D}^{\beta}\right)\right.\\&\left.-\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}\right)\left(\frac{\partial}{\partial u}\hat{D}^{\beta}\right)\Big|_{\gamma=1}\right]\end{aligned} \tag{1.33}$$

with the effective news function

$$\begin{aligned}\frac{\partial}{\partial u}\hat{C}_{\rm eff}(u,\phi)&=\left[1-\frac{\kappa^2}{24\pi}\left(\mathbf{c}-\log 2+\frac{25}{12}\right)\right]\frac{\partial}{\partial u}\hat{C}(u,\phi)\\&-\frac{\kappa^2}{24\pi}\frac{\partial}{\partial u}\int_{-\infty}^{u}d\tau\log[m(u-\tau)]\frac{\partial}{\partial\tau}\hat{C}(\tau,\phi).\end{aligned} \tag{1.34}$$

Here κ^2 is the renormalized coupling constant, $\mathbf{c}$ is the Euler constant and, distinct from Eq. (1.25), the result is not independent of the mass m even in the high-frequency approximation. The quantities $\hat{D}^\alpha$ and $\hat{C}$ in the expressions above pertain to the original *bare* source.

Expression (1.33) is to be compared with the sum of expressions (1.30) and (1.25). It is seen that the pole at $\gamma=1$ gets eliminated but there appears a finite vacuum contribution to the energy of the electromagnetic radiation. The retarded integral along $\mathcal{I}^+$ in Eq. (1.34) represents a nonlocal tail of the electromagnetic radiation caused by the vacuum stress. Technically, when a point tends to $\mathcal{I}^+$, its past light cone becomes a sum of a null hyperplane and a single null generator that merges with $\mathcal{I}^+$ [2]. The nonlocal radiation tail is a contribution of this generator (see Appendix C).

The energy flux in Eq. (1.33) belongs partially to charged particles and partially to the electromagnetic waves. The fact that only the total energy flux is calculable signifies that the separation of the vacuum energy between the charged par-

[4] I am using the terminology of Bondi [7].

ticles and the electromagnetic waves is subject to the quantum uncertainty. However, up to this uncertainty, the two contributions should be separable since they are measurable separately:

$$-\frac{\partial \mathcal{E}_{\rm tot}}{\partial u}=-\frac{\partial \mathcal{E}_{\rm charge}}{\partial u}-\frac{\partial \mathcal{E}_{\rm waves}}{\partial u}. \tag{1.35}$$

A calculation of the flux of charge helps to make this separation. The density of the outgoing flux of charge can be calculated from the expectation-value equations for the electromagnetic field (Sec. III below). The result for this density reads

$$-\frac{\partial}{\partial u}\hat{e}(u,\phi)=-\frac{\kappa^2}{3(4\pi)^2}\frac{\partial}{\partial u}\hat{D}_{||}\Big|_{\gamma=1} \tag{1.36}$$

so that the total released charge is

$$\int_{-\infty}^{\infty}du\int d^2\mathcal{S}(\phi)\left(-\frac{\partial\hat{e}}{\partial u}\right)$$
$$=\frac{\kappa^2}{3(4\pi)^2}\int d^2\mathcal{S}(\phi)[\hat{D}_{||}(u=-\infty)$$
$$-\hat{D}_{||}(u=+\infty)]\big|_{\gamma=1}. \tag{1.37}$$

Hence one may infer that the portion of the total energy flux (1.33) that goes with the charged particles is the one associated with the longitudinal projection of the moment

$$-\frac{\partial \mathcal{E}_{\rm charge}}{\partial u}=-\frac{1}{(4\pi)^2}\frac{1}{3}\int_0^1 d\gamma\,\gamma^2\,{\rm tr}\left(\frac{\partial}{\partial u}\hat{D}_{||}\right)^2+{\rm Q.N.} \tag{1.38}$$

The remaining energy in the total flux (1.33) goes with the electromagnetic radiation. Since expression (1.33) is valid only in the high-frequency approximation anyway, condition (1.1) can be used for its further simplification. It will be recalled that the domain of nonstationarity of the bare source is assumed compact. Its temporal scale (in time u) is a purely classical quantity of order $1/\nu$. Therefore, if in Eq. (1.34) one writes

$$\log[m(u-\tau)]=\log\frac{m}{\nu}+\log[\nu(u-\tau)], \tag{1.39}$$

the contribution of the second term will be of order $O(1)$ whereas the contribution of the first term will be large:

$$\frac{\partial}{\partial u}\hat{C}_{\rm eff}(u,\phi)=\left\{1-\frac{\kappa^2}{24\pi}\log\frac{m}{\nu}+O\left(\frac{m}{\nu}\right)^0\right\}\frac{\partial}{\partial u}\hat{C}(u,\phi). \tag{1.40}$$

As a result, for u in the support of the bare news function, the radiation flux becomes merely a renormalized classical one:

$$-\frac{\partial \mathcal{E}_{\rm waves}}{\partial u}=-Z\frac{\partial \mathcal{E}_{\rm cl}}{\partial u}, \tag{1.41}$$

$$Z=1-\frac{\kappa^2}{12\pi}\log\frac{m}{\nu}+O\left(\frac{m}{\nu}\right)^0. \tag{1.42}$$

Note the sign of the quantum correction. The radiation gets amplified.

The results above pertain to the standard loop. For other models the vacuum fluxes are multiples of the respective fluxes for the standard loop (Sec. VIII). Thus, for the spinor QED, the flux of charge is *twice* the one in Eq. (1.36), and the quantum correction to the flux of energy is *twice* the one in Eq. (1.33). Only the numerical constant which in Eq. (1.34) is 25/12 needs to be calculated anew but this constant is anyway unimportant. The explicit results for the spinor QED are obtained by introducing the said factor of 2 and substituting

$$\hat{\mathcal{R}}_{\mu\nu}=-iqF_{\mu\nu}\hat{1},\quad \hat{e}=-iqe\hat{1},\quad \kappa^2=4q^2,\quad {\rm tr}\hat{1}=4, \tag{1.43}$$

where $F_{\mu\nu}$ is the Maxwell tensor, e is the electric charge of the source, q is the electron's charge, and m in Eq. (1.34) is the electron's mass.

In conclusion it will be noted that the result obtained cannot be the end of the story since, obviously, it violates the energy conservation law. Indeed, the frequency ν is proportional to the energy of the bare source and, since the factor $\log(\nu/m)$ can be arbitrarily large, at a sufficiently large ν the source will radiate more energy than it has initially. In this respect the present case is similar to the case of charged spherical shell considered in Ref. [2]. A charged spherical shell expanding in the self field emits no electromagnetic waves (neither classical nor quantum) but it creates charged particles from the vacuum and, at an ultrarelativistic energy, this radiation violates the energy conservation law [2]. The measure of the violation is in both cases one and the same, $\kappa^2\log\nu$, and the cause is also one and the same: the problem has not been made fully self-consistent. Although in the present case the back reaction of the vacuum on the electromagnetic field is taken into account (otherwise the emitted energy would not even be finite), its reaction on the motion of the source is not. This task remains beyond the scope of the present work but it may be conjectured that the missing back reaction effect is nonanalytic in the coupling constant.

Equations (1.33)–(1.37) and their corollaries are the main results of the present work. Their derivation is given below. A reader not interested in the technical details may still want to read Secs. II–IV. Section II presents the general scheme of the calculation including the important intermediate results and displays the mechanism of the vacuum back reaction. Section III presents the solution of the expectation-value equations and the calculation of the emission of charge. In Sec. IV, creation of massive particles is considered, and a criterion of the high-frequency approximation is derived.

The technical details are presented in Secs. V–VIII. The calculation required in the present work is more complicated than in Ref. [2] because the nonlocal form factors act now on functions having noncompact spatial supports. For a test function X, compactness of the spatial support is equivalent to the following powers of decrease at null infinities $\mathcal{I}^{\pm}$ and spatial infinity i^0:

$$X|_{\mathcal{I}^{\pm}}=O\left(\frac{1}{r^3}\right),\quad X|_{i^0}=O\left(\frac{1}{r^4}\right). \tag{1.44}$$

The behaviors of the form factors derived or quoted in Ref. [2] are valid only under conditions (1.44). For the present calculation, critical is the behavior of the test function at $\mathcal{I}^+$. The test function that does not satisfy condition (1.44) at $\mathcal{I}^+$ will be called *singular* at $\mathcal{I}^+$. The operators $\log(-\Box)$ and $1/\Box$ with test functions singular at $\mathcal{I}^+$ are considered in Appendix C. The behaviors of the third-order form factors at $\mathcal{I}^+$ are obtained in Appendix B. Appendix A summarizes the structure of the one-loop form factors.

II. THE MECHANISM OF THE VACUUM BACK REACTION

For obtaining the vacuum energy-momentum tensor to lowest order in the commutator curvature, one needs the terms in the effective action quadratic in the commutator curvature and linear in the gravitational curvature, i.e., quadratic terms of order $\hat{\mathcal{R}}\times\hat{\mathcal{R}}$ and cubic terms of order $R\times\hat{\mathcal{R}}\times\hat{\mathcal{R}}$. Their general form is [8,9]

$$S_{\rm vac}=S_{\rm vac}(2)+S_{\rm vac}(3)+\text{higher-order terms}, \tag{2.1}$$

$$S_{\rm vac}(2)=\frac{1}{2(4\pi)^2}\int dx g^{1/2}\,{\rm tr}\,\hat{\mathcal{R}}_{\mu\nu}\gamma(-\Box)\hat{\mathcal{R}}^{\mu\nu}, \tag{2.2}$$

$$S_{\rm vac}(3)=\frac{1}{2(4\pi)^2}\int dx g^{1/2} \times{\rm tr}\sum_i \Gamma_i(-\Box_1,-\Box_2,-\Box_3)R_1\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3(i) \tag{2.3}$$

with some form factors $\gamma(-\Box)$ and $\Gamma_i(-\Box_1,-\Box_2,-\Box_3)$. In the basis of nonlocal invariants of third order [9], there are six invariants of the needed type:

$$R_1\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3(7)=R_1\hat{\mathcal{R}}_2^{\mu\nu}\hat{\mathcal{R}}_{3\mu\nu},$$
$$R_1\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3(8)=R_1^{\alpha\beta}\hat{\mathcal{R}}_{2\alpha}{}^{\mu}\hat{\mathcal{R}}_{3\beta\mu},$$
$$R_1\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3(18)=R_{1\alpha\beta}\nabla_\mu\hat{\mathcal{R}}_2^{\mu\alpha}\nabla_\nu\hat{\mathcal{R}}_3^{\nu\beta},$$
$$R_1\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3(19)=R_1^{\alpha\beta}\nabla_\alpha\hat{\mathcal{R}}_2^{\mu\nu}\nabla_\beta\hat{\mathcal{R}}_{3\mu\nu},$$
$$R_1\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3(20)=R_1\nabla_\alpha\hat{\mathcal{R}}_2^{\alpha\mu}\nabla^\beta\hat{\mathcal{R}}_{3\beta\mu},$$
$$R_1\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3(21)=R_1^{\mu\nu}\nabla_\mu\nabla_\lambda\hat{\mathcal{R}}_2^{\lambda\alpha}\hat{\mathcal{R}}_{3\alpha\nu} \tag{2.4}$$

(I preserve the numbers that these invariants have in the full list of Ref. [9].)

For κ^2 in Eq. (1.6) to be the renormalized coupling constant, the form factor $\gamma(-\Box)$ should satisfy the normalization condition $\gamma(0)=0$. The normalized $\gamma(-\Box)$ calculated for the standard loop is

$$\gamma(-\Box)=-\frac{1}{2}\int_0^1 d\alpha\int_0^{\alpha(1-\alpha)}dx\log\left(1-x\frac{\Box}{m^2}\right). \tag{2.5}$$

When applied to a high-frequency field, this operator takes the form

$$\gamma(-\Box)=\frac{1}{12}\left[\frac{8}{3}-\log\left(-\frac{\Box}{m^2}\right)\right]+O(m^2). \tag{2.6}$$

(The high-frequency limit is considered in Sec. IV.) The constant 8/3 in Eq. (2.6) is observable since it accounts for the difference between the static regime in which the total initially stored charge is calculated and the high-frequency regime in which the emission of charge is calculated. Neither this constant nor the term in $\log m^2$ can be discarded when the operator (2.6) acts on a function singular at $\mathcal{I}^+$ (cf. Ref. [2]).

The third-order form factors Γ_i admit the massless limit and, in the high-frequency approximation, can be taken massless from the outset. One can then use the results of Ref. [9] where the massless Γ_i are calculated for all cubic invariants including the ones in Eq. (2.4).

For obtaining $T^{\mu\nu}_{\rm vac}$ at $\mathcal{I}^+$ one does not need the exact form factors. It suffices to have the asymptotic behaviors of $\Gamma_i(-\Box_1,-\Box_2,-\Box_3)$ with one of the arguments small and the others fixed. The difference with Ref. [2] is that these behaviors are now needed including the terms $O(\Box^0)$. The algorithms of extracting the needed terms are derived in Appendix B.

The contribution of the second-order action (2.2) to $T^{\mu\nu}_{\rm vac}$ will be divided into two:

$$\frac{2}{g^{1/2}}\frac{\delta S_{\rm vac}(2)}{\delta g_{\mu\nu}}=T^{\mu\nu}_{\rm vac}(1)+T^{\mu\nu}_{\rm vac}(2) \tag{2.7}$$

with $T^{\mu\nu}_{\rm vac}(2)$ the contribution of the variation of the form factor

$$\int dx g^{1/2}T^{\mu\nu}_{\rm vac}(2)\,\delta g_{\mu\nu}=\frac{1}{(4\pi)^2}\int dx g^{1/2}\,{\rm tr}\,\hat{\mathcal{R}}_{\mu\nu}\delta\gamma(-\Box)\hat{\mathcal{R}}^{\mu\nu}. \tag{2.8}$$

Denoting $T^{\mu\nu}_{\rm vac}(3)$ the contribution of the third-order action (2.3), one has

$$T^{\mu\nu}_{\rm vac}=T^{\mu\nu}_{\rm vac}(1)+T^{\mu\nu}_{\rm vac}(2)+T^{\mu\nu}_{\rm vac}(3). \tag{2.9}$$

The vacuum energy flux in Eq. (1.10) will then also be a sum of the respective three contributions:

$$-\frac{\partial\mathcal{E}_{\rm vac}}{\partial u}=-\frac{\partial\mathcal{E}_{\rm vac}(1)}{\partial u}-\frac{\partial\mathcal{E}_{\rm vac}(2)}{\partial u}-\frac{\partial\mathcal{E}_{\rm vac}(3)}{\partial u}. \quad (2.10)$$

The expectation-value equations are obtained by varying the total action $S_{\rm cl}+S_{\rm vac}$ with respect to the connection field. These are the following equations for the source of the full, quantum-corrected, commutator curvature:

$$\hat{J}^{\mu}_{\rm full}+\frac{\kappa^2}{2\pi}\gamma(-\Box)\hat{J}^{\mu}_{\rm full}=\hat{J}^{\mu}_{\rm bare} \quad (2.11)$$

with the retarded boundary conditions for $\gamma(-\Box)$ [2]. Solving them iteratively one obtains

$$\hat{J}^{\mu}_{\rm full}=\hat{J}^{\mu}_{\rm bare}-\frac{\kappa^2}{2\pi}\gamma(-\Box)\hat{J}^{\mu}_{\rm bare} \quad (2.12)$$

and, hence,

$$\hat{\mathcal{R}}^{\mu\nu}_{\rm full}=\hat{\mathcal{R}}^{\mu\nu}_{\rm bare}-\frac{\kappa^2}{2\pi}\gamma(-\Box)\hat{\mathcal{R}}^{\mu\nu}_{\rm bare}. \quad (2.13)$$

For displaying the mechanism of the vacuum back reaction, it suffices to write down the expressions for $T^{\mu\nu}_{\rm cl}$ and $T^{\mu\nu}_{\rm vac}(1)$:

$$T^{\mu\nu}_{\rm cl}=-\frac{1}{4\pi\kappa^2}{\rm tr}\left(\hat{\mathcal{R}}^{\mu\lambda}\hat{\mathcal{R}}^{\nu}{}_{\lambda}-\frac{1}{4}g^{\mu\nu}\hat{\mathcal{R}}_{\alpha\beta}\hat{\mathcal{R}}^{\alpha\beta}\right), \quad (2.14)$$

$$T^{\mu\nu}_{\rm vac}(1)=-\frac{1}{8\pi^2}{\rm tr}\left(\hat{\mathcal{R}}^{\mu\lambda}\gamma(-\Box)\hat{\mathcal{R}}^{\nu}{}_{\lambda}-\frac{1}{4}g^{\mu\nu}\hat{\mathcal{R}}_{\alpha\beta}\gamma(-\Box)\hat{\mathcal{R}}^{\alpha\beta}\right). \quad (2.15)$$

Using Eq. (2.13) one finds

$$T^{\mu\nu}_{\rm cl}|_{J=J_{\rm full}}=[T^{\mu\nu}_{\rm cl}-2T^{\mu\nu}_{\rm vac}(1)]|_{J=J_{\rm bare}}. \quad (2.16)$$

The total energy-momentum tensor of the commutator curvature $T^{\mu\nu}_{\rm cl}+T^{\mu\nu}_{\rm vac}$ is then

$$\begin{aligned}T^{\mu\nu}_{\rm tot}&=[T^{\mu\nu}_{\rm cl}+T^{\mu\nu}_{\rm vac}(1)+T^{\mu\nu}_{\rm vac}(2)+T^{\mu\nu}_{\rm vac}(3)]|_{J=J_{\rm full}}\\&=[T^{\mu\nu}_{\rm cl}-T^{\mu\nu}_{\rm vac}(1)+T^{\mu\nu}_{\rm vac}(2)+T^{\mu\nu}_{\rm vac}(3)]|_{J=J_{\rm bare}}\end{aligned} \quad (2.17)$$

and hence the total energy flux is

$$-\frac{\partial\mathcal{E}_{\rm tot}}{\partial u}=-\left(\frac{\partial\mathcal{E}_{\rm cl}}{\partial u}-\frac{\partial\mathcal{E}_{\rm vac}(1)}{\partial u}+\frac{\partial\mathcal{E}_{\rm vac}(2)}{\partial u}+\frac{\partial\mathcal{E}_{\rm vac}(3)}{\partial u}\right)\Bigg|_{J=J_{\rm bare}}. \quad (2.18)$$

Thus the effect of the vacuum back reaction is *changing the sign of* $T^{\mu\nu}_{\rm vac}(1)$. As will be seen in a moment, this effect is dramatic.

Note that if the substitution (2.13) was made in the action, then, after varying with respect to $g_{\mu\nu}$, both $T^{\mu\nu}_{\rm vac}(1)$ and $T^{\mu\nu}_{\rm vac}(2)$ would change their signs. This procedure is incorrect because it amounts to varying the action in $g_{\mu\nu}$ at fixed $\hat{\mathcal{R}}_{\rm bare}$ whereas the energy-momentum tensor is obtained by varying the action in $g_{\mu\nu}$ at fixed $\hat{\mathcal{R}}_{\rm full}$.[5] This makes difference since the relation between $\hat{\mathcal{R}}_{\rm full}$ and $\hat{\mathcal{R}}_{\rm bare}$ itself depends on the metric through the operator $\Box$. The correct procedure is making the substitution (2.13) in the energy-momentum tensor.

The dictum that $T^{\mu\nu}_{\rm vac}$ at $\mathcal{I}^+$ is infrared divergent means that expansion (1.10) does not hold. Rather there is an expansion of the form

$$T^{\mu\nu}_{\rm vac}|_{\mathcal{I}^+}={\rm terms}\frac{\log r}{r^2}+{\rm terms}\frac{1}{r^2}+O\left(\frac{1}{r^3}\right). \quad (2.19)$$

If this was the behavior of the total energy-momentum tensor, the expectation-value spacetime would fail to be asymptotically flat. This is not the case but, in the intermediate expressions, the factor $\log r$ will conventionally be included in $\partial\mathcal{E}_{\rm vac}/\partial u$ thereby considering this energy flux as divergent.

The contributions (2.10) to $\partial\mathcal{E}_{\rm vac}/\partial u$ are calculated in Secs. V–VII below. Their main ingredient is the $\gamma=1$ radiation moment

$$\hat{D}^{\alpha}|_{\gamma=1}\equiv\hat{D}^{\alpha}_{\mathbf{1}}. \quad (2.20)$$

The latter notation is used everywhere below. The results are

$$\begin{aligned}-\frac{\partial\mathcal{E}_{\rm vac}(1)}{\partial u}=\frac{1}{(4\pi)^2}\frac{1}{6}{\rm tr}\Bigg\{&-\left(\log mr+2\mathbf{c}-\log 2+\frac{8}{3}\right)\\&\times\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}_{\mathbf{1}}\right)\left(\frac{\partial}{\partial u}\hat{D}_{\mathbf{1}\alpha}\right)-\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}_{\mathbf{1}}\right)\\&\times\frac{\partial}{\partial u}\int_{-\infty}^{u}d\tau\log[m(u-\tau)]\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}\alpha}(\tau)\Bigg\},\end{aligned} \quad (2.21)$$

$$\begin{aligned}-\frac{\partial\mathcal{E}_{\rm vac}(2)}{\partial u}=\frac{1}{(4\pi)^2}\frac{1}{6}{\rm tr}\Bigg\{&\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}_{\mathbf{1}}\right)\frac{\partial}{\partial u}\int_{-\infty}^{u}d\tau\log(u-\tau)\\&\times\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}\alpha}(\tau)-\frac{\partial}{\partial u}\int_{-\infty}^{u}d\tau\log(u-\tau)\\&\times\left(\frac{\partial}{\partial\tau}\hat{D}^{\alpha}_{\mathbf{1}}(\tau)\right)\left(\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}\alpha}(\tau)\right)\Bigg\}+{\rm Q.N.},\end{aligned} \quad (2.22)$$

[5]The effective action does not know bare fields. It is a functional of just one field which after varying and solving the equations (with the appropriate boundary conditions for the resolvents) becomes the full expectation value. Therefore, all currents in the expectation-value equations, including the energy-momentum tensor, are obtained by varying with respect to the full fields keeping the other full fields fixed.

$$-\frac{\partial\mathcal{E}_{\rm vac}(3)}{\partial u}=\frac{1}{(4\pi)^2}\frac{1}{6}{\rm tr}\Bigg\{-\left(\log r+\log 2-\frac{3}{2}\right)\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}_{\mathbf{1}}\right)\times\left(\frac{\partial}{\partial u}\hat{D}_{\mathbf{1}\alpha}\right)+\frac{\partial}{\partial u}\int_{-\infty}^{u}d\tau\log(u-\tau)\times\left(\frac{\partial}{\partial\tau}\hat{D}^{\alpha}_{\mathbf{1}}(\tau)\right)\left(\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}\alpha}(\tau)\right)-2\int_0^1 d\gamma\frac{\gamma^2}{1-\gamma^2}\left[\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}\right)\left(\frac{\partial}{\partial u}\hat{D}_{\alpha}\right)-\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}_{\mathbf{1}}\right)\times\left(\frac{\partial}{\partial u}\hat{D}_{\mathbf{1}\alpha}\right)\right]\Bigg\}+\text{Q.N.} \tag{2.23}$$

Owing to the conservation law (1.27), only the transverse projections of $\hat{D}^{\alpha}_{\mathbf{1}}$ survive in these expressions. Therefore, under the limitation (1.24) the contributions $\partial\mathcal{E}_{\rm vac}(1)/\partial u$ and $\partial\mathcal{E}_{\rm vac}(2)/\partial u$ vanish, and the contribution $\partial\mathcal{E}_{\rm vac}(3)/\partial u$ gives back the result of Ref. [2].

When the limitation (1.24) does not hold, the contribution $\partial\mathcal{E}_{\rm vac}(1)/\partial u$ is infrared divergent. The contribution $\partial\mathcal{E}_{\rm vac}(2)/\partial u$ is not but it has another pathology. The total-derivative term in Eq. (2.22) does not vanish in the integral over time. On the contrary, the behavior of this term at late time is

$$\frac{\partial}{\partial u}\int_{-\infty}^{u}d\tau\log(u-\tau)\left(\frac{\partial}{\partial\tau}\hat{D}^{\alpha}_{\mathbf{1}}(\tau)\right)\left(\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}\alpha}(\tau)\right)\Bigg|_{u\to\infty}=\frac{1}{u}\int_{-\infty}^{\infty}d\tau\left(\frac{\partial}{\partial\tau}\hat{D}^{\alpha}_{\mathbf{1}}(\tau)\right)\left(\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}\alpha}(\tau)\right) \tag{2.24}$$

so that the integrated flux (2.22) diverges:

$$\int_{-\infty}^{\infty}du\frac{\partial\mathcal{E}_{\rm vac}(2)}{\partial u}=\infty. \tag{2.25}$$

The contribution $\partial\mathcal{E}_{\rm vac}(3)/\partial u$ contains the divergences of both types. In the sum of the three contributions the divergence of the integral in time cancels but the infrared divergence *doubles*:

$$-\frac{\partial\mathcal{E}_{\rm vac}}{\partial u}=-\frac{1}{(4\pi)^2}\frac{1}{3}(\log r){\rm tr}\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}_{\mathbf{1}}\right)\left(\frac{\partial}{\partial u}\hat{D}_{\mathbf{1}\alpha}\right)+O(1). \tag{2.26}$$

Only in the total sum (2.18) with the changed sign of $\partial\mathcal{E}_{\rm vac}(1)/\partial u$ both divergences cancel, and the finite result (1.33) emerges.

The cancellations outlined above do not depend on the relative sign and coefficient between $S_{\rm cl}$ and $S_{\rm vac}$ [the κ^2 in Eq. (1.6) is in fact kept arbitrary] but they depend crucially on the balance between $S_{\rm vac}(2)$ and $S_{\rm vac}(3)$. As seen from the expressions (2.21)–(2.23), there is a precise relation between the respective contributions, and this relation maintains for other field models (Sec. VIII) despite the fact that $S_{\rm vac}(2)$ emerges from the purely electromagnetic coupling whereas $S_{\rm vac}(3)$ represents the vertices with the gravitational coupling. Owing to this relation, the final result is rigidly tied to the overall coefficient of the action $S_{\rm vac}(2)$ which is merely the β function. A knowledge of this coefficient is in the end sufficient for obtaining the vacuum radiation fluxes.

III. THE MEAN ELECTROMAGNETIC FIELD AND EMISSION OF CHARGE

Since the quantum correction to the electromagnetic field cancels the infrared divergence in the vacuum energy, it should itself be infrared divergent. This point is clarified below but, before considering the expectation-value equations, it is useful to make a general analysis of the asymptotic properties of the commutator curvature and its source in the case where there is an emission of both waves and charge. To make difference with the notation already used, the quantities in this analysis will be distinguished with boldface.

The existence of a flux of charge at a large distance from the source implies that $\hat{\mathbf{J}}^{\alpha}$ falls off at $\mathcal{I}^+$ as

$$\hat{\mathbf{J}}^{\alpha}|_{\mathcal{I}^+}=\frac{\mathbf{j}^{\alpha}(u,\phi)}{r^2}+O\left(\frac{1}{r^3}\right) \tag{3.1}$$

with some coefficient $\mathbf{j}^{\alpha}(u,\phi)$. This is the most general behavior admissible for an isolated system. Although the support of the source $\hat{\mathbf{J}}^{\alpha}$ is no more confined to a spacetime tube, its domain of nonstationarity must remain compact in time in order that all fluxes die out in the past and future of $\mathcal{I}^+$. More generally, the source should be asymptotically stationary in the past and future. To account for this property in the past, it will be assumed that the domain of nonstationarity of $\hat{\mathbf{J}}^{\alpha}$ is confined to the interior of some future light cone $u=u_-$. Then

$$\mathbf{j}^{\alpha}(u,\phi)|_{u<u_-}=0. \tag{3.2}$$

The density of the flux of charge from a source is expressed through the coefficient in Eq. (3.1) as follows [2]:

$$-\frac{\partial}{\partial u}\hat{\mathbf{e}}(u,\phi)=\frac{1}{8\pi}\nabla_{\alpha}v(r^2\hat{\mathbf{J}}^{\alpha})|_{\mathcal{I}^+}=\frac{1}{8\pi}\nabla_{\alpha}v\,\mathbf{j}^{\alpha}(u,\phi) \tag{3.3}$$

with ∇v in Eq. (1.11). The function

$$\hat{\mathbf{e}}(u)\equiv\hat{\mathbf{e}}(-\infty)+\int_{-\infty}^{u}d\bar{u}\int d^2\mathcal{S}(\phi)\frac{\partial}{\partial\bar{u}}\hat{\mathbf{e}}(\bar{u},\phi) \tag{3.4}$$

defined by Eq. (3.3) up to an additive constant $\hat{\mathbf{e}}(-\infty)$ can be written as an integral over the future light cone [2]

$$\hat{\mathbf{e}}(u)=\frac{1}{4\pi}\int d\bar{x}\,\bar{g}^{1/2}\delta(\bar{u}-u)\bar{\nabla}_{\bar{\mu}}\bar{u}\,\bar{\hat{\mathbf{J}}}^{\bar{\mu}}(\bar{x}) \tag{3.5}$$

provided that the constant

$$\hat{\mathbf{e}}\equiv\hat{\mathbf{e}}(-\infty) \tag{3.6}$$

is taken as a conserved integral over an arbitrary (complete) spacelike hypersurface

$$\hat{\mathbf{e}} = \frac{1}{4\pi}\int d\overline{x}\, \overline{g}^{1/2}\delta(\tau(\overline{x}))\overline{\nabla}_{\overline{\mu}}\tau(\overline{x})\hat{\mathbf{J}}^{\overline{\mu}}(\overline{x}), \quad (\nabla\tau)^2<0. \tag{3.7}$$

The function $\hat{\mathbf{e}}(u)$ may be called the Bondi charge, and the constant $\hat{\mathbf{e}}$ the Arnowitt-Deser-Misner (ADM) charge since their meaning is the same as of the Bondi and ADM masses.[6] The ADM charge is the total charge of the source $\hat{\mathbf{J}}$ in the initial state, i.e. before the beginning of emission. The Bondi charge is the charge that remains in a compact domain by the instant u of retarded time in the process of emission. The ADM charge is conserved because at any instant u it equals a sum of the charge emitted by this instant and the charge remaining by this instant, Eq. (3.4).

Consider the conservation equation

$$\nabla_\alpha \hat{\mathbf{J}}^\alpha = 0. \tag{3.8}$$

Inserting the expansion (3.1) in Eq. (3.8) one obtains

$$\frac{\partial}{\partial u}(\nabla_\alpha u \mathbf{j}^\alpha) = 0, \tag{3.9}$$

whence, in view of Eq. (3.2),

$$\nabla_\alpha u \mathbf{j}^\alpha = 0. \tag{3.10}$$

The latter equation makes it possible to express the flux of charge in Eq. (3.3) through the longitudinal projection of $\mathbf{j}^\alpha$:

$$-\frac{\partial}{\partial u}\hat{\mathbf{e}}(u,\phi) = \frac{1}{4\pi}\nabla_\alpha r \mathbf{j}^\alpha(u,\phi). \tag{3.11}$$

Thus the longitudinal projection $\nabla_\alpha r \mathbf{j}^\alpha$ of the residue in Eq. (3.1) is responsible for the emission of charge, the projection $\nabla_\alpha u \mathbf{j}^\alpha$ vanishes but no conclusion can be made on the transverse projections $m_\alpha \mathbf{j}^\alpha$. Their vanishing does not follow and, at this stage, their role remains unclear.

For obtaining the behavior of the commutator curvature one must first consider the question of convergence of the moment $\hat{\mathbf{D}}_{\mathbf{1}}^\alpha$ of the source $\hat{\mathbf{J}}$. The analysis of convergence is carried out in Ref. [2]. When applied to the present case, it gives the following result. The projection of $\hat{\mathbf{D}}_{\mathbf{1}}^\alpha$ on a basis vector in Eq. (1.26) converges if and only if the *like* projection of the residue $\mathbf{j}^\alpha$ vanishes. It follows that the projection $\nabla_\alpha u \hat{\mathbf{D}}_{\mathbf{1}}^\alpha$ converges, the projection $\nabla_\alpha v \hat{\mathbf{D}}_{\mathbf{1}}^\alpha$ diverges, and the behaviors of the transverse projections $m_\alpha \hat{\mathbf{D}}_{\mathbf{1}}^\alpha$ remain undetermined. Hence using the results for the retarded operator $1/\Box$ in Ref. [2] and Appendix C below one obtains

[6]I continue using the terminology of the theory of asymptotically flat spaces [4].

$$\nabla_\alpha u \frac{1}{\Box}\hat{\mathbf{J}}^\alpha\Big|_{\mathcal{I}^+} = -\frac{1}{r}\nabla_\alpha u \hat{\mathbf{D}}_{\mathbf{1}}^\alpha(u,\phi), \tag{3.12}$$

$$\nabla_\alpha v \frac{1}{\Box}\hat{\mathbf{J}}^\alpha\Big|_{\mathcal{I}^+} = -\frac{\log r}{r}\frac{1}{2}\int_{-\infty}^{u} d\overline{u}\,\nabla_\alpha v \mathbf{j}^\alpha(\overline{u},\phi) \tag{3.13}$$

while for the transverse projections one has two cases:

$$\text{(i)}\quad m_\alpha \mathbf{j}^\alpha = 0, \quad m_\alpha \frac{1}{\Box}\hat{\mathbf{J}}^\alpha\Big|_{\mathcal{I}^+} = -\frac{1}{r}m_\alpha \hat{\mathbf{D}}_{\mathbf{1}}^\alpha(u,\phi), \tag{3.14}$$

$$\text{(ii)}\quad m_\alpha \mathbf{j}^\alpha \neq 0,$$

$$m_\alpha \frac{1}{\Box}\hat{\mathbf{J}}^\alpha\Big|_{\mathcal{I}^+} = -\frac{\log r}{r}\frac{1}{2}\int_{-\infty}^{u} d\overline{u}\, m_\alpha \mathbf{j}^\alpha(\overline{u},\phi). \tag{3.15}$$

When the support of the source is confined to a spacetime tube, the convergent projection $\nabla_\alpha u \hat{\mathbf{D}}_{\mathbf{1}}^\alpha$ equals the total charge of the source [Eq. (1.27)]. This projection remains the conserved ADM charge also in the general case, even when there is an emission of charge and despite the fact that the integration hypersurface in $\hat{\mathbf{D}}_{\mathbf{1}}^\alpha$ is null:

$$\frac{\partial}{\partial u}(\nabla_\alpha u \hat{\mathbf{D}}_{\mathbf{1}}^\alpha) = 0, \quad \nabla_\alpha u \hat{\mathbf{D}}_{\mathbf{1}}^\alpha = \hat{\mathbf{e}}. \tag{3.16}$$

The proof uses the explicit form of the null hyperplane [2] and the stationarity of the source in the past.

The fact that the projection $\nabla_\alpha v \hat{\mathbf{D}}_{\mathbf{1}}^\alpha$ is generally divergent presents no real problem since this projection drops out of both the square of the differentiated moment in Eq. (1.28) and the commutator curvature. Indeed, solving the Jacobi identities to lowest order [2], one obtains for the commutator curvature

$$\hat{\mathbf{R}}_{\mu\nu} = \nabla_\nu \frac{1}{\Box}\hat{\mathbf{J}}_\mu - \nabla_\mu \frac{1}{\Box}\hat{\mathbf{J}}_\nu, \tag{3.17}$$

whence

$$\hat{\mathbf{R}}_{\mu\nu}\big|_{\mathcal{I}^+} = \frac{\partial}{\partial u}\left(\nabla_\nu u \frac{1}{\Box}\hat{\mathbf{J}}_\mu - \nabla_\mu u \frac{1}{\Box}\hat{\mathbf{J}}_\nu\right) + O\left(\frac{1}{r^2}\right). \tag{3.18}$$

The projection (3.13) drops out of this expression by symmetry. Moreover, owing to the conservation law (3.16) one finds

$$\hat{\mathbf{R}}_{\mu\nu}\big|_{\mathcal{I}^+} = \frac{1}{2}\frac{\partial}{\partial u}\left[(\nabla_\nu u m_\mu - \nabla_\mu u m_\nu)\left(m_\alpha^* \frac{1}{\Box}\hat{\mathbf{J}}^\alpha\right) + (\nabla_\nu u m_\mu^* - \nabla_\mu u m_\nu^*)\left(m_\alpha \frac{1}{\Box}\hat{\mathbf{J}}^\alpha\right)\right] + O\left(\frac{1}{r^2}\right), \tag{3.19}$$

and the only projection of $\hat{\mathbf{R}}_{\mu\nu}$ that can behave as $1/r$ (counting only powers) is

$$\tfrac{1}{2}\nabla^{\mu}v m^{\nu}\hat{\mathbf{R}}_{\mu\nu}|_{\mathcal{I}^+}=\frac{\partial}{\partial u}\left(m_{\alpha}\frac{1}{\Box}\hat{\mathbf{J}}^{\alpha}\right)\bigg|_{\mathcal{I}^+}\equiv-\frac{1}{r}\frac{\partial}{\partial u}\hat{\mathbf{C}}+O\left(\frac{1}{r^2}\right), \tag{3.20}$$

where the coefficient $\partial\hat{\mathbf{C}}/\partial u$ at $1/r$ will conventionally be called news function although in the case (ii) above it is infrared divergent. One has either

$$\text{(i)}\quad m_{\alpha}\mathbf{j}^{\alpha}=0,\quad \frac{\partial}{\partial u}\hat{\mathbf{C}}=\frac{\partial}{\partial u}[m_{\alpha}\hat{\mathbf{D}}_{\mathbf{1}}^{\alpha}(u,\phi)] \tag{3.21}$$

or

$$\text{(ii)}\quad m_{\alpha}\mathbf{j}^{\alpha}\neq 0,\quad \frac{\partial}{\partial u}\hat{\mathbf{C}}=(\log r)\frac{1}{2}m_{\alpha}\mathbf{j}^{\alpha}(u,\phi). \tag{3.22}$$

At this stage there appears an argument to make a conclusion on the transverse fluxes $m_{\alpha}\mathbf{j}^{\alpha}$. If one wants the news function to be finite, these fluxes must vanish. However, the only reason for insisting that the news function be finite is making finite the *energy* of the electromagnetic field since by Eqs. (2.14) and (3.20)

$$-\frac{\partial\mathcal{E}_{\rm cl}}{\partial u}=-\frac{1}{16\pi\kappa^2}\mathrm{tr}(r\nabla_{\mu}v m_{\nu}\hat{\mathbf{R}}^{\mu\nu})(r\nabla_{\alpha}v m^{*}_{\beta}\hat{\mathbf{R}}^{\alpha\beta})|_{\mathcal{I}^+}$$
$$\equiv-\frac{1}{4\pi\kappa^2}\mathrm{tr}\left(\frac{\partial}{\partial u}\hat{\mathbf{C}}\right)\left(\frac{\partial}{\partial u}\hat{\mathbf{C}}^{*}\right). \tag{3.23}$$

This is the reason indeed but only if the energy-momentum tensor of the electromagnetic field is given by expression (2.14). For a classical field it is. Therefore, for a classical field one has the case (i) $m_{\alpha}\mathbf{j}^{\alpha}=0$, i.e., the only nonvanishing flux of $\hat{\mathbf{J}}^{\alpha}$ is the flux of charge in Eq. (3.11), and the only divergent projection of the moment is $\nabla_{\alpha}v\hat{\mathbf{D}}_{\mathbf{1}}^{\alpha}$. The flux of the electromagnetic energy is then completely determined by the finite news function in Eq. (3.21). However, if a c-number electromagnetic field is an expectation value rather than the classical field, its energy-momentum tensor is *not* Eq. (2.14). Rather it is a sum $T^{\mu\nu}_{\rm cl}+T^{\mu\nu}_{\rm vac}$, and the same argument that the energy should be finite may now be in favor of the case (ii) $m_{\alpha}\mathbf{j}^{\alpha}\neq 0$ where the news function is divergent.

One is now ready to consider the expectation-value equations. In the high-frequency approximation, Eq. (2.12) takes the form

$$\hat{J}^{\mu}_{\rm full}=\hat{J}^{\mu}_{\rm bare}-\frac{\kappa^2}{24\pi}\left[\frac{8}{3}-\log\left(-\frac{\Box}{m^2}\right)\right]\hat{J}^{\mu}_{\rm bare}. \tag{3.24}$$

Since the bare source has a compact spatial support, one can use the result from Ref. [2]

$$\log(-\Box)X|_{\mathcal{I}^+}=-\frac{2}{r^2}\frac{\partial}{\partial u}D_{\mathbf{1}}(u,\phi|X)+O\left(\frac{1}{r^3}\right) \tag{3.25}$$

which is valid under conditions (1.44) and in which $D_{\mathbf{1}}(u,\phi|X)$ is the $\gamma=1$ moment of the test source X. Since the local terms in Eq. (3.24) are $O(1/r^3)$, one obtains

$$\hat{J}^{\alpha}_{\rm full}|_{\mathcal{I}^+}=-\frac{1}{r^2}\frac{\kappa^2}{12\pi}\frac{\partial}{\partial u}\hat{D}^{\alpha}_{\mathbf{1}\ \rm bare}+O\left(\frac{1}{r^3}\right). \tag{3.26}$$

This is Eq. (3.1) with

$$\mathbf{j}^{\alpha}(u,\phi)=-\frac{\kappa^2}{12\pi}\frac{\partial}{\partial u}\hat{D}^{\alpha}_{\mathbf{1}\ \rm bare}. \tag{3.27}$$

Hence, using the conservation law (3.16) for the bare source, one obtains

$$\nabla_{\alpha}u\mathbf{j}^{\alpha}=-\frac{\kappa^2}{12\pi}\frac{\partial}{\partial u}(\nabla_{\alpha}u\hat{D}^{\alpha}_{\mathbf{1}\ \rm bare})=0 \tag{3.28}$$

and thereby checks condition (3.10). Next, using Eq. (3.11) one calculates the density of the flux of charge

$$-\frac{\partial}{\partial u}\hat{e}_{\rm\ full}(u,\phi)=-\frac{\kappa^2}{3(4\pi)^2}\frac{\partial}{\partial u}(\nabla_{\alpha}r\hat{D}^{\alpha}_{\mathbf{1}\ \rm bare}) \tag{3.29}$$

and thereby obtains the result (1.36). Finally, one calculates the transverse fluxes

$$m_{\alpha}\mathbf{j}^{\alpha}=-\frac{\kappa^2}{12\pi}\frac{\partial}{\partial u}(m_{\alpha}\hat{D}^{\alpha}_{\mathbf{1}\ \rm bare})=-\frac{\kappa^2}{12\pi}\frac{\partial}{\partial u}\hat{C}_{\rm\ bare} \tag{3.30}$$

and discovers that they are proportional to the news function of the bare source. It follows that if the bare source emits waves, then the news function of the full source diverges. By Eq. (3.22),

$$\frac{\partial}{\partial u}\hat{C}_{\rm\ full}=\left(1-(\log r)\frac{\kappa^2}{24\pi}\right)\frac{\partial}{\partial u}\hat{C}_{\rm\ bare}+O(1). \tag{3.31}$$

However, one knows already that this divergence comes to the rescue. One can now check this again. From Eqs. (3.23) and (3.31) one obtains

$$\frac{\partial\mathcal{E}_{\rm cl}}{\partial u}\bigg|_{J_{\rm\ full}}=\frac{1}{4\pi\kappa^2}\left(1-\frac{\kappa^2}{12\pi}\log r+\kappa^2 O(1)\right)\mathrm{tr}\left(\frac{\partial}{\partial u}\hat{C}_{\rm\ bare}\right)$$
$$\times\left(\frac{\partial}{\partial u}\hat{C}^{*}_{\rm bare}\right). \tag{3.32}$$

On the other hand, by Eq. (2.26),

$$\frac{\partial\mathcal{E}_{\rm vac}}{\partial u}=\frac{1}{3(4\pi)^2}(\log r)\mathrm{tr}\left(\frac{\partial}{\partial u}\hat{C}_{\rm\ bare}\right)\left(\frac{\partial}{\partial u}\hat{C}^{*}_{\rm bare}\right)+O(1). \tag{3.33}$$

As a result, the total energy flux (1.32) is finite.

The approximate form (3.24) of the expectation-value equations corresponds to a neglect of the mass of the vacuum particles and is valid only in the region $u>u_-$ where the source is assumed strongly variable. This form can be used for a calculation of the derivative of the Bondi charge in the high-frequency approximation but cannot be used for a calculation of the ADM charge since the latter calculation involves the region $u<u_-$ where the source is static. The *electro*static polarization with massless vacuum particles is infinite. Indeed, for the ADM charge (3.7) to converge, the full source must fall off at *spatial* infinity as

$$\hat{\mathbf{J}}|_{i^0}=O\left(\frac{1}{r^4}\right), \tag{3.34}$$

whereas a calculation with the massless form factor $\log(-\Box)$ in Eq. (3.24) yields the behavior

$$\log(-\Box)\hat{J}_{\text{bare}}|_{i^0}=O\left(\frac{1}{r^3}\right) \tag{3.35}$$

and the divergent result[7]

$$\hat{e}_{\text{full}}=\left(1-(\log r)\frac{\kappa^2}{12\pi}\right)\hat{e}_{\text{bare}}+O(1),\quad m=0. \tag{3.36}$$

The correct result for the ADM charge is obtained with the normalized massive form factor (2.5):

$$\hat{e}_{\text{full}}=\hat{e}_{\text{bare}},\quad m\neq 0. \tag{3.37}$$

IV. CREATION OF MASSIVE PARTICLES AND THE HIGH-FREQUENCY APPROXIMATION

In spite of their apparent similarity, the divergent renormalization (3.36) of the ADM charge and the divergent renormalization (3.31) of the news function have different status. The former is a result of an incorrect use of the high-frequency approximation in the static region whereas the latter is a natural consequence of the intensive pair creation. To show this and to derive a criterion of the high-frequency approximation, the expectation-value equations are considered below with the massive form factor $\gamma(-\Box)$.

The kernel of the operator (2.5) is obtained with the aid of its spectral form

$$\gamma(-\Box)=\frac{1}{12}\left[\frac{8}{3}+\int_{4m^2}^{\infty}d\mu^2\frac{(1-4m^2/\mu^2)^{3/2}}{\mu^2-\Box}-\int_0^{\infty}d\mu^2\frac{1}{\mu^2+m^2}\right] \tag{4.1}$$

in which the threshold $\mu^2=4m^2$ appears explicitly, and, for the convergence at the upper limit, the two spectral integrals are to be considered as a single integral. Outside the support of the source $\hat{J}_{\text{bare}}$, Eq. (2.12) takes the form

$$\hat{J}^{\alpha}_{\text{full}}=-\frac{\kappa^2}{24\pi}\int_{4m^2}^{\infty}d\mu^2\left(1-\frac{4m^2}{\mu^2}\right)^{3/2}\frac{1}{\mu^2-\Box}\hat{J}^{\alpha}_{\text{bare}} \tag{4.2}$$

with the retarded resolvent $(\mu^2-\Box)^{-1}$. Here the order of integrations is important [10]. The spacetime integration implied in $(\mu^2-\Box)^{-1}\hat{J}_{\text{bare}}$ is to be done first, and the spectral-mass integration next.

One is presently interested in the behavior of the full source at a large distance from the support tube of $\hat{J}_{\text{bare}}$. At $r\gg l$ and u fixed, the retarded resolvent acting on a nonstationary source behaves as follows [10]:

$$\frac{1}{\mu^2-\Box}\hat{J}^{\alpha}_{\text{bare}}\propto\frac{1}{r}\exp[-\mu\sqrt{f(u)r}], \tag{4.3}$$

where $f(u)$ is a positive function of time and angles having the dimensions $1/\nu$.

Using Eqs. (4.2) and (4.3) one can estimate the fluxes associated with created particles. For $r\gg l$ one finds

$$\begin{aligned}\hat{J}^{\alpha}_{\text{full}}&\propto\frac{1}{r}\int_{4m^2}^{\infty}d\mu^2\left(1-\frac{4m^2}{\mu^2}\right)^{3/2}\exp[-\mu\sqrt{f(u)r}]\\&=\frac{1}{r^2}\frac{1}{f(u)}\int_0^{\infty}dx^2\frac{x^3}{[x^2+4m^2f(u)r]^{3/2}}\\&\quad\times\exp[-\sqrt{x^2+4m^2f(u)r}].\end{aligned} \tag{4.4}$$

When projected on $\nabla_\alpha r$, the coefficient of $1/r^2$ in the latter expression is the density of the flux of charge through a tube of radius r. It follows that, because of the presence of the threshold, the flux through the tube of radius $r\gg l$ is suppressed by the factor

$$\exp\left(-\frac{2m\sqrt{r}}{\sqrt{\nu}}\right). \tag{4.5}$$

Hence one infers that, although pair creation starts as soon as $\hbar\nu$ reaches the value of order mc^2, the particles are created in the support of the source with small momenta and do not get far away. They stay in a compact spatial domain until $\hbar\nu$ reaches the value

$$\hbar\nu\sim mc^2\left(\frac{mc}{\hbar}l\right). \tag{4.6}$$

[7]The only exception is the case where the bare source has no monopole moment, $\hat{e}_{\text{bare}}=0$. Then one can show that also $\hat{e}_{\text{full}}=0$. An *observable* electric charge cannot be carried by massless particles.

At this value there appears an observable flux of charged particles outside the support tube of $\hat{J}_{\text{bare}}$. The factor $(mc/\hbar)l$ may be interpreted as the number of created particles for which there is room in the spatial support of the source. If the creation is more violent, the particles get out of the tube. Finally, the high-frequency approximation is valid when $\hbar\nu$ is much bigger than the value (4.6):

$$\hbar\nu \gg mc^2\left(\frac{mc}{\hbar}l\right). \tag{4.7}$$

Under this condition the flux of created energy and charge stops depending on the mass of the particles. The mass terms in Eq. (4.4) can then be discarded which is equivalent to replacing the second term of the spectral formula (4.1) with

$$\int_0^\infty d\mu^2 \frac{1}{\mu^2-\Box}. \tag{4.8}$$

Hence the approximation (2.6) for the form factor.

Expression (4.4) holds for the transverse projections of $\hat{J}_{\text{full}}$ as well:

$$m_\alpha \hat{J}^\alpha_{\text{full}}|_{r\gg l} \propto \frac{1}{r^2}, \tag{4.9}$$

and the coefficient of $1/r^2$ in this expression is nonvanishing whenever there is a nonvanishing flux of created particles outside the support of $\hat{J}_{\text{bare}}$. Only in the special case where the electromagnetic radiation of $\hat{J}_{\text{bare}}$ is absent altogether may the transverse projections vanish. Thus the behavior (4.9) is a direct consequence of pair creation. Then, by Eq. (3.22), the news function of the mean field inevitably diverges as $\log r$. Hence relation (3.31). The normalization scale of the $\log r$ in this relation can be read from the kernel of the operator $1/\Box$ (Appendix C). This is $\log(r/l)$.

It will be emphasized once again that there is nothing wrong about the mean electromagnetic field. When pairs are created, its energy is no more governed by its news function since there appears a real vacuum contribution. The news function diverges but the vacuum energy redistributes and keeps the electromagnetic radiation down.

V. CALCULATION OF $T^{\mu\nu}_{\text{vac}}(1)$ AT $\mathcal{I}^+$

There remains to be presented the calculation of the energy fluxes (2.21), (2.22), and (2.23). This is, of course, the main part of the work.

The $T^{\mu\nu}_{\text{vac}}(1)$ is given by expression (2.15) with $\gamma(-\Box)$ in Eq. (2.6). The commutator curvature to be inserted in $T^{\mu\nu}_{\text{vac}}$ is the one generated by the bare source. The bare source has a compact spatial support. Therefore [2],

$$-\frac{1}{\Box}\hat{J}^\alpha_{\text{bare}}|_{\mathcal{I}^+} = \frac{1}{r}\hat{D}^\alpha_{\mathbf{1}} + O\left(\frac{1}{r^2}\right) \tag{5.1}$$

and hence, by Eq. (3.18),

$$\hat{\mathcal{R}}^{\mu\nu}|_{\mathcal{I}^+} = \frac{1}{r}\left(\nabla^\mu u \frac{\partial}{\partial u}\hat{D}^\nu_{\mathbf{1}} - \nabla^\nu u \frac{\partial}{\partial u}\hat{D}^\mu_{\mathbf{1}}\right) + O\left(\frac{1}{r^2}\right). \tag{5.2}$$

It follows that for obtaining $T^{\mu\nu}_{\text{vac}}(1)$ at $\mathcal{I}^+$ one needs to know the behavior of $\log(-\Box)$ with a test function that behaves at $\mathcal{I}^+$ as

$$X|_{\mathcal{I}^+} = \frac{1}{r}A(u,\phi) + O\left(\frac{1}{r^2}\right). \tag{5.3}$$

The needed result is obtained in Appendix C:

$$\begin{aligned} -\log\left(-\frac{\Box}{m^2}\right)X|_{\mathcal{I}^+} &= \frac{A(u,\phi)}{r}(\log mr + 2\mathbf{c} - \log 2) \\ &\quad + \frac{1}{r}\int_{-\infty}^u d\tau \log[m(u-\tau)]\frac{\partial}{\partial\tau}A(\tau,\phi) + O\left(\frac{\log r}{r^2}\right). \end{aligned} \tag{5.4}$$

Substituting Eq. (5.2) for Eq. (5.3) one obtains

$$\begin{aligned} -\hat{\mathcal{R}}^\mu{}_\lambda \log\left(-\frac{\Box}{m^2}\right)\hat{\mathcal{R}}^{\nu\lambda}|_{\mathcal{I}^+} &= \frac{1}{r^2}\nabla^\mu u \nabla^\nu u\Bigg[(\log mr + 2\mathbf{c} - \log 2)\left(\frac{\partial}{\partial u}\hat{D}^\alpha_{\mathbf{1}}\right)\left(\frac{\partial}{\partial u}\hat{D}_{\mathbf{1}\alpha}\right) \\ &\quad + \left(\frac{\partial}{\partial u}\hat{D}^\alpha_{\mathbf{1}}\right)\int_{-\infty}^u d\tau \log[m(u-\tau)]\frac{\partial^2}{\partial\tau^2}\hat{D}_{\mathbf{1}\alpha}(\tau)\Bigg] \\ &\quad + O\left(\frac{1}{r^3}\right). \end{aligned} \tag{5.5}$$

Here and below, use is to be made of the following identity:

$$\frac{d}{du}\int_{-\infty}^u d\tau \log(u-\tau) f(\tau) = \int_{-\infty}^u d\tau \log(u-\tau)\frac{d}{d\tau}f(\tau), \tag{5.6}$$

where $f(\tau)$ is supposed to provide the convergence at the lower limit. The convergence of the integral in Eq. (5.5) and similar integrals is provided by the assumption of asymptotic stationarity of the bare source. Under the simplified assumption that the domain of nonstationarity of the source is compact, there will be time instants u_- and u_+ such that [2]

$$\frac{\partial}{\partial u}\hat{D}^\alpha(u)|_{u<u_-} = 0, \quad \frac{\partial}{\partial u}\hat{D}^\alpha(u)|_{u>u_+} = 0. \tag{5.7}$$

In this way the result (2.21) is obtained.

VI. CALCULATION OF $T^{\mu\nu}_{\rm vac}(2)$ AT $\mathcal{I}^+$

The $T^{\mu\nu}_{\rm vac}(2)$ is defined by Eq. (2.8) with $\gamma(-\Box)$ in Eq. (2.6). Using the spectral formula

$$\log\left(-\frac{\Box}{m^2}\right)=-\int_0^\infty d\mu^2\left(\frac{1}{\mu^2-\Box}-\frac{1}{\mu^2+m^2}\right) \quad (6.1)$$

one finds

$$\int dx g^{1/2}T^{\mu\nu}_{\rm vac}(2)\,\delta g_{\mu\nu} =\frac{1}{12(4\pi)^2}\int dx g^{1/2}\,{\rm tr}\int_0^\infty d\mu^2\left(\frac{1}{\mu^2-\Box}\hat{\mathcal{R}}^{\alpha\beta}\right) \times\delta\Box\left(\frac{1}{\mu^2-\Box}\hat{\mathcal{R}}_{\alpha\beta}\right), \quad (6.2)$$

and the operator $\delta\Box$ can be obtained by calculating

$$\int dx g^{1/2}\,{\rm tr}\,\hat{\mathcal{R}}^{\alpha\beta}(\delta\Box)\hat{\mathcal{R}}_{\alpha\beta} =\int dx g^{1/2}\delta g_{\mu\nu}\,{\rm tr}[p^{\mu\nu}(\nabla_1,\nabla_2)\hat{\mathcal{R}}_1^{\alpha\beta}\hat{\mathcal{R}}_{2\alpha\beta}], \quad (6.3)$$

where $p^{\mu\nu}(\nabla_1,\nabla_2)$ is some polynomial in the derivative ∇_1 acting on $\hat{\mathcal{R}}_1$ and the derivative ∇_2 acting on $\hat{\mathcal{R}}_2$. In terms of this operator polynomial one obtains

$$T^{\mu\nu}_{\rm vac}(2)=-\frac{1}{12(4\pi)^2}\,{\rm tr}\,p^{\mu\nu}(\nabla_1,\nabla_2)\frac{\log(\Box_1/\Box_2)}{\Box_1-\Box_2}\hat{\mathcal{R}}_1^{\alpha\beta}\hat{\mathcal{R}}_{2\alpha\beta}, \quad (6.4)$$

where the nonlocal form factor results from the spectral-mass integration in Eq. (6.2) and, up to higher orders in the curvature, the operators in Eq. (6.4) are commutative.

The explicit form of Eq. (6.3) is

$$\int dx g^{1/2}\,{\rm tr}\,\hat{\mathcal{R}}^{\alpha\beta}(\delta\Box)\hat{\mathcal{R}}_{\alpha\beta} =\int dx g^{1/2}\,\delta g_{\mu\nu}\,{\rm tr}\Big[(\nabla^\mu\hat{\mathcal{R}}^{\alpha\beta})(\nabla^\nu\hat{\mathcal{R}}_{\alpha\beta}) -\frac{1}{4}g^{\mu\nu}\Box(\hat{\mathcal{R}}^{\alpha\beta}\hat{\mathcal{R}}_{\alpha\beta})-2\nabla_\alpha(\hat{\mathcal{R}}^\mu{}_\beta\nabla^\nu\hat{\mathcal{R}}^{\alpha\beta} -\hat{\mathcal{R}}^{\alpha\beta}\nabla^\nu\hat{\mathcal{R}}^\mu{}_\beta)\Big], \quad (6.5)$$

whence

$$T^{\mu\nu}_{\rm vac}(2)=-\frac{1}{12(4\pi)^2}\,{\rm tr}\frac{\log(\Box_1/\Box_2)}{\Box_1-\Box_2}\nabla_1^\mu\hat{\mathcal{R}}_1^{\alpha\beta}\nabla_2^\nu\hat{\mathcal{R}}_{2\alpha\beta} +\text{total derivatives}. \quad (6.6)$$

A detailed analysis shows that the total derivatives in Eq. (6.6) either vanish at $\mathcal{I}^+$ or vanish in the integrated energy flux; i.e., their contribution to $\partial\mathcal{E}_{\rm vac}/\partial u$ is quantum noise [Eq. (1.22)]. The technique used in this analysis is outlined below.

In the remaining term of Eq. (6.6) use will be made of Eq. (3.17) and the conservation equation (3.8) to express the commutator curvature through its source:

$$\hat{\mathcal{R}}_1^{\alpha\beta}\hat{\mathcal{R}}_{2\alpha\beta}=\Box\left(\frac{1}{\Box_1\Box_2}\hat{J}_1^\alpha\hat{J}_{2\alpha}\right)-2\nabla_\alpha\nabla_\beta\left(\frac{1}{\Box_1\Box_2}\hat{J}_1^\alpha\hat{J}_2^\beta\right) -\frac{1}{\Box_1}\hat{J}_1^\alpha\hat{J}_{2\alpha}-\hat{J}_1^\alpha\frac{1}{\Box_2}\hat{J}_{2\alpha}. \quad (6.7)$$

The first two terms of this expression are total derivatives. One obtains

$$T^{\mu\nu}_{\rm vac}(2)=\frac{1}{6(4\pi)^2}\,{\rm tr}\frac{1}{\Box_2}\frac{\log(\Box_1/\Box_2)}{\Box_1-\Box_2}\nabla_1^{(\mu}\hat{J}_{1\alpha}\nabla_2^{\nu)}\hat{J}_2^\alpha +\text{total derivatives}, \quad (6.8)$$

where the indices $\mu\nu$ are symmetrized. Of the new total derivatives, the contribution of the first term in Eq. (6.7) is $O(1/r^3)$ at $\mathcal{I}^+$, and the contribution of the second term is quantum noise. The proof is given below.

The form factor in Eq. (6.8) can be expressed through the operator $\mathcal{H}_q$ introduced in Ref. [2] and Appendix A below:

$$\frac{1}{\Box_2}\frac{\log(\Box_1/\Box_2)}{\Box_1-\Box_2}X_1X_2(x) =-\int_{-\infty}^0 dq\left(\frac{d}{dq}\mathcal{H}_qX_1(x)\right)\left(\int_{-\infty}^q\frac{d\bar{q}}{\bar{q}}\mathcal{H}_{\bar{q}}X_2(x)\right). \quad (6.9)$$

The behavior of this function as $x\to\mathcal{I}^+$ is obtained in the same way as in Ref. [2] by making the replacement of the integration variable

$$q=r(\tau-u),\quad r=r(x)\to\infty, \quad (6.10)$$

where τ is the new integration variable and $u=u(x)$ is the retarded time of the point x at $\mathcal{I}^+$. With q replaced as in Eq. (6.10), one has [2]

$$\mathcal{H}_qX(x)|_{x\to\mathcal{I}^+}=\frac{1}{r}D_{\mathbf{1}}(\tau,\phi|X), \quad (6.11)$$

where the quantity on the right-hand side is the $D_{\mathbf{1}}$ moment of the test source X. As a result, for the function (6.9) one obtains

$$\frac{1}{\Box_2}\frac{\log(\Box_1/\Box_2)}{\Box_1-\Box_2}X_1X_2|_{\mathcal{I}^+}=-\frac{1}{r^2}\int_{-\infty}^{u}d\tau\left(\frac{\partial}{\partial\tau}D_{\mathbf{1}}(\tau,\phi|X_1)\right)\times\left(\int_{-\infty}^{\tau}\frac{d\bar{\tau}}{\bar{\tau}-u}D_{\mathbf{1}}(\bar{\tau},\phi|X_2)\right). \tag{6.12}$$

Using the latter result in Eq. (6.8) one finds

$$T^{\mu\nu}_{\text{vac}}(2)|_{\mathcal{I}^+}=\frac{1}{r^2}\nabla^\mu u\nabla^\nu u\frac{1}{6(4\pi)^2}\text{tr}\left[-\int_{-\infty}^{u}d\tau\left(\frac{\partial^2}{\partial\tau^2}\hat{D}^\alpha_{\mathbf{1}}(\tau)\right)\times\left(\int_{-\infty}^{\tau}\frac{d\bar{\tau}}{\bar{\tau}-u}\frac{\partial}{\partial\bar{\tau}}\hat{D}_{\mathbf{1}\,\alpha}(\bar{\tau})\right)+\text{Q.N.}\right]+O\left(\frac{1}{r^3}\right). \tag{6.13}$$

The integration by parts first in the internal integral

$$\int_{-\infty}^{\tau}\frac{d\bar{\tau}}{\bar{\tau}-u}\frac{\partial}{\partial\bar{\tau}}\hat{D}_{\mathbf{1}\,\alpha}(\bar{\tau})=\log(u-\tau)\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}\,\alpha}(\tau)-\int_{-\infty}^{\tau}d\bar{\tau}\log(u-\bar{\tau})\frac{\partial^2}{\partial\bar{\tau}^2}\hat{D}_{\mathbf{1}\,\alpha}(\bar{\tau}) \tag{6.14}$$

and next in the external integral yields finally the expression (2.22).

It is now convenient to present a proof that the first two terms in Eq. (6.7) can be discarded. Their contributions to $T^{\mu\nu}_{\text{vac}}(2)$ are, respectively,

$$\Delta_1T^{\mu\nu}_{\text{vac}}=-\frac{1}{12(4\pi)^2}\text{tr}\Box\left(\frac{1}{\Box_1\Box_2}\frac{\log(\Box_1/\Box_2)}{\Box_1-\Box_2}\nabla^\mu_1\hat{J}^\alpha_1\nabla^\nu_2\hat{J}_{2\,\alpha}\right) \tag{6.15}$$

and

$$\Delta_2T^{\mu\nu}_{\text{vac}}=\frac{1}{6(4\pi)^2}\text{tr}\nabla_\alpha\nabla_\beta\left(\frac{1}{\Box_1\Box_2}\frac{\log(\Box_1/\Box_2)}{\Box_1-\Box_2}\nabla^\mu_1\hat{J}^\alpha_1\nabla^\nu_2\hat{J}^\beta_2\right). \tag{6.16}$$

Using the same technique as above, one obtains

$$\frac{1}{\Box_1\Box_2}\frac{\log(\Box_1/\Box_2)}{\Box_1-\Box_2}\nabla^\mu_1\hat{J}^\alpha_1\nabla^\nu_2\hat{J}^\beta_2|_{\mathcal{I}^+}=-\frac{1}{2r}\nabla^\mu u\nabla^\nu u\int_{-\infty}^{u}d\tau\left(\int_{-\infty}^{\tau}\frac{d\bar{\tau}}{\bar{\tau}-u}\frac{\partial}{\partial\bar{\tau}}\hat{D}^\alpha_{\mathbf{1}}(\bar{\tau})\right)\times\left(\int_{-\infty}^{\tau}\frac{d\bar{\tau}}{\bar{\tau}-u}\frac{\partial}{\partial\bar{\tau}}\hat{D}^\beta_{\mathbf{1}}(\bar{\tau})\right)+O\left(\frac{1}{r^2}\right). \tag{6.17}$$

Note that this behavior is not even $1/r^2$. It is $1/r$. Nevertheless, one has

$$\Box O\left(\frac{1}{r}\right)\bigg|_{\mathcal{I}^+}=O\left(\frac{1}{r^3}\right) \tag{6.18}$$

(see Appendix B). Therefore, the contribution (6.15) is indeed $O(1/r^3)$ at $\mathcal{I}^+$.

To calculate the contribution (6.16) at $\mathcal{I}^+$, one may use the following result from Ref. [2]. If a symmetric tensor $V^{\alpha\beta}$ is analytic at $\mathcal{I}^+$, and its projections tangential to $\mathcal{I}^+$ vanish:

$$\nabla_\alpha uV^{\alpha\beta}|_{\mathcal{I}^+}=O\left(\frac{1}{r}V\right),\quad \nabla_\alpha u\nabla_\beta uV^{\alpha\beta}|_{\mathcal{I}^+}=O\left(\frac{1}{r^2}V\right), \tag{6.19}$$

then

$$\nabla_\alpha\nabla_\beta V^{\alpha\beta}|_{\mathcal{I}^+}=\frac{1}{r}\frac{\partial}{\partial u}(g_{\alpha\beta}V^{\alpha\beta})+O\left(\frac{1}{r^2}V\right). \tag{6.20}$$

By Eq. (3.16), the tensor (6.17) possesses the properties of $V^{\alpha\beta}$. As a result, for the contribution (6.16) one finds

$$\Delta_2T^{\mu\nu}_{\text{vac}}|_{\mathcal{I}^+}=\frac{1}{r^2}\nabla^\mu u\nabla^\nu u\left(-\Delta_2\frac{\partial\mathcal{E}_{\text{vac}}}{\partial u}\right)+O\left(\frac{1}{r^3}\right), \tag{6.21}$$

where the respective energy flux is a total derivative in time:

$$-\Delta_2\frac{\partial\mathcal{E}_{\text{vac}}}{\partial u}=\frac{-1}{12(4\pi)^2}\text{tr}\frac{\partial}{\partial u}\int_{-\infty}^{u}d\tau\left(\int_{-\infty}^{\tau}\frac{d\bar{\tau}}{\bar{\tau}-u}\frac{\partial}{\partial\bar{\tau}}\hat{D}^\alpha_{\mathbf{1}}(\bar{\tau})\right)\times\left(\int_{-\infty}^{\tau}\frac{d\bar{\tau}}{\bar{\tau}-u}\frac{\partial}{\partial\bar{\tau}}\hat{D}_{\mathbf{1}\,\alpha}(\bar{\tau})\right). \tag{6.22}$$

Since

$$\int_{-\infty}^{u}d\tau\left(\int_{-\infty}^{\tau}\frac{d\bar{\tau}}{\bar{\tau}-u}\frac{\partial}{\partial\bar{\tau}}\hat{D}^\alpha_{\mathbf{1}}(\bar{\tau})\right)\left(\int_{-\infty}^{\tau}\frac{d\bar{\tau}}{\bar{\tau}-u}\frac{\partial}{\partial\bar{\tau}}\hat{D}_{\mathbf{1}\,\alpha}(\bar{\tau})\right)\bigg|_{u\to\infty}=\frac{1}{u}[\hat{D}_{\mathbf{1}}(+\infty)-\hat{D}_{\mathbf{1}}(-\infty)]^2\to0, \tag{6.23}$$

one obtains

$$\int_{-\infty}^{\infty}du\left(\Delta_2\frac{\partial\mathcal{E}_{\text{vac}}}{\partial u}\right)=0. \tag{6.24}$$

Thus the contribution (6.21) is indeed quantum noise.

VII. CALCULATION OF $T^{\mu\nu}_{\text{vac}}(3)$ AT $\mathcal{I}^+$

The $T^{\mu\nu}_{\text{vac}}(3)$ is obtained by varying the third-order action (2.3) with respect to the metric. Only the Ricci curvature that enters the basis invariants (2.4) needs to be varied. The commutator curvatures in these invariants are to be expressed through their sources via Eqs. (3.17) and (3.8). The result may be represented in the form

$$T^{\mu\nu}_{\rm vac}(3)=\frac{1}{(4\pi)^2}{\rm tr}\sum_l \tilde{\Gamma}_l(-\Box,-\Box_2,-\Box_3)\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3{}^{\mu\nu}(l), \tag{7.1}$$

where the form factors $\tilde{\Gamma}_l(-\Box,-\Box_2,-\Box_3)$ are linear combinations of $\Gamma_i(-\Box,-\Box_2,-\Box_3)$, and the structures $\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3{}^{\mu\nu}(l)$ make some nonlocal tensor basis second-order in (the source of) the commutator curvature. The first argument $\Box$ of the form factors $\tilde{\Gamma}_l$ is the operator argument that in the action $S_{\rm vac}(3)$ acts on the Ricci curvature. In the variational derivative (7.1) it becomes an *overall* operator acting at the observation point. In the diagrammatic language, the argument $\Box$ corresponds to the external line of the current (7.1). This is explained in more detail in Appendix B.

It is shown in Appendix B that only the small-$\Box$ expansion of $\tilde{\Gamma}_l(-\Box,-\Box_2,-\Box_3)$ in the argument $\Box$ is relevant to the behavior of the current (7.1) at $\mathcal{I}^+$. This expansion has the form

$$\tilde{\Gamma}_l(-\Box,-\Box_2,-\Box_3)=\log(-\Box)\mathcal{A}_l(\Box_2,\Box_3)+\mathcal{B}_l(\Box_2,\Box_3)+O(\Box),\quad \Box\to 0 \tag{7.2}$$

and the contribution at $\mathcal{I}^+$ of the terms $O(\Box)$ is already $O(1/r^3)$. By the results in Appendix B, both $\mathcal{A}_l(\Box_2,\Box_3)$ and $\mathcal{B}_l(\Box_2,\Box_3)$ can be expressed through the operator $\mathcal{H}_q$ in a way similar to Eq. (6.9). Specifically, all $\mathcal{A}_l(\Box_2,\Box_3)$ are linear combinations of the following operators $F_{mn}(\Box_2,\Box_3)$:

$$F_{mn}(\Box_2,\Box_3)X_2X_3=-2\int_{-\infty}^0 dq\, q^{m+n}\left[\left(\frac{d}{dq}\right)^{m+1}\mathcal{H}_qX_2\right]\times\left[\left(\frac{d}{dq}\right)^{n+1}\mathcal{H}_qX_3\right]. \tag{7.3}$$

Moreover, the term with $\log(-\Box)$ in Eq. (7.2) always appears in the combination

$$\log(-\Box)F_{mn}(\Box_2,\Box_3)+L_{mn}(\Box_2,\Box_3) \tag{7.4}$$

with

$$L_{mn}(\Box_2,\Box_3)X_2X_3=-2\int_{-\infty}^0 dq\, q^{m+n}\left[\log\left(-\frac{q}{2}\right)+2\mathbf{c}\right]\left[\left(\frac{d}{dq}\right)^{m+1}\mathcal{H}_qX_2\right]\times\left[\left(\frac{d}{dq}\right)^{n+1}\mathcal{H}_qX_3\right]. \tag{7.5}$$

All $\mathcal{B}_l(\Box_2,\Box_3)$ in Eq. (7.2) are linear combinations of $L_{mn}(\Box_2,\Box_3)$ and $F_{mn}(\Box_2,\Box_3)$.

Thus one arrives at the ansatz

$$T^{\mu\nu}_{\rm vac}(3)\big|_{\mathcal{I}^+}=\frac{1}{(4\pi)^2}{\rm tr}\sum_l [\log(-\Box)\mathcal{A}_l(\Box_2,\Box_3)+\mathcal{B}_l(\Box_2,\Box_3)]\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3{}^{\mu\nu}(l)+O\left(\frac{1}{r^3}\right) \tag{7.6}$$

in which both the basis elements and the operator coefficients are presently to be determined. One can check that the following nine structures quadratic in the source of the commutator curvature make a basis:

$$\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3{}^{\mu\nu}(1)=\hat{J}^\mu_2\cdot\hat{J}^\nu_3, \tag{7.7}$$

$$\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3{}^{\mu\nu}(2)=\frac{1}{2}\nabla^\mu\nabla^\nu(\hat{J}^\alpha_2\cdot\hat{J}_{3\alpha}), \tag{7.8}$$

$$\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3{}^{\mu\nu}(3)=\hat{J}^\alpha_2\cdot\nabla^\mu_3\nabla^\nu_3\hat{J}_{3\alpha}, \tag{7.9}$$

$$\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3{}^{\mu\nu}(4)=\frac{1}{2}\nabla^\mu\nabla^\nu\nabla_\alpha\nabla_\beta(\hat{J}^\alpha_2\cdot\hat{J}^\beta_3), \tag{7.10}$$

$$\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3{}^{\mu\nu}(5)=\nabla_\alpha\nabla_\beta(\hat{J}^\alpha_2\cdot\nabla^\mu_3\nabla^\nu_3\hat{J}^\beta_3), \tag{7.11}$$

$$\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3{}^{\mu\nu}(6)=\nabla^\alpha(\nabla^{(\mu}_2\hat{J}_{2\alpha}\cdot\hat{J}^{\nu)}_3), \tag{7.12}$$

$$\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3{}^{\mu\nu}(7)=\nabla^\alpha(\hat{J}_{2\alpha}\cdot\nabla^{(\mu}_3\hat{J}^{\nu)}_3), \tag{7.13}$$

$$\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3{}^{\mu\nu}(8)=g^{\mu\nu}\hat{J}^\alpha_2\cdot\hat{J}_{3\alpha}, \tag{7.14}$$

$$\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3{}^{\mu\nu}(9)=g^{\mu\nu}\nabla_\alpha\nabla_\beta(\hat{J}^\alpha_2\cdot\hat{J}^\beta_3). \tag{7.15}$$

The last two structures will be omitted since they cannot contribute to the energy flux through $\mathcal{I}^+$.

The respective coefficients $\mathcal{A}_l(\Box_2,\Box_3)$ and $\mathcal{B}_l(\Box_2,\Box_3)$ are obtained using the algorithms of Appendix B and the table of the third-order form factors in Ref. [9]. Only the basis element (7.10) with $l=4$ has a nonvanishing $\mathcal{A}_l(\Box_2,\Box_3)$:

$$\mathcal{A}_l(\Box_2,\Box_3)=0,\quad l\neq 4, \tag{7.16}$$

$$\mathcal{A}_4(\Box_2,\Box_3)=-\frac{1}{3}\frac{1}{\Box_2\Box_3}[F_{22}(\Box_2,\Box_3)-2F_{11}(\Box_2,\Box_3)]. \tag{7.17}$$

This agrees with Ref. [2] where only the term with $\log(-\Box)$ in Eq. (7.6) was considered. The results for $\mathcal{B}_l(\Box_2,\Box_3)$ are as follows:

$$\mathcal{B}_1(\Box_2,\Box_3)=0, \tag{7.18}$$

$$\mathcal{B}_2(\Box_2,\Box_3)=-\frac{1}{6}\frac{1}{\Box_2\Box_3}+\frac{1}{12}\left(\frac{1}{\Box_2}+\frac{1}{\Box_3}\right)F_{11}(\Box_2,\Box_3), \tag{7.19}$$

$$\mathcal{B}_3(\Box_2,\Box_3)=\frac{1}{12}\left(\frac{1}{\Box_2}-\frac{1}{\Box_3}\right)[F_{11}(\Box_2,\Box_3)-F_{00}(\Box_2,\Box_3)], \tag{7.20}$$

$$\mathcal{B}_4(\Box_2,\Box_3)=\mathcal{B}_4'(\Box_2,\Box_3)+\mathcal{B}_4''(\Box_2,\Box_3), \tag{7.21}$$

$$\mathcal{B}_4'(\Box_2,\Box_3)=-\frac{1}{3}\frac{1}{\Box_2\Box_3}[L_{22}(\Box_2,\Box_3)-2L_{11}(\Box_2,\Box_3)], \tag{7.22}$$

$$\mathcal{B}_4''(\Box_2,\Box_3)=\frac{2}{3}\frac{1}{\Box_2\Box_3}F_{11}(\Box_2,\Box_3), \tag{7.23}$$

$$\mathcal{B}_5(\Box_2,\Box_3)=\frac{1}{6}\frac{1}{\Box_2\Box_3}[F_{12}(\Box_2,\Box_3)-F_{21}(\Box_2,\Box_3)-F_{11}(\Box_2,\Box_3)-F_{00}(\Box_2,\Box_3)], \tag{7.24}$$

$$\mathcal{B}_6(\Box_2,\Box_3)=-\frac{1}{6}\left(\frac{1}{\Box_2}-3\frac{1}{\Box_3}\right)F_{11}(\Box_2,\Box_3)-\frac{1}{6}\frac{1}{\Box_2}F_{00}(\Box_2,\Box_3), \tag{7.25}$$

$$\mathcal{B}_7(\Box_2,\Box_3)=\frac{1}{3}\frac{1}{\Box_3}\left[F_{11}(\Box_2,\Box_3)-\frac{1}{2}F_{00}(\Box_2,\Box_3)\right]. \tag{7.26}$$

Only $\mathcal{B}_4$ contains the contribution of $L_{mn}(\Box_2,\Box_3)$ because the latter can appear only in the combination (7.4).

The kernels for the superpositions of the operators $F_{mn}(\Box_2,\Box_3)$ with $1/\Box_2$ and $1/\Box_3$ are given in Eqs. (A.15) and (A.16) of Appendix A. By the same calculation as in Eqs. (6.9)–(6.12) their behaviors at $\mathcal{I}^+$ are expressed through the moments $D_{\mathbf{1}}$ of the test functions:

$$\frac{1}{\Box_3}F_{mn}(\Box_2,\Box_3)X_2X_3|_{\mathcal{I}^+}=-\frac{1}{r^2}\int_{-\infty}^{u}d\tau(\tau-u)^{m+n}\left(\frac{\partial^{m+1}}{\partial\tau^{m+1}}D_{\mathbf{1}}(\tau,\phi|X_2)\right)\times\left(\frac{\partial^n}{\partial\tau^n}\int_{-\infty}^{\tau}\frac{d\bar{\tau}}{\bar{\tau}-u}D_{\mathbf{1}}(\bar{\tau},\phi|X_3)\right), \tag{7.27}$$

$$\frac{1}{\Box_2\Box_3}F_{mn}(\Box_2,\Box_3)X_2X_3|_{\mathcal{I}^+}=-\frac{1}{2r}\int_{-\infty}^{u}d\tau(\tau-u)^{m+n}\left(\frac{\partial^{m}}{\partial\tau^{m}}\int_{-\infty}^{\tau}\frac{d\bar{\tau}}{\bar{\tau}-u}D_{\mathbf{1}}(\bar{\tau},\phi|X_2)\right)\times\left(\frac{\partial^n}{\partial\tau^n}\int_{-\infty}^{\tau}\frac{d\bar{\tau}}{\bar{\tau}-u}D_{\mathbf{1}}(\bar{\tau},\phi|X_3)\right), \tag{7.28}$$

$$\frac{1}{\Box_2\Box_3}X_2X_3|_{\mathcal{I}^+}=\frac{1}{r^2}D_{\mathbf{1}}(u,\phi|X_2)D_{\mathbf{1}}(u,\phi|X_3). \tag{7.29}$$

With these behaviors, the calculation of the $\mathcal{B}_l$ terms in Eq. (7.6) essentially repeats the calculation in Sec. VI. The $\mathcal{B}_l$ terms that involve the form factors (7.27) are analogous to Eq. (6.8), and the $\mathcal{B}_l$ terms that involve the form factors (7.28) are analogous to Eq. (6.16). This concerns all $\mathcal{B}_l$ except $\mathcal{B}_4'$.

For $l\neq 4$ the results are as follows. Since $\mathcal{B}_1=0$, there remain two basis structures, $l=6$ and $l=7$, in which the indices of the energy-momentum tensor do not belong to derivatives. Their contributions at $\mathcal{I}^+$ vanish by virtue of the conservation law (3.16). This may be exemplified with just one term

$$\frac{1}{\Box_3}F_{11}(\Box_2,\Box_3)\hat{J}_{2\alpha}\nabla_3^\alpha\hat{J}_3^\nu|_{\mathcal{I}^+}=-\frac{1}{r^2}\nabla^\alpha u\int_{-\infty}^{u}d\tau(\tau-u)\left(\frac{\partial^2}{\partial\tau^2}\hat{D}_{\mathbf{1}\alpha}(\tau)\right)\left(\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}}^\nu(\tau)\right)+O\left(\frac{1}{r^3}\right)=O\left(\frac{1}{r^3}\right). \tag{7.30}$$

The contributions of the remaining structures are of the form

$$\mathcal{B}_l(\Box_2,\Box_3)\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3^{\mu\nu}(l)|_{\mathcal{I}^+}=\frac{1}{r^2}\nabla^\mu u\nabla^\nu u\left[-c|_l\left(\frac{\partial}{\partial u}\hat{D}_{\mathbf{1}}^\alpha\right)\left(\frac{\partial}{\partial u}\hat{D}_{\mathbf{1}\alpha}\right)+\text{Q.N.}\right]+O\left(\frac{1}{r^3}\right) \tag{7.31}$$

with

$$c|_{l=2}=\frac{1}{12},\quad c|_{l=3}=\frac{1}{6},\quad c|_{l=5}=\frac{1}{4}. \tag{7.32}$$

The contribution of $\mathcal{B}_4''$ is also of the form (7.31) with $c|_{l=4}=0$. Thus the effect of all structures induced by the third-order action except the basis structure with $l=4$ boils down to a finite renormalization of the classical news function.

The main contribution comes from the basis structure (7.10) with $l=4$. One may write

$$[\log(-\Box)\mathcal{A}_4(\Box_2,\Box_3)+\mathcal{B}_4'(\Box_2,\Box_3)]\hat{\mathcal{R}}_2\hat{\mathcal{R}}_3^{\mu\nu}(4)=\nabla^\mu\nabla^\nu[\log(-\Box)\hat{I}(x)+\hat{N}(x)], \tag{7.33}$$

where

$$\hat{I}(x)=-\frac{1}{6}\nabla_\alpha\nabla_\beta\frac{1}{\Box_2\Box_3}[F_{22}(\Box_2,\Box_3)-2F_{11}(\Box_2,\Box_3)]\hat{J}_2^\alpha\hat{J}_3^\beta, \tag{7.34}$$

$$\hat{N}(x) = -\frac{1}{6}\nabla_\alpha\nabla_\beta\frac{1}{\Box_2\Box_3}[L_{22}(\Box_2,\Box_3) - 2L_{11}(\Box_2,\Box_3)]\hat{J}_2^\alpha\hat{J}_3^\beta. \tag{7.35}$$

The scalar $\mathrm{tr}\hat{I}(x)$ is the central object in Ref. [2]. In the same way as above one obtains

$$\hat{I}(x)|_{\mathcal{I}^+} = \frac{1}{6r^2}\int_{-\infty}^{u} d\tau(u-\tau)\left(\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}}^\alpha(\tau)\right)\left(\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}\alpha}(\tau)\right) + O\left(\frac{1}{r^3}\right), \tag{7.36}$$

$$\begin{aligned}\hat{N}(x)|_{\mathcal{I}^+} = {}& \frac{\log r}{6r^2}\int_{-\infty}^{u} d\tau(u-\tau)\left(\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}}^\alpha(\tau)\right)\left(\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}\alpha}(\tau)\right)\\ &+\frac{1}{6r^2}\int_{-\infty}^{u} d\tau(u-\tau)\left(\log(u-\tau) - \log 2 + 2\mathbf{c} + \frac{1}{2}\right)\left(\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}}^\alpha(\tau)\right)\left(\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}\alpha}(\tau)\right)\\ &+\frac{1}{12r^2}\hat{D}_{\mathbf{1}}^\alpha(u)\hat{D}_{\mathbf{1}\alpha}(u) + O\left(\frac{1}{r^3}\right),\end{aligned} \tag{7.37}$$

where Eq. (7.36) reproduces the result in Ref. [2]. However, the scalar $\mathrm{tr}\hat{I}(x)$ is needed at one more limit which in [2] is called i^+. This is the limit $r\to\infty$ along the timelike geodesic that, when traced to the future, reaches infinity at the point ϕ of the celestial sphere with the energy $E=(1-\gamma^2)^{-1/2}$ per unit rest mass. The result obtained in Ref. [2] for this limit is

$$\hat{I}(x)|_{i^+} = \frac{\gamma(1-\gamma^2)}{12r}\int_{-\infty}^{\infty} du\left(\frac{\partial}{\partial u}\hat{D}^\alpha\right)\left(\frac{\partial}{\partial u}\hat{D}_\alpha\right), \tag{7.38}$$

where $\hat{D}^\alpha$ is the full (γ-dependent) radiation moment of the source $\hat{J}$.

It follows from the properties of $\hat{I}(x)$ above that, for the calculation of the function (7.33) at $\mathcal{I}^+$, one needs to know the behavior of $\log(-\Box)$ with a scalar test function that behaves at $\mathcal{I}^+$ as

$$X|_{\mathcal{I}^+} = \frac{1}{r^2}A(u,\phi) \tag{7.39}$$

and at i^+ as

$$X|_{i^+} = \frac{\gamma(1-\gamma^2)}{r}Q(\gamma,\phi), \quad Q(1,\phi)\neq 0, \tag{7.40}$$

where $A(u,\phi)$ and $Q(\gamma,\phi)$ are some coefficients. The needed result is obtained in Appendix C:

$$-\log(-\Box)X|_{\mathcal{I}^+} = 2\frac{A(u,\phi)}{r^2}(\log r + \mathbf{c}) + \frac{B(u,\phi)}{r^2} + O\left(\frac{\log r}{r^3}\right) \tag{7.41}$$

and

$$\begin{aligned}\int d^2\mathcal{S}(\phi)B(u,\phi)|_{u\to\infty} &= u\int d^2\mathcal{S}(\phi)\Bigg[-6Q(1,\phi) + 4\int_0^1 d\gamma\frac{\gamma^2}{1-\gamma^2}\\ &\quad\times[Q(\gamma,\phi)-Q(1,\phi)]\Bigg] + O(\log u).\end{aligned} \tag{7.42}$$

The behavior of the function (7.33) at $\mathcal{I}^+$ is obtained by substituting Eq. (7.36) for Eq. (7.39) and using Eqs. (7.41) and (7.37). Summarizing the calculation above one has

$$T^{\mu\nu}_{\mathrm{vac}}(3)|_{\mathcal{I}^+} = \frac{1}{r^2}\nabla^\mu u\nabla^\nu u\left(-\frac{\partial\mathcal{E}_{\mathrm{vac}}(3)}{\partial u}\right) + O\left(\frac{1}{r^3}\right), \tag{7.43}$$

$$\begin{aligned}-\frac{\partial\mathcal{E}_{\mathrm{vac}}(3)}{\partial u} = \frac{1}{(4\pi)^2}\mathrm{tr}\Bigg[&-\frac{1}{6}\left(\log r + \log 2 + \frac{3}{2}\right)\left(\frac{\partial}{\partial u}\hat{D}_{\mathbf{1}}^\alpha\right)\\ &\times\left(\frac{\partial}{\partial u}\hat{D}_{\mathbf{1}\alpha}\right) + \frac{1}{6}\frac{\partial}{\partial u}\int_{-\infty}^{u} d\tau\log(u-\tau)\\ &\times\left(\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}}^\alpha(\tau)\right)\left(\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}\alpha}(\tau)\right)\\ &-\frac{\partial^2}{\partial u^2}\hat{B}(u,\phi) + \mathrm{Q.N.}\Bigg].\end{aligned} \tag{7.44}$$

The contribution of the latter total-derivative term to the radiation energy

$$\int_{-\infty}^{\infty} du\int d^2\mathcal{S}(\phi)\left(-\frac{\partial\mathcal{E}_{\mathrm{vac}}(3)}{\partial u}\right) \tag{7.45}$$

is

$$-\frac{1}{(4\pi)^2}\mathrm{tr}\int d^2\mathcal{S}(\phi)\frac{\partial}{\partial u}\hat{B}(u,\phi)|_{u\to\infty}. \tag{7.46}$$

Substituting Eq. (7.38) for Eq. (7.40) and using Eq. (7.42) one obtains

$$-\int d^2\mathcal{S}(\phi)\frac{\partial}{\partial u}\hat{B}(u,\phi)|_{u\to\infty}$$

$$=\int_{-\infty}^{\infty}du\int d^2\mathcal{S}(\phi)\Bigg\{\frac{1}{2}\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}_{\mathbf{1}}\right)\left(\frac{\partial}{\partial u}\hat{D}_{\mathbf{1}\,\alpha}\right)$$

$$-\frac{1}{3}\int_0^1 d\gamma\frac{\gamma^2}{1-\gamma^2}\Bigg[\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}\right)\left(\frac{\partial}{\partial u}\hat{D}_{\alpha}\right)-\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}_{\mathbf{1}}\right)$$

$$\times\left(\frac{\partial}{\partial u}\hat{D}_{\mathbf{1}\,\alpha}\right)\Bigg]\Bigg\}. \tag{7.47}$$

In this way the result (2.23) emerges.

VIII. OTHER MODELS

The results for quantum-field models other than the standard loop can be obtained by combining the results for the standard loop [3]. However, the results for the standard loop should then be known in full, including the contributions of the potential $\hat{P}$. The contribution of the potential to the vacuum energy flux is given in expression (1.21) but this expression implies that the potential is regular at $\mathcal{I}^+$:

$$\hat{P}|_{\mathcal{I}^+}=O\left(\frac{1}{r^3}\right). \tag{8.1}$$

If condition (8.1) does not hold, the contribution of the potential should be calculated anew. For the starting point one may take the expression for $S_{\rm vac}(2)$ with the form factors in the high-frequency approximation [2]

$$S_{\rm vac}(2)=\frac{1}{2(4\pi)^2}\int dx g^{1/2}\,{\rm tr}\Bigg[-\frac{1}{2}\hat{P}\log\left(-\frac{\Box}{m^2}\right)\hat{P}$$

$$-\frac{1}{12}\hat{\mathcal{R}}_{\mu\nu}\log\left(-\frac{\Box}{m^2}\right)\hat{\mathcal{R}}^{\mu\nu}+{\rm const}\times\hat{P}\hat{P}$$

$$+{\rm const}\times\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu}\Bigg] \tag{8.2}$$

and the expression for $T^{\mu\nu}_{\rm vac}(3)|_{\mathcal{I}^+}$ calculated in Ref. [2] up to terms $O(\Box^0)$ in the argument $\Box$ of the external line:

$$T^{\mu\nu}_{\rm vac}(3)|_{\mathcal{I}^+}=\frac{1}{(4\pi)^2}\nabla^{\mu}\nabla^{\nu}{\rm tr}\log(-\Box)\hat{I}(x)+O(\Box^0), \tag{8.3}$$

$$\hat{I}(x)=\frac{1}{2}\left(F_{11}(\Box_2,\Box_3)-\frac{1}{3}F_{00}(\Box_2,\Box_3)\right)\hat{P}_2\hat{P}_3$$

$$-\frac{1}{6}\nabla_{\alpha}\nabla_{\beta}\frac{1}{\Box_2\Box_3}(F_{22}(\Box_2,\Box_3)$$

$$-2F_{11}(\Box_2,\Box_3))\hat{J}^{\alpha}_2\hat{J}^{\beta}_3. \tag{8.4}$$

Of the missing terms $O(\Box^0)$, the important ones can easily be restored. These are the terms in L_{mn}. Since the $\log(-\Box)$ in Eq. (8.3) originates from the expansion of the third-order form factors (Appendix B), each F_{mn} in Eq. (8.4) should be accompanied by the respective L_{mn} to form the combination (7.4). With the terms in L_{mn} added, expression (8.3) becomes analogous to Eq. (7.33). The remaining terms $O(\Box^0)$ and the unspecified constants in Eq. (8.2) contribute only to a numerical renormalization of the news function (Sec. VII).

For the first example, consider the spinor QED. The effective action generated by the fermion loop in this model is (-1) times the action for the standard loop with

$$\hat{P}=\frac{1}{2}\gamma^{\mu}\gamma^{\nu}\hat{\mathcal{R}}_{\mu\nu},\quad \gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=2g^{\mu\nu}\hat{1} \tag{8.5}$$

and

$$\hat{\mathcal{R}}_{\mu\nu}=-iqF_{\mu\nu}\hat{1},\quad {\rm tr}\,\hat{1}=4, \tag{8.6}$$

where $F_{\mu\nu}$ is the Maxwell tensor, and q is the electron's charge.

Since the potential in Eq. (8.5) is singular at $\mathcal{I}^+$, one has to resort to Eqs. (8.2)–(8.4). One obtains

$${\rm tr}\hat{P}_2\hat{P}_3=-\frac{1}{2}{\rm tr}\,\hat{\mathcal{R}}^{\mu\nu}_2\hat{\mathcal{R}}_{3\mu\nu} \tag{8.7}$$

which is valid with any insertion of the form $f(\nabla_2,\nabla_3)$. Then, by Eq. (6.7),

$${\rm tr}\hat{P}_2\hat{P}_3={\rm tr}\Bigg[-\frac{1}{2}\Box\left(\frac{1}{\Box_2\Box_3}\hat{J}^{\alpha}_2\hat{J}_{3\alpha}\right)+\nabla_{\alpha}\nabla_{\beta}\left(\frac{1}{\Box_2\Box_3}\hat{J}^{\alpha}_2\hat{J}^{\beta}_3\right)$$

$$+\frac{1}{2}\left(\frac{1}{\Box_2}\hat{J}^{\alpha}_2\hat{J}_{3\alpha}+\hat{J}^{\alpha}_2\frac{1}{\Box_3}\hat{J}_{3\alpha}\right)\Bigg]. \tag{8.8}$$

When this expression is inserted in Eqs. (8.4),(8.3), the contribution of the first term vanishes at $\mathcal{I}^+$ because of the presence of the overall $\Box$ (see Appendix B), and the contribution of the second term is pure quantum noise because it has the same structure as the $\hat{J}\hat{J}$ term in Eq. (8.4) but with no form factor F_{22}.[8] As a result, one is left with

$$\hat{I}(x)=\frac{1}{2}\frac{1}{\Box_3}F_{11}(\Box_2,\Box_3)\hat{J}^{\alpha}_2\hat{J}_{3\alpha}$$

$$-\frac{1}{6}\nabla_{\alpha}\nabla_{\beta}\frac{1}{\Box_2\Box_3}F_{22}(\Box_2,\Box_3)\hat{J}^{\alpha}_2\hat{J}^{\beta}_3+{\rm Q.N.}, \tag{8.9}$$

where the first term is the contribution of the potential.

[8] In each sum of F_{nn} in Eq. (8.4), only the F_{nn} with the highest n is to be retained since the junior F_{nn} contribute only to the quantum noise [2].

For $\hat{I}(x)$ in Eq. (8.9) the technique of Ref. [2] yields straight away

$$\hat{I}(x)|_{\mathcal{I}^+} = -\frac{1}{3r^2}\int_{-\infty}^{u} d\tau\,(u-\tau)\left(\frac{\partial}{\partial\tau}\hat{D}^{\alpha}_{\mathbf{1}}(\tau)\right)\left(\frac{\partial}{\partial\tau}\hat{D}_{\mathbf{1}\,\alpha}(\tau)\right) \tag{8.10}$$

which is (-2) times the expression (7.36), and

$$\hat{I}(x)|_{i^+} = -\frac{\gamma(1-\gamma^2)}{6r}\int_{-\infty}^{\infty} du\left(\frac{\partial}{\partial u}\hat{D}^{\alpha}\right)\left(\frac{\partial}{\partial u}\hat{D}_{\alpha}\right) \tag{8.11}$$

which is (-2) times the expression (7.38). In addition, there is the overall (-1) appropriate for fermions. It follows that $\partial\mathcal{E}_{\text{vac}}(3)/\partial u$ for QED is *twice* the result for the standard loop. On the other hand, using Eq. (8.7) in Eq. (8.2) and changing the overall sign, one finds that $S_{\text{vac}}(2)$ for QED is also *twice* the result for the standard loop. Thus, up to a numerical addition to the renormalization of the news function, all the results for QED are obtained by doubling the respective results for the standard loop and making the substitution (8.6). Note that, since the balance between $S_{\text{vac}}(2)$ and $S_{\text{vac}}(3)$ is maintained, the fact of doubling can be read just from the β function.

Both the standard loop and the spinor QED have the ''zero-charge'' [11] sign of the static vacuum polarization. It is interesting to see what will be the results in the case of the ''asymptotically free'' sign. For that, consider creation of the Yang-Mills quanta in the external Yang-Mills field. In this consideration, it is convenient to refer to the standard loop with the commutator curvature

$$\hat{\mathcal{R}}_{\mu\nu} = \mathcal{R}^{\,a}_{\,b\mu\nu} = C^{a}_{fb}F^{f}_{\mu\nu}, \tag{8.12}$$

where C^a_{fb} are the group structure constants, and $F^f_{\mu\nu}$ is the strength of the external Yang-Mills field.

In the minimal [3] gauge, the effective action generated by the ghost loop is (-2) times the action for the standard loop with $\hat{P}=0$ and $\hat{\mathcal{R}}_{\mu\nu}$ in Eq. (8.12). The quantities pertaining to the loop of the gauge field will be distinguished with a tilde and expressed through the quantity in Eq. (8.12). The loop of the gauge field is the standard loop with [3]

$$\tilde{P} = P^{(a\alpha)}_{(b\beta)} = -2\mathcal{R}^{\,a}_{\,b\alpha\gamma}g^{\gamma\beta}, \tag{8.13}$$

$$\tilde{\mathcal{R}}_{\mu\nu} = \mathcal{R}^{(a\alpha)}_{(b\beta)\mu\nu} = \mathcal{R}^{\,a}_{\,b\mu\nu}\delta^{\beta}_{\alpha}. \tag{8.14}$$

Hence, in terms of the $\hat{\mathcal{R}}_{\mu\nu}$ in Eq. (8.12),

$$\text{tr}\,\tilde{P}_2\tilde{P}_3 = -4\,\text{tr}\,\hat{\mathcal{R}}^{\mu\nu}_2\hat{\mathcal{R}}_{3\mu\nu}, \tag{8.15}$$

$$\text{tr}\,\tilde{\mathcal{R}}^{\mu\nu}_2\tilde{\mathcal{R}}_{3\mu\nu} = 4\,\text{tr}\,\hat{\mathcal{R}}^{\mu\nu}_2\hat{\mathcal{R}}_{3\mu\nu}. \tag{8.16}$$

Relation (8.15) differs from Eq. (8.7) only in the coefficient. Therefore, the calculation of the contribution of the potential repeats literally the one above; only the result should be multiplied by 8. The contribution of the potential $\tilde{P}$ to $\partial\mathcal{E}_{\text{vac}}(3)/\partial u$ is then (-24) times the result for the standard loop. The contribution of $\tilde{\mathcal{R}}_{\mu\nu}$ is, by Eq. (8.16), four times the result for the standard loop. Since the contribution of ghosts is (-2) times the result for the standard loop, the grand total is (-22) times the result for the standard loop. The total action $S_{\text{vac}}(2)$ for the Yang-Mills field is also (-22) times the result for the standard loop as follows immediately from using Eqs. (8.15) and (8.16) in Eq. (8.2) and adding the ghost contribution.

Thus, also for the Yang-Mills coupling, all vacuum fluxes are multiples of the respective fluxes for the standard loop [with the substitution (8.12)], and the multiplicity is (-22) in accord with the β function but the price for the asymptotic freedom is that the radiation energy is *negative*.[9] This is not surprising. Because the Yang-Mills quanta are exactly massless, a source of the Yang-Mills field would cause initially an infinite static polarization. The Yang-Mills charge is unobservable at infinity.

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APPENDIX A: THE ONE-LOOP FORM FACTORS

The basic building element for all one-loop form factors [9] is the operator[10]

$$\mathcal{H}_q = \sqrt{\frac{2q}{\Box}}K_1(\sqrt{2q\Box}), \quad q<0 \tag{A1}$$

depending on the parameter q, with K_1 the order-1 Macdonald function. By the properties of the Macdonald functions one has also

$$\frac{d}{dq}\mathcal{H}_q = K_0(\sqrt{2q\Box}) \tag{A2}$$

and

$$2q\frac{d^2}{dq^2}\mathcal{H}_q = \Box\mathcal{H}_q. \tag{A3}$$

Despite its scaring appearances the operator (A1) has a simple kernel. Its retarded kernel is

$$\mathcal{H}_qX(x) = \frac{1}{4\pi}\int_{\text{past of }x} d\bar{x}\bar{g}^{1/2}\delta(\sigma(x,\bar{x})-q)X(\bar{x}), \tag{A4}$$

[9] This does not contradict the proof of positivity in Ref. [2] since the proof has been given only for the case where the potential is regular at $\mathcal{I}^+$, Eq. (8.1).

[10] All operator functions are originally defined in the Euclidean domain $\Box<0$.

where $\sigma(x,\bar{x})$ is the world function [6], and the integration is over the past sheet of the hyperboloid of equal geodetic distance from x. The derivation of Eq. (A4) is based on the spectral representation for the operator (A1):

$$\mathcal{H}_q = -\frac{1}{\Box} - \sqrt{-2q}\int_0^\infty d\mu \frac{J_1(\mu\sqrt{-2q})}{\mu^2-\Box}, \qquad \text{(A5)}$$

where J_1 is the Bessel function. Inserting in Eq. (A5) the kernel of the retarded resolvent [10]

$$\frac{1}{\mu^2-\Box}X(x) = \frac{1}{4\pi}\int_{\text{past of } x} d\bar{x}\bar{g}^{1/2} \times\left(\delta(\sigma)-\theta(-\sigma)\frac{\mu J_1(\mu\sqrt{-2\sigma})}{\sqrt{-2\sigma}}\right)\bar{X}, \qquad \text{(A6)}$$

doing the spectral-mass integrations

$$\int_0^\infty d\mu\, J_1(\mu\sqrt{-2q}) = \frac{1}{\sqrt{-2q}},$$

$$\int_0^\infty d\mu\, \mu\, J_1(\mu\sqrt{-2q})J_1(\mu\sqrt{-2\sigma}) = \delta(\sigma-q), \qquad \text{(A7)}$$

and using that

$$-\frac{1}{\Box}X(x) = \frac{1}{4\pi}\int_{\text{past of } x} d\bar{x}\bar{g}^{1/2}\delta(\sigma)\bar{X} \qquad \text{(A8)}$$

one obtains Eq. (A4).

The kernel (A4) was used in Ref. [2] without pointing out its relation to Eq. (A1). This relation and the technique in Ref. [9] make it possible to obtain the kernels of all one-loop form factors in the expectation-value equations. Thus, for the second-order and third-order form factors [9]

$$F_{mn}(\Box_1,\Box_2) = \left(\frac{\partial}{\partial j_1}\right)^m\left(\frac{\partial}{\partial j_2}\right)^n \frac{\log(j_1\Box_1/j_2\Box_2)}{j_1\Box_1-j_2\Box_2}\bigg|_{j_1=j_2=1}, \qquad \text{(A9)}$$

$$\Gamma_{kmn}(\Box_1,\Box_2,\Box_3) = -\int_{\alpha>0} d\alpha_1 d\alpha_2 d\alpha_3\, \delta(1-\alpha_1-\alpha_2-\alpha_3) \times \frac{\alpha_1{}^k\alpha_2{}^m\alpha_3{}^n}{\alpha_2\alpha_3\Box_1+\alpha_1\alpha_3\Box_2+\alpha_1\alpha_2\Box_3} \qquad \text{(A10)}$$

one has [9]

$$F_{mn}(\Box_1,\Box_2) = -2\int_{-\infty}^0 dq\, q^{m+n}\frac{d^m}{dq^m}K_0(\sqrt{2q\Box_1}) \times\frac{d^n}{dq^n}K_0(\sqrt{2q\Box_2}), \qquad \text{(A11)}$$

$$\Gamma_{kmn}(\Box_1,\Box_2,\Box_3) = \frac{4(-1)^{k+m+n}}{(k+m+n)!}\int_{-\infty}^0 dq\, q^{k+m+n} \times\frac{d^k}{dq^k}K_0(\sqrt{2q\Box_1})\frac{d^m}{dq^m}K_0(\sqrt{2q\Box_2}) \times\frac{d^n}{dq^n}K_0(\sqrt{2q\Box_3}), \qquad \text{(A12)}$$

and, therefore,

$$F_{mn}(\Box_1,\Box_2)X_1X_2(x) = -2\int_{-\infty}^0 dq\, q^{m+n}\left[\left(\frac{d}{dq}\right)^{m+1}\mathcal{H}_qX_1(x)\right] \times\left[\left(\frac{d}{dq}\right)^{n+1}\mathcal{H}_qX_2(x)\right], \qquad \text{(A13)}$$

$$\Gamma_{kmn}(\Box_1,\Box_2,\Box_3)X_1X_2X_3(x) = \frac{4(-1)^{k+m+n}}{(k+m+n)!}\int_{-\infty}^0 dq\, q^{k+m+n}\left[\left(\frac{d}{dq}\right)^{k+1}\mathcal{H}_qX_1(x)\right] \times\left[\left(\frac{d}{dq}\right)^{m+1}\mathcal{H}_qX_2(x)\right]\left[\left(\frac{d}{dq}\right)^{n+1}\mathcal{H}_qX_3(x)\right] \qquad \text{(A14)}$$

with $\mathcal{H}_qX(x)$ in Eq. (A4).

Equation (A3) makes it possible to obtain easily the superpositions of the kernels above with $1/\Box$. For example,

$$\frac{1}{\Box_2}F_{mn}(\Box_1,\Box_2)X_1X_2 = -\int_{-\infty}^0 dq\, q^{m+n}\left(\frac{d^{m+1}}{dq^{m+1}}\mathcal{H}_qX_1\right)\left(\frac{d^n}{dq^n}\int_{-\infty}^q\frac{d\bar{q}}{\bar{q}}\mathcal{H}_{\bar{q}}X_2\right), \qquad \text{(A15)}$$

$$\frac{1}{\Box_1\Box_2}F_{mn}(\Box_1,\Box_2)X_1X_2 = -\frac{1}{2}\int_{-\infty}^0 dq\, q^{m+n}\left(\frac{d^m}{dq^m}\int_{-\infty}^q\frac{d\bar{q}}{\bar{q}}\mathcal{H}_{\bar{q}}X_1\right) \times\left(\frac{d^n}{dq^n}\int_{-\infty}^q\frac{d\bar{q}}{\bar{q}}\mathcal{H}_{\bar{q}}X_2\right) \qquad \text{(A16)}$$

which is valid including the cases $n=0$, $m=0$ (cf. Ref. [2]). Similarly for the third-order form factors. The convergence of the integrals in q at the upper limit is controlled by the behaviors

$$\mathcal{H}_q\big|_{q=0} = -\frac{1}{\Box}, \quad \frac{d}{dq}\mathcal{H}_q\big|_{q\to 0} = O(\log q) \qquad \text{(A17)}$$

following from Eq. (A1), and the convergence at the lower limit should be provided by the properties of the test functions [2]. Equation (A3) can also be obtained directly by acting with the operator $\Box$ on Eq. (A4) and neglecting the curvature in σ, $\Box\sigma=4+O[R]$.

One might have introduced the kernel even more elementary than the one in Eq. (A4):

$$\Theta_q X(x)=\frac{1}{4\pi}\int_{\text{past of } x} d\bar{x}\bar{g}^{1/2}\theta(q-\sigma(x,\bar{x}))X(\bar{x}),\quad q<0, \tag{A18}$$

$$\frac{d}{dq}\Theta_q=\mathcal{H}_q,\quad \Theta_q\big|_{q=0}=\frac{2}{\Box^2}, \tag{A19}$$

whence

$$\Theta_q=\frac{2q}{\Box}K_2(\sqrt{2q\Box}). \tag{A20}$$

The initial condition in Eq. (A19) implies that

$$\frac{1}{\Box^2}X(x)=\frac{1}{8\pi}\int_{\text{past of } x} d\bar{x}\bar{g}^{1/2}\theta(-\sigma)\bar{X} \tag{A21}$$

and

$$\frac{1}{(4\pi)^2}\int_{\text{past of } x} d\bar{x}\bar{g}^{1/2}\delta(\sigma(x,\bar{x}))\int_{\text{past of } \bar{x}} d\bar{\bar{x}}\,\bar{\bar{g}}^{1/2}\delta(\sigma(\bar{x},\bar{\bar{x}}))\bar{\bar{X}}$$
$$=\frac{1}{8\pi}\int_{\text{past of x}} d\bar{x}\bar{g}^{1/2}\theta(-\sigma(x,\bar{x}))\bar{X}. \tag{A22}$$

Equation (A21) can be obtained by acting with the operator $\Box$ on Eq. (A18) and using Eq. (A8). It is also a limiting case of the formula [2]

$$\frac{1}{(m^2-\Box)^2}X(x)=\frac{1}{8\pi}\int_{\text{past of } x} d\bar{x}\bar{g}^{1/2}\theta(-\sigma)J_0(m\sqrt{-2\sigma})\bar{X} \tag{A23}$$

for the massive operator. The kernels (A18) and (A21) do not decrease at the future infinity and can be used for a direct determination of the moments. Thus,

$$\frac{\partial}{\partial u}\left(\frac{1}{\Box^2}X\bigg|_{\mathcal{I}^+}\right)=\frac{1}{2}D_{\mathbf{1}}(u,\phi|X), \tag{A24}$$

where the quantity on the right-hand side is the $D_{\mathbf{1}}$ moment of the source X.

APPENDIX B: THE THIRD-ORDER FORM FACTORS AT $\mathcal{I}^+$

The form factors in the third-order action (2.3) are linear combinations of the functions $\Gamma_{kmn}(\Box_1,\Box_2,\Box_3)$ introduced in Appendix A. The typical contribution of such a form factor to the energy-momentum tensor at point x has the form

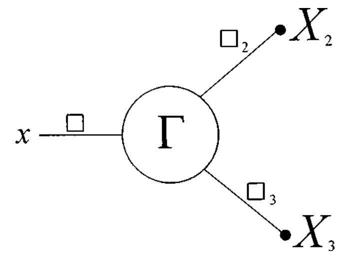

FIG. 1. The diagram for the contribution (B1) to the expectation-value current at point x. The argument $\Box$ of the form factor Γ corresponds to the external line.

$$\Gamma_{kmn}(\Box,\Box_2,\Box_3)X_2X_3(x), \tag{B1}$$

where the X's are the commutator curvatures or their derivatives, and it is assumed that first $\Box_2$ acts on $X_2=X(x_2)$, and $\Box_3$ on $X_3=X(x_3)$ with subsequently making the points x_2 and x_3 coincident with the observation point x, and next the first argument $\Box$ of the form factor acts on the thus obtained function of the observation point. This nonlocal structure corresponds to the diagram in Fig. 1.

By the results in Appendix A, expression (B1) can be represented as follows:

$$\Gamma_{kmn}(\Box,\Box_2,\Box_3)X_2X_3(x)$$
$$=\frac{4(-1)^{k+m+n}}{(k+m+n)!}\int_{-\infty}^{0}dq\left(q^k\frac{d^k}{dq^k}K_0(\sqrt{2q\Box})\right)\mathcal{F}(q,x), \tag{B2}$$

where the operator $\Box$ acts to the right on the function of x, and this function is

$$\mathcal{F}(q,x)=q^{m+n}\left[\left(\frac{d}{dq}\right)^{m+1}\mathcal{H}_q X_2(x)\right]\left[\left(\frac{d}{dq}\right)^{n+1}\mathcal{H}_q X_3(x)\right]. \tag{B3}$$

When the X's are expressed as in Eq. (6.7) through the sources J *having compact spatial supports*, there occur two essentially different cases. An example of the first case is

$$X_2X_3=J_2^\alpha J_3^\beta, \tag{B4}$$

and examples of the second case are

$$X_2X_3=\frac{1}{\Box_2}J_2^\alpha J_3^\beta\quad\text{or}\quad J_2^\alpha\frac{1}{\Box_3}J_3^\beta\quad\text{or}\quad\nabla_\alpha\nabla_\beta\left(\frac{1}{\Box_2\Box_3}J_2^\alpha J_3^\beta\right). \tag{B5}$$

The difference between the two cases is in the behaviors of integrals with the function $\mathcal{F}(q,x)$ as $x\to\mathcal{I}^+$. These behaviors are readily obtained by the technique in Ref. [2] (see also Sec. VI above). In the first case one has

$$\int_{-\infty}^{0}dq\,\mathcal{F}(q,x)\big|_{x\to\mathcal{I}^+}=O(r^{-3}), \tag{B6}$$

and in the second case

$$\int_{-\infty}^{0} dq\, \mathcal{F}(q,x)|_{x\to\mathcal{I}^+} = O(r^{-2}). \tag{B7}$$

The second case is our main concern here since the function (B7) is singular at $\mathcal{I}^+$.

Our present goal is obtaining the behavior of the current (B2) as $x\to\mathcal{I}^+$. The principal assertion is that this behavior is determined by the first few terms of the small-$\Box$ expansion of the form factor Γ in the argument $\Box$ of the external line. For the proof it suffices to consider two generic terms of the small-$\Box$ expansion of the function $K_0(\sqrt{2q\Box})$ in Eq. (B2):

$$(q\Box)^p \quad \text{and} \quad (q\Box)^p \log q\Box. \tag{B8}$$

It will be recalled that the behaviors (B6) and (B7) are obtained by making the replacement (6.10) of the integration variable q. From the form of this replacement, it follows that in the case (B7) one has also

$$\int_{-\infty}^{0} dq\, q^p\, \mathcal{F}(q,x)|_{x\to\mathcal{I}^+} = r^{p-2} a_p(u,\phi), \tag{B9}$$

$$\int_{-\infty}^{0} dq\, q^p \log(-q) \mathcal{F}(q,x)|_{x\to\mathcal{I}^+} = r^{p-2}\log r\, a_p(u,\phi) + O(r^{p-2}), \tag{B10}$$

and then, by the result in Appendix C,

$$\log(-\Box)\left(\int_{-\infty}^{0} dq\, q^p\, \mathcal{F}(q,x)\right)\Bigg|_{x\to\mathcal{I}^+} = -r^{p-2}\log r\, a_p(u,\phi) + O(r^{p-2}), \quad p\geqslant 1 \tag{B11}$$

with one and the same coefficient $a_p(u,\phi)$ in all the expressions (B9) to (B11).

Moreover, using the following form of the operator $\Box$ at $\mathcal{I}^+$ [2],

$$\Box X|_{\mathcal{I}^+} = -\frac{2}{r}\frac{\partial}{\partial u}X - 2\frac{\partial^2}{\partial u\,\partial r}X + O\left(\frac{1}{r^2}X\right), \tag{B12}$$

one obtains

$$\Box O(r^p)|_{\mathcal{I}^+} = O(r^{p-1}), \quad p\neq -1 \tag{B13}$$

and, in the exceptional case $p=-1$,

$$\Box O(r^{-1})|_{\mathcal{I}^+} = O(r^{-3}). \tag{B14}$$

Owing to the latter fact, one has

$$\Box^p O(r^{p-2})|_{\mathcal{I}^+} = O\left(\frac{1}{r^3}\right), \quad p\geqslant 1. \tag{B15}$$

The relations above make it possible to obtain the contributions to the current (B2) of the expansion terms (B8):

$$\Box^p \int_{-\infty}^{0} dq\, q^p\, \mathcal{F}(q,x)|_{x\to\mathcal{I}^+} = O\left(\frac{1}{r^3}\right), \quad p\geqslant 1, \tag{B16}$$

$$\Box^p\left[\log(-\Box)\left(\int_{-\infty}^{0} dq\, q^p\, \mathcal{F}(q,x)\right) + \int_{-\infty}^{0} dq\, q^p \log(-q)\mathcal{F}(q,x)\right]\Bigg|_{x\to\mathcal{I}^+} = O\left(\frac{1}{r^3}\right), \quad p\geqslant 1. \tag{B17}$$

It follows that, in the case (B7), the function of $\Box$ in Eq. (B2) can be truncated as follows:

$$q^k \frac{d^k}{dq^k} K_0(\sqrt{2q\Box}) = \begin{cases} \frac{1}{2}(-1)^k (k-1)!, & k>0, \\ -\frac{1}{2}\log\frac{q\Box}{2} - \mathbf{c}, & k=0, \end{cases} + \text{irrelevant terms}, \tag{B18}$$

where the irrelevant terms are the terms whose contributions to the current (B2) are $O(1/r^3)$ at $\mathcal{I}^+$. By a similar analysis, in the case (B6) this function can be truncated even more:

$$q^k \frac{d^k}{dq^k} K_0(\sqrt{2q\Box}) = \begin{cases} 0, & k>0, \\ -\frac{1}{2}\log(-\Box), & k=0, \end{cases} + \text{irrelevant terms}, \tag{B19}$$

and one recovers the algorithm used in Ref. [2]. Thus the amendment needed in the case (B7) as compared to the case (B6) is retaining the terms $O(\Box^0)$ of the form factors.

In addition to the contributions of the form (B1), the vacuum energy-momentum tensor contains contributions in which the form factors $\Gamma_{kmn}(\Box,\Box_2,\Box_3)$ are superposed with $1/\Box$ in the argument of the external line [9]:

$$\frac{1}{\Box}\Gamma_{kmn}(\Box,\Box_2,\Box_3) X_2 X_3(x). \tag{B20}$$

These contributions occur only at $k\geqslant 1$ [9] and only in the case (B6).[11] By the same consideration as above, the operator function in Eq. (B2) can then be truncated as follows:

[11]This fact is a matter of a direct calculation [9] but it is also a necessary condition for the expectation-value spacetime to be asymptotically flat.

$$\frac{1}{\Box}q^k\frac{d^k}{dq^k}K_0(\sqrt{2q\Box})$$
$$=\frac{1}{2}(-1)^k(k-1)!\frac{1}{\Box}$$
$$+\begin{cases}-\frac{1}{4}(-1)^k(k-2)!q, & k>1,\\ -\frac{1}{4}\left(\log\frac{q\Box}{2}+2\mathbf{c}-1\right)q, & k=1,\end{cases}$$
$$+\text{irrelevant terms.} \tag{B21}$$

The effect of these truncations is that the third-order form factors boil down to the second-order form factors. The latter are the functions F_{mn} and L_{mn} introduced in Sec. VII, and similar functions originating from expansion (B21) and differing from F_{mn} and L_{mn} by an extra power of q:

$$G_{mn}(\Box_2,\Box_3)X_2X_3$$
$$=-2\int_{-\infty}^{0}dq\, q^{m+n+1}\left[\left(\frac{d}{dq}\right)^{m+1}\mathcal{H}_qX_2\right]$$
$$\times\left[\left(\frac{d}{dq}\right)^{n+1}\mathcal{H}_qX_3\right], \tag{B22}$$

$$M_{mn}(\Box_2,\Box_3)X_2X_3$$
$$=-2\int_{-\infty}^{0}dq\, q^{m+n+1}\left[\log\left(-\frac{q}{2}\right)+2\mathbf{c}\right]$$
$$\times\left[\left(\frac{d}{dq}\right)^{m+1}\mathcal{H}_qX_2\right]\left[\left(\frac{d}{dq}\right)^{n+1}\mathcal{H}_qX_3\right]. \tag{B23}$$

Using Eq. (A3), the latter functions can be expressed through F_{mn} and L_{mn}:

$$G_{mn}(\Box_2,\Box_3)$$
$$=\frac{2}{\Box_2}[F_{m+2,n}(\Box_2,\Box_3)+(m+1)F_{m+1,n}(\Box_2,\Box_3)]$$
$$=\frac{2}{\Box_3}[F_{m,n+2}(\Box_2,\Box_3)+(n+1)F_{m,n+1}(\Box_2,\Box_3)], \tag{B24}$$

$$M_{mn}(\Box_2,\Box_3)$$
$$=\frac{2}{\Box_2}[L_{m+2,n}(\Box_2,\Box_3)+(m+1)L_{m+1,n}(\Box_2,\Box_3)]$$
$$=\frac{2}{\Box_3}[L_{m,n+2}(\Box_2,\Box_3)+(n+1)L_{m,n+1}(\Box_2,\Box_3)]. \tag{B25}$$

Equivalence of the two forms in Eq. (B24) follows from the identities for F_{mn} in Ref. [2]. Similar identities can be derived for L_{mn}, G_{mn}, and M_{mn}. All of them are based on Eq. (A3) and the integration by parts in the q integrals.

The consideration above can be summarized as follows. For the case (B4) one has

$$\Gamma_{kmn}(\Box,\Box_2,\Box_3)J_2J_3\big|_{\mathcal{I}^+}=O\left(\frac{1}{r^3}\right),\quad k>0, \tag{B26}$$

$$\Gamma_{\mathbf{0}mn}(\Box,\Box_2,\Box_3)J_2J_3\big|_{\mathcal{I}^+}$$
$$=\frac{(-1)^{m+n}}{(m+n)!}\log(-\Box)[F_{mn}(\Box_2,\Box_3)J_2J_3]+O\left(\frac{1}{r^3}\right). \tag{B27}$$

For all subcases in Eq. (B5) one has

$$\Gamma_{kmn}(\Box,\Box_2,\Box_3)X_2X_3\big|_{\mathcal{I}^+}$$
$$=-\frac{(-1)^{m+n}(k-1)!}{(k+m+n)!}F_{mn}(\Box_2,\Box_3)X_2X_3+O\left(\frac{1}{r^3}\right),$$
$$k>0, \tag{B28}$$

$$\Gamma_{\mathbf{0}mn}(\Box,\Box_2,\Box_3)X_2X_3\big|_{\mathcal{I}^+}$$
$$=\frac{(-1)^{m+n}}{(m+n)!}\{\log(-\Box)[F_{mn}(\Box_2,\Box_3)X_2X_3]$$
$$+L_{mn}(\Box_2,\Box_3)X_2X_3\}+O\left(\frac{1}{r^3}\right). \tag{B29}$$

For the superpositions of Γ with $1/\Box$ one has

$$\frac{1}{\Box}\Gamma_{kmn}(\Box,\Box_2,\Box_3)J_2J_3\big|_{\mathcal{I}^+}$$
$$=-\frac{(-1)^{m+n}(k-1)!}{(k+m+n)!}\frac{1}{\Box}[F_{mn}(\Box_2,\Box_3)J_2J_3]$$
$$+\frac{1}{2}\frac{(-1)^{m+n}(k-2)!}{(k+m+n)!}G_{mn}(\Box_2,\Box_3)J_2J_3+O\left(\frac{1}{r^3}\right),$$
$$k>1, \tag{B30}$$

$$\frac{1}{\Box}\Gamma_{\mathbf{1}mn}(\Box,\Box_2,\Box_3)J_2J_3\big|_{\mathcal{I}^+}$$
$$=-\frac{(-1)^{m+n}}{(m+n+1)!}\frac{1}{\Box}[F_{mn}(\Box_2,\Box_3)J_2J_3]$$
$$-\frac{1}{2}\frac{(-1)^{m+n}}{(m+n+1)!}\{\log(-\Box)$$
$$\times[G_{mn}(\Box_2,\Box_3)J_2J_3]+M_{mn}(\Box_2,\Box_3)J_2J_3$$
$$-G_{mn}(\Box_2,\Box_3)J_2J_3\}+O\left(\frac{1}{r^3}\right). \tag{B31}$$

The senior terms of the latter expressions, proportional to $1/\Box$, cancel in the energy-momentum tensor [1]. Another useful relation

$$\frac{\Box}{\Box_2\Box_3}\Gamma_{kmn}(\Box,\Box_2,\Box_3)J_2J_3|_{\mathcal{I}^+}=O\left(\frac{1}{r^3}\right) \qquad \text{(B32)}$$

is a consequence of Eq. (B14).

Finally, the relations (B26)–(B32) remain unchanged when multiplied by $\Box_2/\Box_3$ or $\Box_3/\Box_2$. Indeed, replacing the function $\mathcal{F}(q,x)$ with $(\Box_2/\Box_3)\mathcal{F}(q,x)$ or $(\Box_3/\Box_2)\mathcal{F}(q,x)$ does not change the behaviors (B6) and (B7). Using Eq. (A3), the multiplier $\Box_2/\Box_3$ or $\Box_3/\Box_2$ can be absorbed in any second-order or third-order form factor.

APPENDIX C: THE OPERATORS $\log(-\Box)$ AND $1/\Box$ FOR TEST FUNCTIONS SINGULAR AT $\mathcal{I}^+$

The behavior (4.3) of the resolvent is valid only for test functions having compact spatial supports [10]. Therefore, the behaviors at $\mathcal{I}^+$ of all form factors, used in Ref. [2], are also valid only under conditions (1.44). For $\log(-\Box)$ this is the behavior (3.25). In the general case, the support of the test function may conventionally be divided into a compact domain and asymptotic domain. The $1/r^2$ behavior in Eq. (3.25) is a contribution of the compact domain. Any behavior of $\log(-\Box)X|_{\mathcal{I}^+}$ more singular than $1/r^2$ (call it just singular) can only be a contribution of the asymptotic domain, i.e., of $X|_{\mathcal{I}^+}$ itself. Similarly, the regular behavior of $(1/\Box)X|_{\mathcal{I}^+}$ is $1/r$ and is a contribution of the compact domain. A key to obtaining the contributions of the asymptotic domain is the fact that the null hyperplane reaches $\mathcal{I}^+$ at only one point of the celestial sphere [2]. Therefore, the singular contributions are always local in the angles although possibly nonlocal in time. To see why they may be nonlocal in time recall that, when a point tends to $\mathcal{I}^+$, one generator of its past light cone merges with $\mathcal{I}^+$ entirely [2]. The retarded time ranges along this generator to $-\infty$ whereas the whole generator is labeled by a single value of the angles.

Let $\mathcal{L}(x,\bar{x})$ be the retarded kernel of $\log(-\Box)$,

$$\log(-\Box)X(x)=\int d\bar{x}\,\bar{g}^{1/2}\mathcal{L}(x,\bar{x})\bar{X}. \qquad \text{(C1)}$$

By the argument above,

$$\log(-\Box)X(x)|_{x=(u,\phi,r\to\infty)} = \int d\bar{x}\,\bar{g}^{1/2}\,\mathcal{L}(x,\bar{x})(\bar{X}|_{\bar{\phi}=\phi})+O\left(\frac{1}{r^2}\right), \qquad \text{(C2)}$$

i.e., for obtaining the singular terms at $\mathcal{I}^+$, the test function can be taken at the angles of the observation point, $\bar{\phi}=\phi$. The angle integrations in $d\bar{x}$ can then be done explicitly, and, as a result, the kernel becomes spherically symmetric. Thus, for obtaining the singular contributions at $\mathcal{I}^+$, it suffices to consider the spherically symmetric kernel.

The spherically symmetric kernel suffices for obtaining also the regular contributions provided that X is a scalar, and

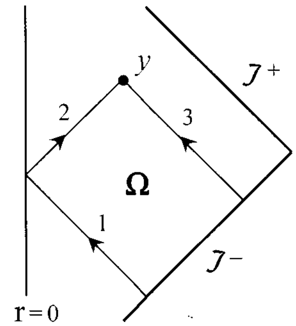

FIG. 2. Penrose diagram for the Lorentzian section of a spherically symmetric spacetime. The timelike line $r=0$ is the central geodesic. The union of paths 1 and 2, and path 3 are the two radial light rays that come to the two-dimensional observation point y. Ω is the domain bounded by the paths 1,2,3.

one needs only the integral of Eq. (C2) over the two-sphere. Indeed, to lowest order in the curvature, the scalar kernel of $\log(-\Box)$ can depend on the angles only through the arc length between the points ϕ and $\bar{\phi}$ on the two-sphere. Therefore,

$$\int d^2\mathcal{S}(\phi)[\log(-\Box)X]=\int d\bar{x}\,\bar{g}^{1/2}\mathcal{L}(x,\bar{x})\left(\int d^2\mathcal{S}(\bar{\phi})\bar{X}\right), \qquad \text{(C3)}$$

where the angle integrations in $d\bar{x}$ concern only the kernel $\mathcal{L}(x,\bar{x})$ and convert it into a spherically symmetric kernel.

One case of the singular behavior considered below is where $X|_{\mathcal{I}^+}=O(1/r)$, and $\log(-\Box)X|_{\mathcal{I}^+}$ is needed up to the regular terms $O(1/r^2)$. In this case Eq. (C2) works. Another case is where $X|_{\mathcal{I}^+}=O(1/r^2)$, and $\log(-\Box)X|_{\mathcal{I}^+}$ is needed including the regular terms $1/r^2$. This case is more difficult but is encountered only in $T^{\mu\nu}_{\rm vac}(3)$ (Sec. VII) where the limitations implied in Eq. (C3) are fulfilled. Therefore, in both cases one may use the spherically symmetric kernel.

Below, y is a point of the two-dimensional Lorentzian section of a spherically symmetric spacetime, and $Y(y)$ is a test function restricted to this section. The spherically symmetric retarded kernel of the operator $\log(-\Box)$ is of the form [8,5]

$$\begin{aligned}
-\log\left(-\frac{\Box}{m^2}\right)Y(y)={}&\frac{1}{r}\int_\infty^0 d\bar{r}\frac{\bar{r}}{\bar{r}+r}\bar{Y}|_{\text{path 1}}\\
&+\frac{1}{r}\int_0^r d\bar{r}\log[m(r-\bar{r})]\frac{d}{d\bar{r}}(\bar{r}\bar{Y}|_{\text{path 2}})\\
&+\frac{1}{r}\int_\infty^r d\bar{r}\log[m(\bar{r}-r)]\frac{d}{d\bar{r}}(\bar{r}\bar{Y}|_{\text{path 3}})\\
&+2\mathbf{c}Y(y),
\end{aligned} \qquad \text{(C4)}$$

where r is the luminosity coordinate of the observation point y, and the integrations are along the null paths 1,2,3 shown in Fig. 2. In Eq. (C4), each of the paths is parametrized with the luminosity coordinate $\bar{r}$. The retarded time labeling the radial future light cones and normalized in Eq. (1.11) will be denoted u as above. In the coordinates $y=(u,r)$, $\overline{y}=(\overline{u},\overline{r})$, and with the curvature neglected, path 1 is $\overline{u}+2\overline{r}=u$, path 2 is $\overline{u}=u$, and path 3 is $\overline{u}+2\overline{r}=u+2r$.

The spherically symmetric retarded kernel of the operator $1/\Box$ is of the form

$$-\frac{1}{\Box}Y(y)=\frac{1}{2r}\int_{\Omega}d^2\overline{y}g^{1/2}(\overline{y})\overline{rY}, \tag{C5}$$

where Ω is the domain bounded by the paths 1,2,3, and $d^2\overline{y}g^{1/2}(\overline{y})$ is the induced volume element. Hence, in the coordinates $y=(u,r)$,

$$-\frac{\partial}{\partial u}\frac{1}{\Box}Y(u,r)=\frac{1}{2r}\int_{\infty}^{0}d\overline{r}\,\overline{rY}\Big|_{\text{path 1}}+\frac{1}{2r}\int_{0}^{r}d\overline{r}\,\overline{rY}\Big|_{\text{path 2}}-\frac{1}{2r}\int_{\infty}^{r}d\overline{r}\,\overline{rY}\Big|_{\text{path 3}}. \tag{C6}$$

Denoting the contributions of the paths 1,2,3 in Eq. (C4) $\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3$, one has

$$-\log\left(-\frac{\Box}{m^2}\right)Y(y)=\mathcal{P}_1(y)+\mathcal{P}_2(y)+\mathcal{P}_3(y)+2\mathbf{c}Y(y). \tag{C7}$$

The contribution of path 2 can be rewritten identically as follows:

$$\mathcal{P}_2(y)=\frac{1}{r^{n+1}}\int_0^r d\overline{r}\,\overline{r}\frac{\overline{r}^n-r^n}{\overline{r}-r}\overline{Y}\Big|_{\text{path 2}}+(\log mr)Y(y)+\frac{1}{r^n}\int_0^1 d\xi\log(1-\xi)f(r\xi) \tag{C8}$$

with n arbitrary, and

$$f(\overline{r})\equiv\frac{d}{d\overline{r}}(\overline{r}^{n+1}\overline{Y}|_{\text{path 2}}). \tag{C9}$$

If one chooses n equal to the power of decrease of Y at $\mathcal{I}^+$

$$Y(y)|_{\mathcal{I}^+}=\frac{A(u)}{r^n}, \tag{C10}$$

the last integral in Eq. (C8) will have a finite limit:

$$\int_0^1 d\xi\log(1-\xi)f(r\xi)\Big|_{r\to\infty}=-A(u). \tag{C11}$$

In this way one obtains for $n=1$

$$n=1,\quad \mathcal{P}_2(y)|_{\mathcal{I}^+}=\frac{\log(mr)}{r}A(u)+O\left(\frac{\log r}{r^2}\right) \tag{C12}$$

(with no pure $1/r$ term), and for $n=2$

$$n=2,\quad \mathcal{P}_2(y)|_{\mathcal{I}^+}=2\frac{\log(mr)}{r^2}A(u)-\frac{1}{r^2}\int_0^\infty d\overline{r}\log(m\overline{r})\frac{d}{d\overline{r}}(\overline{r}^2\overline{Y}|_{\text{path 2}})+O\left(\frac{\log r}{r^3}\right). \tag{C13}$$

As $y\to\mathcal{I}^+$, path 3 shifts entirely to $\mathcal{I}^+$. Introducing the retarded time $\overline{u}$ as a parameter along path 3, one can easily calculate the limit

$$\mathcal{P}_3(y)|_{\mathcal{I}^+}=\int_{-\infty}^{u}d\overline{u}\log\left(m\frac{u-\overline{u}}{2}\right)\frac{d}{d\overline{u}}(\overline{Y}|_{\mathcal{I}^+}). \tag{C14}$$

Hence for the behavior (C10) one obtains

$$\mathcal{P}_3(y)|_{\mathcal{I}^+}=\frac{1}{r^n}\int_{-\infty}^{u}d\overline{u}\log\left(m\frac{u-\overline{u}}{2}\right)\frac{d}{d\overline{u}}A(\overline{u})+O\left(\frac{1}{r^{n+1}}\right). \tag{C15}$$

Finally, as $y\to\mathcal{I}^+$, path 1 remains fixed. Therefore, its contribution is always regular:

$$\mathcal{P}_1(y)|_{\mathcal{I}^+}=\frac{1}{r^2}\int_{\infty}^{0}d\overline{r}\,\overline{r\,Y}\Big|_{\text{path 1}}. \tag{C16}$$

The contributions of the paths 1,2,3 in Eq. (C6) are considered similarly.

In the case $n=2$ above, the total result is

$$-\log(-\Box)Y(y)|_{\mathcal{I}^+}=2\frac{A(u)}{r^2}(\log r+\mathbf{c})+\frac{B(u)}{r^2}+O\left(\frac{\log r}{r^3}\right), \tag{C17}$$

where

$$B(u)=B_1(u)+B_2(u), \tag{C18}$$

$$B_1(u)=\int_\infty^0 d\overline{r}\,\overline{r\,Y}\Big|_{\text{path 1}}-\int_0^\infty d\overline{r}\log\overline{r}\frac{d}{d\overline{r}}(\overline{r}^2\overline{Y}|_{\text{path 2}}), \tag{C19}$$

$$B_2(u)=\int_{-\infty}^{u}d\overline{u}\log\left(\frac{u-\overline{u}}{2}\right)\frac{d}{d\overline{u}}A(\overline{u}), \tag{C20}$$

and the next task is obtaining the behavior of the coefficient $B(u)$ as $u\to\infty$.

The analysis of the behavior of $B_1(u)$ at late time essentially repeats the one in Ref. [2]. The dominant contribution to this behavior comes from $Y(y)$ at the limit $y\to i^+$ which in the present case is the limit $r\to\infty$ along the radial timelike geodesic that reaches the future infinity with the energy $E=(1-\gamma^2)^{-1/2}$ per unit rest mass:

$$y\to i^+:\quad u=\frac{1-\gamma}{\gamma}r,\quad r\to\infty. \tag{C21}$$

The variables γ and r may be used as coordinates of the point y:

$$Y(y)=Y(\gamma,r). \tag{C22}$$

Then the definition of the limit i^+ is

$$Y|_{i^+}=Y(\gamma,r\to\infty). \tag{C23}$$

Of interest is the following behavior of Y at i^+ (see Sec. VII):

$$Y|_{i^+}=\frac{\gamma(1-\gamma^2)}{r}Q(\gamma),\quad Q(1)\neq 0, \tag{C24}$$

where $Q(\gamma)$ is some regular function of γ.

The limits i^+ and $\mathcal{I}^+$ are related [2]. For an analytic function, the sequence of limits i^+ and $\gamma\to 1$ coincides with the future of $\mathcal{I}^+$. Hence, using Eq. (C21), one obtains

$$(Y|_{\mathcal{I}^+})_{u\to\infty}=(Y|_{i^+})_{\gamma\to 1}=\frac{2u}{r^2}Q(1). \tag{C25}$$

Therefore, the behavior (C24) implies a linear growth of the coefficient in Eq. (C10) at late time:

$$A(u)|_{u\to\infty}=2uQ(1). \tag{C26}$$

The late-time behavior of $B_1(u)$ in Eq. (C19) is obtained by introducing γ as an integration variable in both integrals and restricting[12] both integrations to the interval $0<\gamma<1$. One obtains

$$B_1(u)|_{u\to\infty}=-u\int_0^1\frac{d\gamma}{(1+\gamma)^2}h_1\left(\gamma,r=\frac{\gamma u}{1+\gamma}\to\infty\right)-u\int_0^1\frac{d\gamma}{(1-\gamma)^2}h_2\left(\gamma,r=\frac{\gamma u}{1-\gamma}\to\infty\right), \tag{C27}$$

where

$$h_1(\gamma,r)=rY(\gamma,r), \tag{C28}$$

$$h_2(\gamma,r)=\log r\left(\frac{\partial}{\partial r}+\frac{\gamma(1-\gamma)}{r}\frac{\partial}{\partial\gamma}\right)r^2Y(\gamma,r). \tag{C29}$$

With the behavior (C24) of Y at i^+ this yields the result

$$B_1(u)|_{u\to\infty}=-u\int_0^1 d\gamma\,\gamma\frac{1-\gamma}{1+\gamma}Q(\gamma)-u\int_0^1 d\gamma\left(\log\frac{u\gamma}{1-\gamma}\right)\frac{\partial}{\partial\gamma}[\gamma^2(1+\gamma)Q(\gamma)]+O(\log u). \tag{C30}$$

The integration by parts brings this expression to the final form

$$B_1(u)|_{u\to\infty}=-2u(\log u-\log 2+2)Q(1)+4u\int_0^1 d\gamma\frac{\gamma^2}{1-\gamma^2}[Q(\gamma)-Q(1)]+O(\log u) \tag{C31}$$

in which the coefficient of the linear growth is convergent. In this way the pole at $\gamma=1$ is eliminated [see Eq. (1.21) and the discussion of this problem in Ref. [2]].

The behavior of $B_2(u)$ at late time is obtained by rewriting Eq. (C20) identically as follows:

$$B_2(u)=\int_{-\infty}^{u_0}d\bar{u}\log\left(\frac{u-\bar{u}}{2}\right)\frac{d}{d\bar{u}}A(\bar{u})+\left(\log\frac{u}{2}\right)[A(u)-A(u_0)]+u\int_{u_0/u}^{1}d\xi\log(1-\xi)g(u\xi), \tag{C32}$$

where $u_0<u$, and

$$g(\bar{u})\equiv\frac{d}{d\bar{u}}A(\bar{u}). \tag{C33}$$

As $u\to\infty$, the first term in Eq. (C32) is $O(\log u)$, and the remaining terms are determined by the behavior (C26). In this way one obtains

$$B_2(u)|_{u\to\infty}=2u\left(\log\frac{u}{2}-1\right)Q(1)+O(\log u). \tag{C34}$$

In the sum (C18) the senior terms $u\log u$ cancel, and the final result is

$$B(u)|_{u\to\infty}=u\left[-6Q(1)+4\int_0^1 d\gamma\frac{\gamma^2}{1-\gamma^2}[Q(\gamma)-Q(1)]\right]+O(\log u). \tag{C35}$$

Taking into account Eqs. (C2) and (C3), one can summarize the calculations above as follows. For any function $X(x)$ in four dimensions that behaves at $\mathcal{I}^+$ as

$$X|_{\mathcal{I}^+}=\frac{A(u,\phi)}{r^n},\quad n<2 \tag{C36}$$

one has

[12]The integration limits $0<\gamma<1$ emerge after one restricts the support of $\bar{Y}$ to the interior of some future light cone $\bar{u}=$const and the exterior of some tube $\bar{r}=$const. The complementary portions of the support of $\bar{Y}$ contribute negligibly as $u\to\infty$ [2].

$$-\frac{1}{\Box}X|_{\mathcal{I}^+}=\frac{1}{2(2-n)}\frac{1}{r^{n-1}}\int_{-\infty}^{u}d\tau A(\tau,\phi)+\frac{\mathcal{O}}{r^{n-1}} \tag{C37}$$

and

$$-\log\left(-\frac{\Box}{m^2}\right)X|_{\mathcal{I}^+}=\frac{A(u,\phi)}{r^n}\left(\log mr+2\mathbf{c}-\log 2-1+\int_0^1\frac{d\xi}{\xi^{n-1}}\frac{1-\xi^n}{1-\xi}\right)+\frac{1}{r^n}\int_{-\infty}^{u}d\tau\log[m(u-\tau)]\frac{\partial}{\partial\tau}A(\tau,\phi)+\frac{\mathcal{O}}{r^n}, \tag{C38}$$

where $\mathcal{O}|_{\mathcal{I}^+}=0$. For a function $X(x)$ that behaves at $\mathcal{I}^+$ as

$$X|_{\mathcal{I}^+}=\frac{A(u,\phi)}{r^2}, \tag{C39}$$

one has

$$-\frac{1}{\Box}X|_{\mathcal{I}^+}=\frac{1}{2}\frac{\log r}{r}\int_{-\infty}^{u}d\tau A(\tau,\phi)+O\left(\frac{1}{r}\right) \tag{C40}$$

and

$$-\log(-\Box)X|_{\mathcal{I}^+}=2\frac{A(u,\phi)}{r^2}(\log r+\mathbf{c})+\frac{B(u,\phi)}{r^2}+O\left(\frac{\log r}{r^3}\right) \tag{C41}$$

with some coefficient $B(u,\phi)$. If in the latter case the function $X(x)$ is a scalar that behaves at i^+ as

$$X|_{i^+}=\frac{\gamma(1-\gamma^2)}{r}Q(\gamma,\phi), \tag{C42}$$

then

$$\int d^2\mathcal{S}(\phi)B(u,\phi)|_{u\to\infty}=u\int d^2\mathcal{S}(\phi)\left[-6Q(1,\phi)+4\int_0^1 d\gamma\frac{\gamma^2}{1-\gamma^2}[Q(\gamma,\phi)-Q(1,\phi)]\right]+O(\log u). \tag{C43}$$

Note that the term with $\log r$ in Eq. (C41) is doubled as compared to Eq. (C38).

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