Second-order rotational effects on the *r*-modes of neutron stars

Lee Lindblom
Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena, California 91125

Gregory Mendell
Department of Physics and Astronomy, University of Wyoming, Laramie, Wyoming 82071

Benjamin J. Owen
Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena, California 91125
and Albert Einstein Institut, Max Planck Institut für Gravitationsphysik, Am Mühlenberg 5, D-14476 Golm, Germany

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Techniques are developed here for evaluating the *r*-modes of rotating neutron stars through second order in the angular velocity of the star. Second-order corrections to the frequencies and eigenfunctions for these modes are evaluated for neutron star models. The second-order eigenfunctions for these modes are determined here by solving an unusual inhomogeneous hyperbolic boundary-value problem. The numerical techniques developed to solve this unusual problem are somewhat non-standard and may well be of interest beyond the particular application here. The bulk-viscosity coupling to the *r*-modes, which appears first at second order, is evaluated. The bulk-viscosity time scales are found here to be longer than previous estimates for normal neutron stars, but shorter than previous estimates for strange stars. These new time scales do not substantially affect the current picture of the gravitational radiation driven instability of the *r*-modes either for neutron stars or for strange stars. [S0556-2821(99)01518-0]

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I. INTRODUCTION

Recently the *r*-modes have been found to play an interesting and important role in the evolution of hot young rapidly rotating neutron stars. Andersson [1] and Friedman and Morsink [2] were the first to show that gravitational radiation tends to drive the *r*-modes unstable in all rotating stars. Lindblom, Owen, and Morsink [3] then showed that the coupling of gravitational radiation to the *r*-modes is sufficiently strong to overcome internal fluid dissipation effects and so drive these modes unstable in hot young neutron stars. This result has been verified by Andersson, Kokkotas, and Schutz [4]. This result seemed somewhat surprising at first because the dominant coupling of gravitational radiation to the *r*-modes is through the current multipoles rather than the more familiar and usually dominant mass multipoles. But it is now generally accepted that gravitational radiation does drive unstable any hot young neutron star with angular velocity greater than about 5% of the maximum (the angular velocity where mass shedding occurs). This instability therefore provides a natural explanation for the lack of observed very fast pulsars associated with young supernova remnants.

The *r*-mode instability is also interesting as a possible source of gravitational radiation. In the first few minutes after the formation of a hot young rapidly rotating neutron star in a supernova, gravitational radiation will increase the amplitude of the *r*-mode (with spherical harmonic index $m=2$) to levels where non-linear hydrodynamic effects become important in determining its subsequent evolution. While the non-linear evolution of these modes is not well understood as yet, Owen *et al.* [5] have developed a simple non-linear evolution model to describe it approximately. This model predicts that within about one year the neutron star spins down (and cools down) to an angular velocity (and temperature) low enough that the instability is again suppressed by internal fluid dissipation. All of the excess angular momentum of the neutron star is radiated away via gravitational radiation. Owen *et al.* [5] estimate the detectability of the gravitational waves emitted during this spindown, and find that neutron stars spinning down in this manner may be detectable by the second-generation (''enhanced'') LIGO interferometers out to the Virgo cluster. Bildsten [6] and Andersson, Kokkotas, and Stergioulas [7] have raised the possibility that the *r*-mode instability may also operate in older colder neutron stars spun up by accretion in low-mass x-ray binaries. The gravitational waves emitted by some of these systems (e.g. Sco X-1) may also be detectable by enhanced LIGO [8]. Thus, the *r*-modes of rapidly rotating neutron stars have become a topic of considerable interest in relativistic astrophysics.

The purpose of this paper is to explore further the properties of the *r*-modes of rotating neutron stars. The initial analyses of the *r*-mode instability [1–3] were based on a small angular-velocity expansion for these modes developed originally by Papaloizou and Pringle [9]. This expansion in powers of the angular velocity kept only the lowest-order terms in the expressions for the various quantities associated with the mode: the frequency, velocity perturbation, etc. This lowest-order expansion is sufficient to explore many of the interesting physical properties of these modes, including the gravitational radiation instability. However, some important physical quantities vanish at lowest order and hence a second-order analysis is needed [10]. For example the coupling of the *r*-modes to bulk viscosity vanishes in the lowest-order expansion. Estimates of this important bulk-viscosity coupling to the *r*-modes have been given by Lindblom, Owen, and Morsink [3], Andersson, Kokkotas, and Schutz [4,11], and Kokkotas and Stergioulas [12]. But (as discussed in more detail in Sec. VI below) none of these is based on the fully self-consistent second-order calculation needed to evaluate this coupling properly. Since bulk viscosity is expected to be the dominant internal fluid dissipation mechanism in hot young neutron stars, it is important to extend the analysis so that this important physical effect can be evaluated accurately.

The dominant internal fluid dissipation mechanism in neutron stars colder than about 10^9 K is thought to be a superfluid effect called mutual friction [13] caused by the scattering of electrons off the magnetic fields in the cores of vortices. Levin [14] has shown that the importance of the r-mode instability in low-mass x-ray binaries depends crucially on the details of the mutual friction damping of these modes. Unfortunately the mutual friction dissipation also vanishes at lowest order in a small angular velocity expansion of the superfluid r-modes. Thus in order to evaluate this effect properly, it is also necessary to determine the structure of the r-modes of superfluid neutron stars through second order in the angular velocity. This provides another motivation then for developing the tools needed to evaluate the second-order rotational effects in the r-modes.

In this paper we develop a new formalism for exploring the higher-order rotational effects in the r-modes. Our analysis is based on the two-potential formalism [15] in which all physical properties of a mode of a rotating star are expressed in terms of two scalar potentials: a hydrodynamic potential δU and the gravitational potential $\delta\Phi$. We define a small angular velocity expansion for the r-modes in terms of these potentials, and derive the equations explicitly for the second-order terms. This expansion provides a straightforward and relatively simple way to determine the second-order effects, such as the bulk viscosity coupling, which are of interest to us here. The equations that determine the second-order terms in the r-modes form an inhomogeneous hyperbolic boundary value problem that is not amenable to solution by standard numerical techniques. Therefore we have developed new numerical techniques which could well have applications beyond the present problem. In particular these techniques will also be needed to solve the analogous superfluid pulsation equations that determine the effects of mutual friction on these modes.

The time scales derived here for the bulk viscosity damping of the r-modes differ considerably from earlier estimates. We find the bulk-viscosity coupling to these modes to be weaker for normal neutron stars than any previous estimates. Consequently the gravitational radiation-driven instability is somewhat more effective at driving unstable the r-modes in hot young neutron stars than earlier estimates suggested. Although quantitatively different from earlier estimates, our new values for the bulk-viscosity damping time do not substantially alter the expected spindown scenario in hot young neutron stars. We re-evaluate the critical angular velocity curve (above which the r-mode instability sets in) and find no qualitative change from earlier estimates. Our new value for the minimum critical angular velocity is somewhat lower than earlier estimates: about 5% compared to about 8% of the maximum. In very hot young neutron stars there is the possibility that bulk viscosity could re-heat the neutron star (due in part to non-linear effects in the bulk viscosity) and so suppress the instability to some extent [16]. This could result in a significant increase in the time scale required to spin down young neutron stars, and could therefore decrease significantly the detectability of the gravitational radiation emitted. Our new calculation of the bulk viscosity time scale indicates that reheating will not be a major factor in the evolution of young neutron stars. Our calculations also show that the bulk-viscosity coupling in strange stars is somewhat stronger than the initial estimates by Madsen [17]. We find that bulk viscosity completely suppresses the r-mode instability in strange stars hotter than $T \gtrsim 5\times 10^8$ K, in good qualitative agreement with Madsen.

In Sec. II we review the structure of equilibrium stellar models through second order in the angular velocity of the star. In Sec. III we review the two-potential formalism for describing the modes of rotating stars, and derive the small angular velocity expansion of these equations through second order. In Sec. IV we focus our attention on the ''classical'' r-modes, the modes found previously to be subject to the gravitational radiation driven instability. We obtain analytical expressions for the second-order corrections to the frequencies of these modes, and present numerical results for polytropes and for more realistic neutron star models. In Sec. V we develop the numerical techniques needed to find the second-order eigenfunctions for the r-modes; we use those techniques to find those eigenfunctions, and we present the results graphically. In Sec. VI we use our new second-order expressions for the r-modes to compute the effects of bulk viscosity on the evolution of these modes. In the Appendix we discuss the convergence of the numerical relaxation technique used in Sec. V to solve the unusual hyperbolic boundary value problem for the second-order eigenfunctions.

II. SLOWLY ROTATING STELLAR MODELS

Our analysis of the r-modes of rotating stars is based on expanding the equations as power series in the angular velocity Ω of the star. The first step therefore in obtaining these equations is to find the structures of equilibrium stellar models in a similar power series expansion. This section describes how to solve the equilibrium structure equations for uniformly rotating barotropic stars in such a slow rotation expansion. The solutions will be obtained here up to and including the terms of order Ω^2.

Let $h(p)$ denote the thermodynamic enthalpy of the barotropic fluid:

$$h(p)=\int_0^p \frac{dp'}{\rho(p')}, \tag{2.1}$$

where p is the pressure and ρ is the density of the fluid. This definition can always be inverted to determine $p(h)$. The barotropic equation of state, $\rho=\rho(p)$, then determines $\rho(h)=\rho[p(h)]$. The equations which determine the family of stationary, axisymmetric uniformly rotating barotropic stellar models are Euler's equation, which for this case has the simple form

$$0=\nabla_a[h-\tfrac{1}{2}r^2(1-\mu^2)\Omega^2-\Phi], \tag{2.2}$$

and the gravitational potential equation,

$$\nabla^a\nabla_a\Phi=-4\pi G\rho. \tag{2.3}$$

In these expressions r and $\mu=\cos\theta$ are the standard spherical coordinates, and Φ is the gravitational potential.

We seek solutions to Eqs. (2.2) and (2.3) as power series in the angular velocity Ω. To that end, we define

$$h(r,\mu)=h_0(r)+h_2(r,\mu)\,\frac{\Omega^2}{\pi G\overline{\rho_0}}+\mathcal{O}(\Omega^4), \tag{2.4}$$

$$\rho(r,\mu)=\rho_0(r)+\rho_2(r,\mu)\,\frac{\Omega^2}{\pi G\overline{\rho_0}}+\mathcal{O}(\Omega^4), \tag{2.5}$$

$$\Phi(r,\mu)=\Phi_0(r)+\Phi_2(r,\mu)\,\frac{\Omega^2}{\pi G\overline{\rho_0}}+\mathcal{O}(\Omega^4), \tag{2.6}$$

where $\overline{\rho_0}$ is the average density of the non-rotating star in the family. Using these expressions then, the first two terms in the solution to Eq. (2.2) are given by

$$C_0=h_0(r)-\Phi_0(r), \tag{2.7}$$

$$C_2=h_2(r,\mu)-\tfrac{1}{2}\pi G\overline{\rho_0}r^2(1-\mu^2)-\Phi_2(r,\mu), \tag{2.8}$$

where C_0 and C_2 are constants. The non-rotating model can be determined in the usual way by solving the gravitational potential equation,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_0}{dr}\right)=-4\pi G\rho_0, \tag{2.9}$$

together with Eq. (2.7). The integration constant, C_0, can be shown to be $C_0=-GM_0/R_0$ by evaluating Eq. (2.7) at the surface of the star. The constants M_0 and R_0 are the mass and radius of the non-rotating star.

The second-order contributions to the stellar structure are determined by solving the gravitational potential, Eq. (2.3), together with Eq. (2.8). The second-order density perturbation ρ_2 is related to h_2 by

$$\rho_2(r,\mu)=\left(\frac{d\rho}{dh}\right)_0 h_2(r,\mu). \tag{2.10}$$

Thus using Eq. (2.8), the equation for the second-order gravitational potential can be written in the form

$$\begin{aligned}\nabla^a\nabla_a\Phi_2+4\pi G\left(\frac{d\rho}{dh}\right)_0\Phi_2\\ =-4\pi G\left(\frac{d\rho}{dh}\right)_0\{C_2+\tfrac{1}{3}\pi G\overline{\rho_0}r^2[1-P_2(\mu)]\},\end{aligned} \tag{2.11}$$

where $P_2(\mu)=\frac{1}{2}(3\mu^2-1)$. We note that the right side of Eq. (2.11) splits into a function depending only on r plus a function of r multiplied by $P_2(\mu)$. Since the operator on the left side of Eq. (2.11) acting on $P_2(\mu)$ gives a function of r multiplied by $P_2(\mu)$, it follows that the second-order gravitational potential Φ_2 must have a similar splitting:

$$\Phi_2(r,\mu)=\Phi_{20}(r)+\Phi_{22}(r)P_2(\mu). \tag{2.12}$$

Thus the partial differential equation (2.11) for Φ_2 reduces to a pair of ordinary differential equations for the potentials Φ_{20} and Φ_{22}:

$$\begin{aligned}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_{20}}{dr}\right)+4\pi G\left(\frac{d\rho}{dh}\right)_0\Phi_{20}\\ =-4\pi G\left(\frac{d\rho}{dh}\right)_0(C_2+\tfrac{1}{3}r^2\pi G\overline{\rho_0}),\end{aligned} \tag{2.13}$$

$$\begin{aligned}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_{22}}{dr}\right)-\frac{6}{r^2}\Phi_{22}+4\pi G\left(\frac{d\rho}{dh}\right)_0\Phi_{22}\\ =\tfrac{4}{3}\pi^2G^2\overline{\rho_0}r^2\left(\frac{d\rho}{dh}\right)_0.\end{aligned} \tag{2.14}$$

Appropriate boundary conditions are needed to select the unique physically relevant solutions to Eqs. (2.13) and (2.14). In order to ensure that the gravitational potential is non-singular at the center of the star, $r=0$, we must require that Φ_{20} and Φ_{22} satisfy the following boundary conditions there:

$$0=\left(\frac{d\Phi_{20}}{dr}\right)_{r=0}=\Phi_{22}(0). \tag{2.15}$$

The potential Φ_{22} must also fall to zero as $r\to\infty$. We can ensure this by requiring that Φ_{22} match smoothly at the surface of the star to a potential that in the exterior of the star is proportional to $P_2(\mu)/r^3$. It is sufficient therefore to require that Φ_{22} satisfy the condition

$$\left(\frac{d\Phi_{22}}{dr}\right)_{r=R_0}=-\frac{3\Phi_{22}(R_0)}{R_0}. \tag{2.16}$$

An additional condition is also needed to fix Φ_{20}. It is customary to consider families of rotating stars which have the same total mass. In this case the monopole part of the exterior gravitational potential is the same for all members of the family. To ensure this, we must require that the potential Φ_{20} and its derivative vanish on the surface of the star:

$$0=\Phi_{20}(R_0)=\left(\frac{d\Phi_{20}}{dr}\right)_{r=R_0}. \tag{2.17}$$

It might appear that Eqs. (2.15) and (2.17) now over constrain the potential Φ_{20}. This would be the case, except that the constant C_2 that appears on the right side of Eq. (2.13) is still undetermined. The boundary conditions, Eqs. (2.15)–(2.17), are just sufficient, however, to fix uniquely the potentials Φ_{20} and Φ_{22} together with the integration constant C_2 as solutions to Eqs. (2.13) and (2.14). We also note that these boundary conditions ensure that

$$0=\int_{-1}^{1}\int_{0}^{R_0} r^2\rho_2(r,\mu)\,dr\,d\mu. \tag{2.18}$$

In summary, then, the thermodynamic functions $h(r,\mu)$ and $\rho(r,\mu)$ in slowly rotating barotropic stars are given by Eqs. (2.4) and (2.5), where $\rho_2(r,\mu)$ and $h_2(r,\mu)$ are given by

$$\begin{aligned}\rho_2(r,\mu)&=\left(\frac{d\rho}{dh}\right)_0 h_2(r,\mu)\\&=\left(\frac{d\rho}{dh}\right)_0\{C_2+\Phi_{20}(r)+\tfrac{1}{3}\pi G\overline{\rho}_0 r^2\\&\quad+[\Phi_{22}(r)-\tfrac{1}{3}\pi G\overline{\rho}_0 r^2]P_2(\mu)\}.\end{aligned} \tag{2.19}$$

These expressions for $h(r,\mu)$ and $\rho(r,\mu)$ depend only on the structures of the non-rotating star through the functions $h_0(r)$, $\rho_0(r)$, and $(d\rho/dh)_0$, the potentials Φ_{20} and Φ_{22} from Eqs. (2.13) and (2.14), and the constant C_2.

It is also instructive to work out an expression for the surface $r=R(\mu,\Omega)$ of the rotating star. This surface occurs where the thermodynamic potential $h[R(\mu,\Omega),\mu]=0$. Solving this equation, we find

$$R(\mu,\Omega)=R_0+R_2(\mu)\frac{\Omega^2}{\pi G\overline{\rho}_0}+\mathcal{O}(\Omega^4), \tag{2.20}$$

where $R_2(\mu)$ is given by

$$\begin{aligned}R_2(\mu)&=R_{20}+R_{22}P_2(\mu)\\&=\frac{3}{4\pi G\overline{\rho}_0 R_0}\{C_2+\tfrac{1}{3}\pi G\overline{\rho}_0 R_0^2\\&\quad+[\Phi_{22}(R_0)-\tfrac{1}{3}\pi G\overline{\rho}_0 R_0^2]P_2(\mu)\}.\end{aligned} \tag{2.21}$$

We have developed a computer code that solves these equations numerically for stars with an arbitrary equation of state. We have tested this code against analytical expressions which can be obtained for a polytropic neutron star equation of state, $p=K\rho^2$, with K chosen so that a $1.4M_\odot$ model has a radius of $R_0=12.533$ km. We find that the constants that determine the slowly rotating model for this polytropic case have the values $C_2=.09802C_0$, $R_{20}=.15198R_0$, and $R_{22}=-.37995R_0$. Our numerical results agree with the analytical to floating-point precision.

III. THE PULSATION EQUATIONS

The modes of any uniformly rotating barotropic stellar model can be described completely in terms of two scalar potentials δU and $\delta\Phi$ [15]. The potential $\delta\Phi$ is the Newtonian gravitational potential, while δU determines the hydrodynamic perturbation of the star:

$$\delta U=\frac{\delta p}{\rho}-\delta\Phi, \tag{3.1}$$

where δp is the Eulerian pressure perturbation, and ρ is the unperturbed density of the equilibrium stellar model. We assume here that the time dependence of the mode is $e^{i\omega t}$ and that its azimuthal angular dependence is $e^{im\varphi}$, where ω is the frequency of the mode and m is an integer. The velocity perturbation δv^a is determined in this case by

$$\delta v^a=iQ^{ab}\nabla_b\delta U. \tag{3.2}$$

The tensor Q^{ab} depends on the frequency of the mode and the angular velocity of the equilibrium star:

$$Q^{ab}=\frac{1}{(\omega+m\Omega)^2-4\Omega^2}\times\left[(\omega+m\Omega)\delta^{ab}-\frac{4\Omega^2}{\omega+m\Omega}z^az^b-2i\nabla^a v^b\right]. \tag{3.3}$$

In Eq. (3.3) the unit vector z^a points along the rotation axis of the equilibrium star, δ^{ab} is the Euclidean metric tensor (the identity matrix in Cartesian coordinates), and v^a is the velocity of the equilibrium stellar model.

In general, the potentials δU and $\delta\Phi$ are solutions of the following system of equations [15]:

$$\nabla_a(\rho Q^{ab}\nabla_b\delta U)=-(\omega+m\Omega)\frac{d\rho}{dh}(\delta U+\delta\Phi), \tag{3.4}$$

$$\nabla^a\nabla_a\delta\Phi=-4\pi G\frac{d\rho}{dh}(\delta U+\delta\Phi), \tag{3.5}$$

subject to the appropriate boundary conditions at the surface of the star for δU and at infinity for $\delta\Phi$. In order to discuss these boundary conditions in more detail we let Σ denote a function that vanishes on the surface of the star, and which has been normalized so that its gradient, $n_a=\nabla_a\Sigma$, is the outward directed unit normal vector there, $n^an_a=1$. The boundary condition on the function δU at the surface of the star, $\Sigma=0$, is to require that the Lagrangian perturbation in the enthalpy h vanish there, $\Delta h=0$. This condition can be written in terms of the variables used here by noting that

$$\Delta h=\delta h+\left(\frac{\delta v^a}{i\kappa\Omega}\right)\nabla_a h, \tag{3.6}$$

where κ is related to the frequency of the mode by

$$\kappa\Omega = \omega + m\Omega. \tag{3.7}$$

Thus using Eqs. (3.1) and (3.2) the boundary condition can be written in terms of δU and $\delta\Phi$ as

$$0 = [\kappa\Omega(\delta U + \delta\Phi) + Q^{ab}\nabla_a h \nabla_b \delta U]_{\Sigma\uparrow 0}. \tag{3.8}$$

The perturbed gravitational potential $\delta\Phi$ must vanish at infinity, $\lim_{r\to\infty}\delta\Phi = 0$. In addition $\delta\Phi$ and its first derivative must be continuous at the surface of the star. The problem of finding the modes of uniformly rotating barotropic stars is reduced therefore to finding the solutions to Eqs. (3.4) and (3.5) subject to the boundary condition in Eq. (3.8).

The equation for the hydrodynamic potential δU, Eq. (3.4), has a complicated dependence on the frequency of the mode and the angular velocity of the star through Q^{ab} as given in Eq. (3.3). In the analysis that follows it will be necessary to have those dependences displayed more explicitly. To that end, we re-write Eq. (3.4) and the boundary condition, Eq. (3.8), in the following equivalent forms:

$$\nabla^a[\rho(\kappa^2\delta^{ab} - 4z^a z^b)\nabla_b \delta U] + \frac{2m\kappa}{\varpi}\varpi^a\nabla_a\rho\ \ \delta U$$
$$= -\kappa^2(\kappa^2-4)\Omega^2\frac{d\rho}{dh}(\delta U + \delta\Phi), \tag{3.9}$$

$$\left[(\kappa^2\delta^{ab} - 4z^a z^b)\nabla_a h \nabla_b \delta U + \frac{2m\kappa}{\varpi}\varpi^a\nabla_a h\ \ \delta U + \kappa^2(\kappa^2-4)\Omega^2(\delta U + \delta\Phi)\right]_{\Sigma\uparrow 0}$$
$$= 0. \tag{3.10}$$

Here we use the notation ϖ for the cylindrical radial coordinate, $\varpi = r\sqrt{1-\mu^2}$, and ϖ^a to denote the unit vector in the ϖ direction.

Our purpose now will be to derive solutions to Eqs. (3.5) and (3.9) as power series in the angular velocity of the star. To that end we define the expansions of the potentials δU and $\delta\Phi$ as

$$\delta U = R_0^2\Omega^2\left[\delta U_0 + \delta U_2\frac{\Omega^2}{\pi G\bar{\rho}_0} + \mathcal{O}(\Omega^4)\right], \tag{3.11}$$

$$\delta\Phi = R_0^2\Omega^2\left[\delta\Phi_0 + \delta\Phi_2\frac{\Omega^2}{\pi G\bar{\rho}_0} + \mathcal{O}(\Omega^4)\right]. \tag{3.12}$$

The normalizations of δU and $\delta\Phi$ have been chosen to make the δU_i and $\delta\Phi_i$ dimensionless under the assumption that the lowest order terms scale as Ω^2. Here we have limited our consideration to the generalized r-modes [18]: modes which are dominated by rotational effects and whose frequencies vanish linearly therefore in the angular velocity of the star. In this case κ [as defined in Eq. (3.7)] is finite in the small angular velocity limit, and so we may expand

$$\kappa = \kappa_0 + \kappa_2\frac{\Omega^2}{\pi G\bar{\rho}_0} + \mathcal{O}(\Omega^4). \tag{3.13}$$

Using these expansions, together with those for the structure of the equilibrium star from Eqs. (2.4) and (2.5), it is straightforward to write down order by order the equations for the mode. The lowest order terms in the expansions of Eqs. (3.9) and (3.5) are the following,

$$\nabla^a[\rho_0(\kappa_0^2\delta^{ab} - 4z^a z^b)\nabla_b\delta U_0] + \frac{2m\kappa_0}{\varpi}\varpi^a\nabla_a\rho_0\ \ \delta U_0 = 0, \tag{3.14}$$

$$\nabla^a\nabla_a\delta\Phi_0 = -4\pi G\left(\frac{d\rho}{dh}\right)_0(\delta U_0 + \delta\Phi_0). \tag{3.15}$$

Similarly, the lowest order term in the expansion of the boundary condition is

$$\left[(\kappa_0^2\delta^{ab} - 4z^a z^b)\nabla_a h_0\nabla_b\delta U_0 + \frac{2m\kappa_0}{\varpi}\varpi^a\nabla_a h_0\ \ \delta U_0\right]_{r=R_0}$$
$$= 0. \tag{3.16}$$

Continuing on to second order, the equations for the potentials are

$$\nabla_a[\rho_0(\kappa_0^2\delta^{ab} - 4z^a z^b)\nabla_b\delta U_2] + \frac{2m\kappa_0}{\varpi}\varpi^a\nabla_a\rho_0\ \ \delta U_2$$
$$+ \nabla_a[\rho_2(\kappa_0^2\delta^{ab} - 4z^a z^b)\nabla_b\delta U_0 + 2\kappa_0\kappa_2\rho_0\nabla^a\delta U_0]$$
$$+ \frac{2m}{\varpi}\varpi^a(\kappa_2\nabla_a\rho_0 + \kappa_0\nabla_a\rho_2)\delta U_0$$
$$= -\kappa_0^2(\kappa_0^2-4)\pi G\bar{\rho}_0\left(\frac{d\rho}{dh}\right)_0(\delta U_0 + \delta\Phi_0), \tag{3.17}$$

$$\nabla^a\nabla_a\delta\Phi_2 = -4\pi G\left(\frac{d\rho}{dh}\right)_0(\delta U_2 + \delta\Phi_2)$$
$$-4\pi G\left(\frac{d^2\rho}{dh^2}\right)_0 h_2(\delta U_0 + \delta\Phi_0). \tag{3.18}$$

The second-order boundary condition is somewhat more complicated; it must include two types of terms. The first type comes from the second-order terms in the expansion of Eq. (3.8) in powers of the angular velocity. The second type comes from the fact that the boundary condition is to be imposed on the actual surface of the rotating star, not the surface $r = R_0$. This second type of term is the correction to the lowest-order boundary condition, Eq. (3.16), needed to impose it on the actual boundary of the star (to second order in the angular velocity). Hence the terms of the second type are proportional to R_2, the second-order change in the radius of the star from Eq. (2.21). The resulting boundary condition is

$$\Big\{(\kappa_0^2\delta^{ab}-4z^az^b)\nabla_a h_0\nabla_b\delta U_2+\frac{2m\kappa_0}{\varpi}\varpi^a\nabla_a h_0\delta U_2+(\kappa_0^2\delta^{ab}-4z^az^b)\nabla_a h_2\nabla_b\delta U_0+\frac{2m\kappa_0}{\varpi}\varpi^a\nabla_a h_2\delta U_0$$
$$+2\kappa_0\kappa_2\nabla^a h_0\nabla_a\delta U_0+\frac{2m\kappa_2}{\varpi}\varpi^a\nabla_a h_0\delta U_0+\kappa_0^2(\kappa_0^2-4)\pi G\overline{\rho_0}(\delta U_0+\delta\Phi_0)$$
$$+R_2 r^c\nabla_c\left[(\kappa_0^2\delta^{ab}-4z^az^b)\nabla_a h_0\nabla_b\delta U_0+\frac{2m\kappa_0}{\varpi}\delta U_0\varpi^a\nabla_a h_0\right]\Big\}_{r=R_0}=0. \tag{3.19}$$

In summary, then, Eqs. (3.17) and (3.18) together with the boundary condition, Eq. (3.19), determine the second-order terms in the structure of any generalized r-mode.

IV. CLASSICAL r-MODES

There exists a large class of modes in rotating barotropic stellar models whose properties are determined primarily by the rotation of the star [18,19]. We refer to these as generalized r-modes. In this section we restrict our attention however to those modes which contribute primarily to the gravitational radiation driven instability. These ''classical'' r-modes (which were studied first by Papaloizou and Pringle [9]) are generated by hydrodynamic potentials of the form (see e.g. Lindblom and Ipser [18])

$$\delta U_0=\alpha\left(\frac{r}{R_0}\right)^{m+1}P^m_{m+1}(\mu)e^{im\varphi}. \tag{4.1}$$

It is straightforward to verify that this δU_0 is a solution to Eq. (3.14) if the eigenvalue κ_0 has the value

$$\kappa_0=\frac{2}{m+1}. \tag{4.2}$$

This δU_0 and κ_0 also satisfy the boundary condition, Eq. (3.16), without further restriction at the boundary (and at every point within the star as well). The gravitational potential $\delta\Phi_0$ must have the same angular dependence as δU_0. Thus, $\delta\Phi_0$ must (through a slight abuse of notation) have the form

$$\delta\Phi_0=\alpha\,\delta\Phi_0(r)P^m_{m+1}(\mu)e^{im\varphi}. \tag{4.3}$$

The gravitational potential, Eq. (3.15), reduces to an ordinary differential equation then for $\delta\Phi_0(r)$:

$$\frac{d^2\delta\Phi_0}{dr^2}+\frac{2}{r}\frac{d\delta\Phi_0}{dr}+\left[4\pi G\left(\frac{d\rho}{dh}\right)_0-\frac{(m+1)(m+2)}{r^2}\right]\delta\Phi_0$$
$$=-4\pi G\left(\frac{d\rho}{dh}\right)_0\left(\frac{r}{R_0}\right)^{m+1}. \tag{4.4}$$

Once δU_0 and $\delta\Phi_0$ are known, it is straightforward to evaluate the perturbations in other thermodynamic quantities to this order. For example $\delta p_0=\rho_0\delta h_0=\rho_0(\delta U_0+\delta\Phi_0)$. And it is straightforward to evaluate then the velocity perturbation to this order using Eq. (3.2).

We next consider the second-order contributions to the r-modes. First, let us analyze the second-order equation for the potential δU, Eq. (3.17). This equation contains two types of terms: those proportional to δU_2 and those that are not. We will consider those terms not proportional to δU_2 as source terms, and we evaluate them now. It is convenient to break these source terms into three groups. The first group is proportional to ρ_2. These terms can be simplified by recalling that δU_0 satisfies Eq. (3.14) for *any* spherically symmetric density distribution. Then, using the fact from Eq. (2.19) that $\rho_2(r,\mu)=\rho_{20}(r)+\rho_{22}(r)P_2(\mu)$, we find

$$\nabla_a[\rho_2(\kappa_0^2\delta^{ab}-4z^az^b)\nabla_b\delta U_0]+\frac{2m\kappa_0}{\varpi}\varpi^a\nabla_a\rho_2\delta U_0$$
$$=-\frac{12m(m+2)}{(m+1)^2}\frac{\rho_{22}}{r^2}\delta U_0. \tag{4.5}$$

The second group of terms is proportional to κ_2. These terms have the following simplified form:

$$\nabla^a(2\kappa_0\kappa_2\rho_0\nabla_a\delta U_0)+\frac{2m\kappa_2}{\varpi}\varpi^a\nabla_a\rho_0\delta U_0$$
$$=2(m+2)\kappa_2\frac{1}{r}\frac{d\rho_0}{dr}\delta U_0. \tag{4.6}$$

Thus, combining together these terms with those on the right side of Eq. (3.17), we obtain the following expression for the equation that determines δU_2 for the classical r-modes:

$$\nabla_a\left\{\rho_0\left[\frac{4\delta^{ab}}{(m+1)^2}-4z^az^b\right]\nabla_b\delta U_2\right\}+\frac{4m\varpi^a\nabla_a\rho_0}{(m+1)\varpi}\delta U_2$$
$$=\frac{12m(m+2)}{(m+1)^2}\frac{\rho_{22}}{r^2}\delta U_0-2(m+2)\kappa_2\frac{1}{r}\frac{d\rho_0}{dr}\delta U_0$$
$$+16\pi G\overline{\rho_0}\frac{m(m+2)}{(m+1)^4}\left(\frac{d\rho}{dh}\right)_0(\delta U_0+\delta\Phi_0). \tag{4.7}$$

A similar reduction can also be made on the second-order boundary condition, Eq. (2.19). We collect similar terms together to obtain the following simplifications:

$$(\kappa_0^2\delta^{ab}-4z^az^b)\nabla_a h_2\nabla_b\delta U_0+\frac{2m\kappa_0}{\varpi}\varpi^a\nabla_a h_2\ \delta U_0 = -\frac{12m(m+2)}{(m+1)^2}\frac{h_{22}}{r^2}\delta U_0, \tag{4.8}$$

$$R_2 r^c\nabla_c\left[(\kappa_0^2\delta^{ab}-4z^az^b)\nabla_a h_0\nabla_b\delta U_0+\frac{2m\kappa_0}{\varpi}\delta U_0\varpi^a\nabla_a h_0\right] = 0. \tag{4.9}$$

The latter follows from the fact that the expression in Eq. (3.16) is zero everywhere if δU_0 is given by Eq. (4.1). Combining these simplified expressions together gives the following form for the boundary condition that constrains δU_2:

$$\left\{\left[\frac{4\delta^{ab}}{(m+1)^2}-4z^az^b\right]\nabla_a h_0\nabla_b\delta U_2+\frac{4m\varpi^a\nabla_a h_0}{(m+1)\varpi}\delta U_2 - \frac{12m(m+2)}{(m+1)^2}\frac{h_{22}}{r^2}\delta U_0+2(m+2)\kappa_2\frac{1}{r}\frac{dh_0}{dr}\delta U_0 - \frac{16m(m+2)}{(m+1)^4}\pi G\bar{\rho}_0(\delta U_0+\delta\Phi_0)\right\}_{r=R_0}=0. \tag{4.10}$$

We note that the operator on the left side of Eq. (4.7) which acts on δU_2 is identical to the operator that acts on δU_0 from the lowest-order equation (3.14). We also note that the right side of Eq. (4.7) is a function of r multiplied by the angular function $P^m_{m+1}(\mu)e^{im\varphi}$. These facts allow us to derive a simple formula for the second-order eigenvalue κ_2 in terms of known quantities. Multiply the left side of Eq. (4.7) by δU_0^* and integrate over the interior of the star. This integral vanishes because this operator is symmetric and δU_0^* also satisfies Eq. (3.14). This implies that the integral of δU_0^* multiplied by the right side of Eq. (4.7) also vanishes. This integral gives the following expression for the eigenvalue κ_2 once the angular integrals are performed:

$$\kappa_2\int_0^{R_0}\left(\frac{r}{R_0}\right)^{2m+2} r\frac{d\rho_0}{dr}dr = \frac{6m}{(m+1)^2}\int_0^{R_0}\rho_{22}\left(\frac{r}{R_0}\right)^{2m+2}dr+\frac{8\pi G\bar{\rho}_0 m}{(m+1)^4} \times\int_0^{R_0}r^2\left(\frac{r}{R_0}\right)^{m+1}\left[\left(\frac{r}{R_0}\right)^{m+1}+\delta\Phi_0(r)\right]\left(\frac{d\rho}{dh}\right)_0 dr. \tag{4.11}$$

We have evaluated Eq. (4.11) numerically to determine κ_2 for a variety of equations of state. Table I presents the values of κ_2 for the classical r-modes with $2\leqslant m\leqslant 6$ of stars with polytropic equations of state. We also present in Fig. 1 a graph of the frequency $\omega/\Omega=\kappa-2$ of the $m=2$ classical r-modes computed for $1.4M_\odot$ stellar models based on seven realistic equations of state [20]. The dashed line in Fig. 1 corresponds to the lowest-order approximation of the r-mode frequency $\omega/\Omega=\kappa_0-2$, which is the same for any equation of state. The solid curves are based on the second-order formula $\omega/\Omega=\kappa_0-2+\kappa_2\Omega^2/\pi G\bar{\rho}_0$. It is interesting to see in Fig. 1 that the higher-order terms make only small (up to about 12% at the highest angular velocities) corrections to the frequencies of these modes for stars with realistic equations of state. We also note that the general tendency of the frequency of these modes to be smaller than that predicted by the lowest-order expression is consistent with the results found by Lindblom and Ipser [18] for the Maclaurin spheroids. An analytical expression for κ_2 can be obtained from Eq. (4.11) for the uniform density case by performing the indicated integrals analytically. The resulting expression is equivalent to Eq. (6.10) of Lindblom and Ipser [18].

TABLE I. The second-order eigenvalues κ_2 of the classical r-modes for stars with polytropic equations of state $p=K\rho^{1+1/n}$.

n	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
0.0	.57407	.59766	.54720	.49074	.44044
0.5	.41718	.43861	.40415	.36406	.32782
1.0	.29883	.32054	.29946	.27250	.24729
1.5	.21183	.23426	.22369	.20693	.19019
2.0	.14777	.17084	.16846	.15961	.14942
2.5	.10091	.12426	.12808	.12532	.12016
3.0	.06716	.09024	.09859	.10039	.09905
3.5	.04334	.06556	.07699	.08210	.08357
4.0	.02692	.04773	.06102	.06839	.07186
4.5	.01589	.03487	.04896	.05768	.06252

V. NUMERICAL SOLUTIONS FOR δU_2

In this section we discuss the numerical solution of the equations that determine the second-order corrections to the

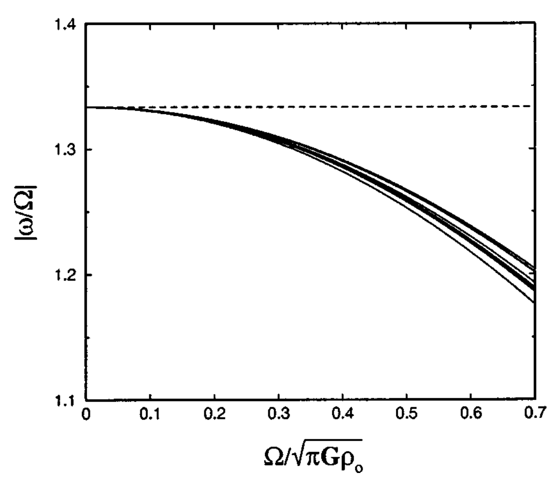

FIG. 1. Angular velocity dependence of the frequencies of the classical $m=2$ r-modes for $1.4M_\odot$ stellar models based on seven realistic neutron star equations of state. The dashed curve is based on the lowest-order expression for $\omega/\Omega=\kappa_0-2$, while the solid curves are based on the second-order expression $\omega/\Omega=\kappa_0-2+\kappa_2\Omega^2/\pi G\bar{\rho}_0$.

eigenfunctions δU_2 and $\delta\Phi_2$ of the classical r-modes. Once δU_2 is known, the solution of Eq. (3.18) to determine $\delta\Phi_2$ is straightforward. Thus our discussion will concentrate on the more difficult problem of solving Eq. (4.7) for δU_2. It will be convenient to introduce the notation

$$D(\delta U_2)=\nabla_a\left\{\rho_0\left[\frac{4\delta^{ab}}{(m+1)^2}-4z^a z^b\right]\nabla_b\delta U_2\right\}+\frac{4m\varpi^a\nabla_a\rho_0}{(m+1)\varpi}\delta U_2, \tag{5.1}$$

for the differential operator that appears on the left side of Eq. (4.7). Thus Eq. (4.7) can be written in the form

$$D(\delta U_2)=F, \tag{5.2}$$

where

$$F=\frac{12m(m+2)}{(m+1)^2}\frac{\rho_{22}}{r^2}\delta U_0-2(m+2)\kappa_2\frac{1}{r}\frac{d\rho_0}{dr}\delta U_0+16\pi G\bar{\rho}_0\frac{m(m+2)}{(m+1)^4}\left(\frac{d\rho}{dh}\right)_0(\delta U_0+\delta\Phi_0). \tag{5.3}$$

The problem of solving Eq. (5.2) numerically is a somewhat non-standard problem that is made difficult by two facts. First, the operator D has a non-trivial kernel: $D(\delta U_0)=0$. Many of the straightforward numerical techniques fail in this case. Second, the operator D is hyperbolic. There appears to be little previous work on solving hyperbolic boundary value problems of this type.

We solve Eq. (5.2) here using a variation of the standard relaxation method commonly used to solve elliptic partial differential equations [21]. To that end we introduce a fictitious time parameter λ and convert Eq. (5.2) into an evolution equation

$$\partial_\lambda\delta U_2=D(\delta U_2)-F. \tag{5.4}$$

The idea is to impose as initial data for Eq. (5.4) a guess for δU_2, and then to evolve these data (as a function of λ) until a stationary ($\partial_\lambda\delta U_2=0$) state is reached. If successful, the late time solution ($\lim_{\lambda\to\infty}\delta U_2$) to Eq. (5.4) will also be a solution to Eq. (5.2).

We implement the relaxation method to solve Eq. (5.2) by using a discrete representation of the functions and differential operators. Let u_n^i denote the discrete representation of the function δU_2 evaluated at the fictitious time λ_n. The index i (and later j and k as well) takes on values from 1 to N where N is the dimension of the particular discretization used. Similarly the discrete representation of the differential operator D of Eq. (5.2) is denoted $D^j{}_i$, and the representation of the right side of Eq. (5.2) is denoted F^j. Thus the discrete representation of Eq. (5.2) is simply

$$D^j{}_i u^i=F^j. \tag{5.5}$$

Summation is implied for pairs of repeated indices [e.g. i on the left side of Eq. (5.5)]. The algebraic equation (5.5) cannot be solved by the most straightforward direct numerical techniques because the operator D has a nontrivial kernel (mentioned above). Consequently the matrix $D^j{}_i$ has no inverse.

Thus we are lead to introduce the evolution equation (5.4). We use the "implicit" form of the discrete representation of Eq. (5.4):

$$O^j{}_i u^i_{n+1}\equiv\left(D^j{}_i-\frac{1}{\Delta\lambda}I^j{}_i\right)u^i_{n+1}=F^j-\frac{1}{\Delta\lambda}u^j_n. \tag{5.6}$$

In Eq. (5.6), $I^j{}_i$ is the N dimensional identity matrix, and $\Delta\lambda$ is the relaxation time step. Given u^i_n we solve Eq. (5.6) for u^i_{n+1} by direct solution of the linear algebraic equation. We use the band-diagonal linear equation solver LINSIS from EISPACK to compute $(O^{-1})^i{}_j(F^j-u^j_n/\Delta\lambda)$. We find that this can be computed stably and accurately for almost any value of the relaxation timestep $\Delta\lambda$.

Unfortunately solving Eq. (5.6) iteratively does not yield the desired solution to Eq. (5.5) in the limit of large n. Instead the solution grows exponentially, becoming in the limit of large n closer and closer to a non-trivial solution to the homogeneous equation, δU_0. Fortunately, this malady is easily corrected. Let $\overline{u^i}$ denote the discrete representation of δU_0; thus $D^j{}_i\overline{u^i}\approx 0$ since δU_0 is in the kernel of D. Also we let u_i denote the discrete representation of the co-vector associated with $\overline{u^i}$. In particular choose $\overline{u_i}$ so that $\overline{u^i}\,\overline{u_i}$ is the discrete representation of the integral of $|\delta U_0|^2$. Then the matrix

$$P^j{}_i=I^j{}_i-\frac{\overline{u^j}\,\overline{u_i}}{\overline{u^k}\,\overline{u_k}} \tag{5.7}$$

is the discrete representation of the operator that projects functions into the subspace orthogonal to δU_0. We use this projection in conjunction with Eq. (5.6) to define a modified relaxation scheme to determine u^i_{n+1} iteratively:

$$u^i_{n+1}=P^i{}_j(O^{-1})^j{}_k\left(F^k-\frac{u^k_n}{\Delta\lambda}\right). \tag{5.8}$$

By applying the projection $P^i{}_j$ after each relaxation step we ensure that the exponentially growing kernel is removed from the solution. We find that the iteration scheme defined in Eq. (5.8) does converge quickly and stably to a solution of Eq. (5.5). In the Appendix we discuss the reason this numerical relaxation method works even in the case of the unusual hyperbolic boundary-value problem considered here. We show that convergence is guaranteed for sufficiently large values of the relaxation time step $\Delta\lambda$, and that it also converges for either sign of $\Delta\lambda$.

In order to implement this inversion scheme we need explicit discrete representations of these operators. We find it convenient to work in spherical coordinates r and $\mu=\cos\theta$. In terms of these coordinates then, the differential operator D has the form

$$D(\delta U_2)=\frac{4\rho_0}{(m+1)^2}\left[\frac{\partial^2\delta U_2}{\partial r^2}+\frac{1-\mu^2}{r^2}\frac{\partial^2\delta U_2}{\partial\mu^2}+\frac{2}{r}\frac{\partial\delta U_2}{\partial r}-\frac{2\mu}{r^2}\frac{\partial\delta U_2}{\partial\mu}-\frac{m^2\delta U_2}{r^2(1-\mu^2)}\right]$$
$$-4\rho_0\left[\mu^2\frac{\partial^2\delta U_2}{\partial r^2}+\frac{2\mu(1-\mu^2)}{r}\frac{\partial^2\delta U_2}{\partial r\partial\mu}+\frac{(1-\mu^2)^2}{r^2}\frac{\partial^2\delta U_2}{\partial\mu^2}+\frac{1-\mu^2}{r}\frac{\partial\delta U_2}{\partial r}-\frac{3\mu(1-\mu^2)}{r^2}\frac{\partial\delta U_2}{\partial\mu}\right]$$
$$+\frac{4}{(m+1)^2}\left(\frac{d\rho}{dh}\right)_0\frac{dh_0}{dr}\left[[1-(m+1)^2\mu^2]\frac{\partial\delta U_2}{\partial r}-(m+1)^2\frac{\mu(1-\mu^2)}{r}\frac{\partial\delta U_2}{\partial\mu}+m(m+1)\frac{\delta U_2}{r}\right]. \tag{5.9}$$

A similar spherical representation is also needed for the boundary condition, Eq. (3.11),

$$\left\{[1-(m+1)^2\mu^2]\frac{dh_0}{dr}\frac{\partial\delta U_2}{\partial r}+m(m+1)\frac{1}{r}\frac{dh_0}{dr}\delta U_2-(m+1)^2\mu(1-\mu^2)\frac{1}{r}\frac{dh_0}{dr}\frac{\partial\delta U_2}{\partial\mu}\right.$$
$$\left.+(m+2)\left[\tfrac{1}{2}(m+1)^2\kappa_2\frac{1}{r}\frac{dh_0}{dr}-3m\frac{h_{22}}{r^2}\right]\delta U_0-\frac{4m(m+2)}{(m+1)^2}\pi G\overline{\rho}_0(\delta U_0+\delta\Phi_0)\right\}_{r=R_0}=0. \tag{5.10}$$

The operators in Eqs. (5.9) and (5.10) are transformed into the discrete matrix representation of the operator $D^i{}_j$ using the techniques discussed in Ipser and Lindblom [15]. In particular we use a grid of points (r_j,μ_k) where the r_j are equally spaced in the radial direction and the μ_k are the zeros of one of the odd-order Legendre polynomials [22]. We use standard three-point difference formulas for the derivatives in the radial direction and the higher-order multi-point formulas for the angular derivatives described in Ipser and Lindblom [15]. Using this discretization of the operator D, we find that the iteration scheme described in Eq. (5.8) converges rapidly. We begin the iteration by setting $u_0^i=0$ and find that after about five steps with $\Delta\lambda=-10^6R_0^2/\rho_c$ (where ρ_c is the central density) the changes in u_n^i from one iteration to the next become negligible.

Since the eigenfunction δU_2 is somewhat complicated we present several different representations of it graphically. Figure 2 depicts the functions $\delta U_2(r,\mu_k)$ with μ_k located at the grid points used in our integration: the roots of $P_{19}(\mu_k)=0$ in this case. We present in Fig. 3 another representation of this δU_2 in which we graph the functions $\delta U_2(r_k,\mu)$ for $r_k=(k/5)R_0$. This graph gives a clearer picture of the angular structure of δU_2. Finally we give in Fig. 4 another representation of the function δU_2 in which we decompose the angular structure of δU_2 into spherical harmonics by defining the functions $f_k(r)$:

$$\delta U_2(r,\mu)=\sum_{k\geq 1} f_k(r)P^m_{m+2k-1}(\mu). \tag{5.11}$$

We find numerically that the $f_k(r)$ are negligibly small except for the smallest few values of k. In Fig. 4 we graph the first three $f_k(r)$.

To measure the degree to which δU_2 satisfies the original differential equation, we define

$$\epsilon=\frac{\int|D(\delta U_2)-F|^2r^2drd\mu}{\int|F|^2r^2drd\mu}. \tag{5.12}$$

We find that the value of ϵ achieved by a given solution is approximately $\epsilon\approx(4.3/N_r)^4$ where N_r is the number of radial grid points used in the discretization [23]. It is instructive to compare ϵ to the quantity

$$\epsilon_0=\frac{\int|D(\delta U_0)|^2r^2drd\mu}{\int|F|^2r^2drd\mu}. \tag{5.13}$$

which measures the degree to which δU_0 is in the kernel of D. Since analytically $D(\delta U_0)=0$, the deviation of ϵ_0 from zero is a measure of the accuracy of our discrete representa-

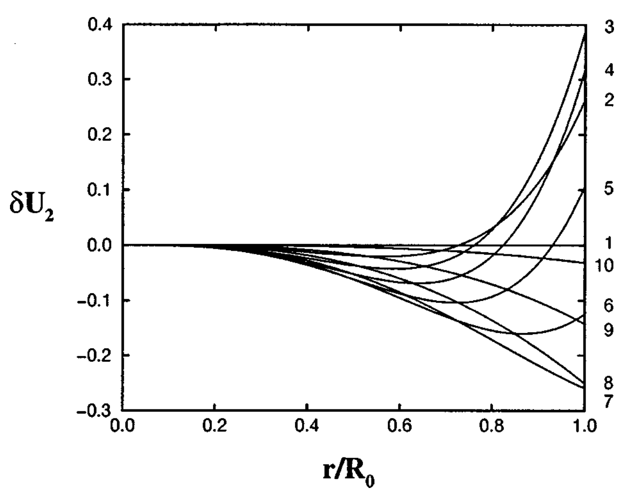

FIG. 2. Functions $\delta U_2(r,\mu_k)$ for a range of values of the angular coordinate μ_k. The numbers along the right vertical axis are the values of k. These range sequentially from $\mu_1=0$ at the equator of the star, to $\mu_{10}\approx 0.992$ near the rotation axis. The equation of state is the polytrope discussed in the text.

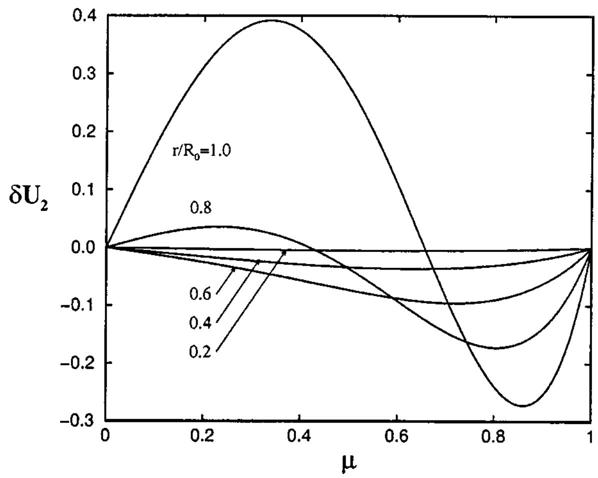

FIG. 3. Functions $\delta U_2(r_k,\mu)$ for a range of values of r_k/R_0.

tion of D. We find that ϵ_0 is approximately $\epsilon_0 \approx (7.7/N_r)^4$ in our numerical solutions. This scaling is what is expected from the truncation errors involved in the three-point difference formulas used to construct D^i_j. Since $\epsilon < \epsilon_0$ in our numerical solutions, we see that δU_2 is a good solution to Eq. (5.2).

VI. BULK VISCOSITY TIME SCALES

One of our primary interests in evaluating these modes through second order in the angular velocity is our desire to obtain the lowest-order expression for the bulk viscosity damping of these modes. Bulk viscosity is driven by the expansion, $\delta\sigma = \nabla_a \delta v^a$, of the fluid perturbation which can be expressed in terms of the scalar perturbation quantities δU and $\delta\Phi$ as

$$\delta\sigma = -i\kappa\Omega \frac{\Delta\rho}{\rho}$$

$$= -\frac{i}{\rho}\frac{d\rho}{dh}[Q^{ab}\nabla_a h \nabla_b \delta U + \kappa\Omega(\delta U + \delta\Phi)]. \quad (6.1)$$

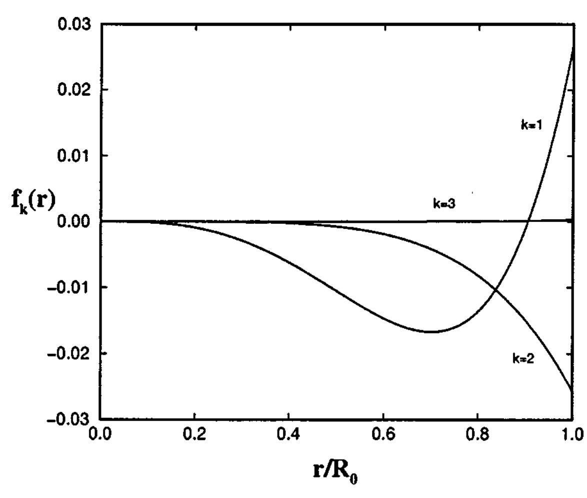

FIG. 4. Functions $f_k(r)$ that determine the spherical harmonic decomposition of δU_2 as defined in Eq. (5.11).

Each of the terms on the right side of Eq. (6.1) is proportional to Ω^3 at lowest order. And thus the second-order quantities δU_2, κ_2 etc. that we have evaluated in the preceding sections are needed to evaluate $\delta\sigma$.

Bulk viscosity causes the energy associated with a perturbation to be dissipated according to the formula

$$\left(\frac{d\tilde{E}}{dt}\right)_B = -\int \zeta\, \delta\sigma\, \delta\sigma^* d^3x, \quad (6.2)$$

where ζ is the bulk viscosity coefficient, and $\tilde{E}$ is the energy of the perturbation as measured in the co-rotating frame of the fluid. The energy $\tilde{E}$ can be expressed as an integral of the fluid perturbations:

$$\tilde{E} = \tfrac{1}{2}\int (\rho\, \delta v^a \delta v^*_a + \delta U \delta\rho^*) d^3x. \quad (6.3)$$

Bulk viscosity causes the energy in a mode to decay (or grow) exponentially with time. We can evaluate the imaginary part of the frequency of a mode that results from bulk-viscosity effects by combining Eqs. (6.2) and (6.3). The result, which defines the bulk-viscosity damping time, τ_B, is given by

$$\frac{1}{\tau_B} = -\frac{1}{2\tilde{E}}\left(\frac{d\tilde{E}}{dt}\right)_B. \quad (6.4)$$

In order to evaluate $1/\tau_B$ we need to have explicit expressions for the various terms that appear in the integrands of Eqs. (6.2) and (6.3). The energy $\tilde{E}$, for example, can be expressed as the integral

$$\tilde{E} = \frac{\alpha^2\pi}{2m}(m+1)^3(2m+1)!\, R_0^4\Omega^2 \int_0^{R_0} \rho_0(r)\left(\frac{r}{R_0}\right)^{2m+2} dr$$
$$+ \mathcal{O}(\Omega^4), \quad (6.5)$$

by performing the angular integrals indicated in Eq. (6.3) [24].

In the dissipation integral, Eq. (6.2), an explicit expression for the bulk viscosity ζ is needed. In standard neutron-star matter the dominant form of bulk viscosity is due to the emission of neutrinos via the modified URCA process [25]. An approximate expression for this form of the bulk-viscosity coefficient is [26]

$$\zeta = 6.0\times 10^{-59}\frac{\rho^2 T^6}{\kappa^2\Omega^2}, \quad (6.6)$$

where all quantities are expressed in cgs units. For the case of the classical r-modes the expansion $\delta\sigma$ that appears in Eq. (6.2) can be expressed explicitly in terms of the potentials δU_0, $\delta\Phi_0$, and δU_2:

$$\delta\sigma = \frac{i}{\rho_0}\left(\frac{d\rho}{dh}\right)_0 \frac{R_0^2\Omega^3}{\pi G\bar{\rho}_0}\frac{m+1}{2m(m+2)}$$
$$\times\Bigg\{\frac{m(m+1)}{r}\frac{dh_0}{dr}\delta U_2 + [1-(m+1)^2\mu^2]\frac{dh_0}{dr}\frac{\partial\delta U_2}{\partial r}$$
$$-(m+1)^2\frac{\mu(1-\mu^2)}{r}\frac{dh_0}{dr}\frac{\partial\delta U_2}{\partial\mu} - 3m(m+2)h_{22}\frac{\delta U_0}{r^2}$$
$$+\kappa_2\frac{(m+2)(m+1)^2}{2r}\frac{dh_0}{dr}\delta U_0$$
$$-4\pi G\bar{\rho}_0\frac{m(m+2)}{(m+1)^2}(\delta U_0+\delta\Phi_0)\Bigg\} + \mathcal{O}(\Omega^5). \qquad (6.7)$$

It will be of some interest to evaluate the accuracy of some previously published approximations for the bulk-viscosity time scale. One of these [3] is based on an approximation $\delta\sigma \approx \delta s$ for the expansion of the mode, where

$$\delta s \equiv -i\kappa\Omega\frac{\delta\rho}{\rho}$$
$$= -\frac{2i}{m+1}\frac{1}{\rho_0}\left(\frac{d\rho}{dh}\right)_0(\delta U_0+\delta\Phi_0)R_0^2\Omega^3 + \mathcal{O}(\Omega^5). \qquad (6.8)$$

We note that δs is just the last term in the expression given in Eq. (6.7) for $\delta\sigma$. It is the only term in Eq. (6.7) that depends only on the lowest-order perturbation quantities: the others depend on higher-order corrections through δU_2, κ_2 or h_{22}. We define the approximate bulk-viscosity time scale $\tau_{\tilde{B}}$ in analogy with Eq. (6.4) by replacing $\delta\sigma$ with δs in Eq. (6.2).

The bulk-viscosity contribution to the imaginary part of the frequency, $1/\tau_B$, is proportional to Ω^2. This follows from Eqs. (6.2) and (6.4) because $\tilde{E}$ scales as Ω^2 from Eq. (6.5), ζ as Ω^{-2} from Eq. (6.6), and $\delta\sigma$ as Ω^3 from Eq. (6.7). (We note that a previous calculation of this bulk-viscosity time scale by Andersson, Kokkotas, and Schutz [4] reported that $1/\tau_B$ was proportional to $\Omega^{1.77}$.) The bulk-viscosity damping time, $1/\tau_B$, also scales with temperature as T^6. Thus it is convenient to define $\tilde{\tau}_B$: the bulk viscosity time scale evaluated at $\Omega^2 = \pi G\bar{\rho}_0$ and $T = 10^9$ K,

$$\frac{1}{\tau_B} = \frac{1}{\tilde{\tau}_B}\left(\frac{T}{10^9\ \mathrm{K}}\right)^6\left(\frac{\Omega^2}{\pi G\bar{\rho}_0}\right) + \mathcal{O}(\Omega^4). \qquad (6.9)$$

We have evaluated $\tilde{\tau}_B$ numerically for the $m=2$ r-mode (the one most unstable to gravitational radiation) of a $1.4M_\odot$ stellar model with the polytropic equation of state discussed in Sec. II, and find $\tilde{\tau}_B = 2.01\times 10^{11}$ s. This value is longer by the factor $3.7(\sqrt{\pi G\bar{\rho}_0}/\Omega)^{0.23}$ than that found by Andersson, Kokkotas and Schutz [4]. This discrepancy may be due to the fact that their calculation was based on the second-order formalism of Saio which was reported to contain errors by Smeyers and Martens [11].

For comparison, we have also re-evaluated the approximate timescale $\tau_{\tilde{B}}$ described above and find $\tilde{\tau}_{\tilde{B}} = 7.04\times 10^9$ s. This approximation was based on the assumption that the average value of the Lagrangian change in the density $\Delta\rho$ [which appears on the right side of Eq. (6.1)] would be similar in magnitude to the average value of the Eulerian change in the density $\delta\rho$ [which appears on the right side of Eq. (6.8)]. Our results here show that $\delta\rho$ is on average a factor of 5.3 larger than $\Delta\rho$ for the $m=2$ r-mode. We also note that the value of the approximate time scale $\tilde{\tau}_{\tilde{B}}$ found here is about 10 times the value reported by Lindblom, Owen, and Morsink [3]. This discrepancy is due to an error in Eq. (4) of that paper: the right side of that equation should be multiplied by an additional overall factor of $(2l+1)/[l(l+1)\sqrt{2l+3}]$. Consequently the numerical bulk viscosity damping time estimate $\tau_{\tilde{B}}$ given there must be multiplied by $l^2(l+1)^2(2l+3)/(2l+1)^2$, or about 10.08 for the $l=2$ case, in agreement with our present calculation.

The most interesting application of these dissipative timescales is to evaluate the stability of rotating neutron stars to the gravitational radiation driven instability in the r-modes [3]. The imaginary part of the frequency of the r-modes, $1/\tau$, includes contributions from gravitational radiation $\tilde{\tau}_{GR}$ in addition to shear $\tilde{\tau}_S$ and bulk $\tilde{\tau}_B$ viscosity effects. The general expression for $\tau(\Omega,T)$, a function of the temperature T and angular velocity Ω of the star, is given by

$$\frac{1}{\tau(\Omega,T)} = \frac{1}{\tilde{\tau}_{GR}}\left(\frac{\Omega^2}{\pi G\bar{\rho}}\right)^3 + \frac{1}{\tilde{\tau}_S}\left(\frac{10^9\ \mathrm{K}}{T}\right)^2$$
$$+\frac{1}{\tilde{\tau}_B}\left(\frac{T}{10^9\ \mathrm{K}}\right)^6\left(\frac{\Omega^2}{\pi G\bar{\rho}}\right). \qquad (6.10)$$

The bulk viscosity time scale $\tilde{\tau}_B = 2.01\times 10^{11}$ s has been evaluated in this paper, while the gravitational radiation time scale, $\tilde{\tau}_{GR} = -3.26$ s, and the shear viscosity timescale $\tilde{\tau}_S = 2.52\times 10^8$ s, were obtained by Lindblom, Owen and Morsink [3] for the polytropic stellar model discussed in Sec. II. It is interesting to determine from this expression the critical angular velocity Ω_c:

$$\frac{1}{\tau(\Omega_c,T)} = 0. \qquad (6.11)$$

For stars at a given temperature, those with $\Omega > \Omega_c$ are unstable to the gravitational radiation driven instability in the r-modes, while those rotating more slowly are stable. Figure 5 depicts Ω_c for a range of temperatures relevant for hot young neutron stars. For stars cooler than about 10^9 K, superfluid effects change the dissipation processes completely and the analysis presented here is no longer relevant. The dashed curve in Fig. 5 represents the critical angular velocities computed using the approximate bulk viscosity damping time $\tilde{\tau}_{\tilde{B}} = 7.05\times 10^9$ s instead of the exact value. We see that even though $\tilde{\tau}_B \approx 29\tilde{\tau}_{\tilde{B}}$, the qualitative shape of the Ω_c curve is not affected. The minimum of the $\Omega_c(T)$ curve depicted in

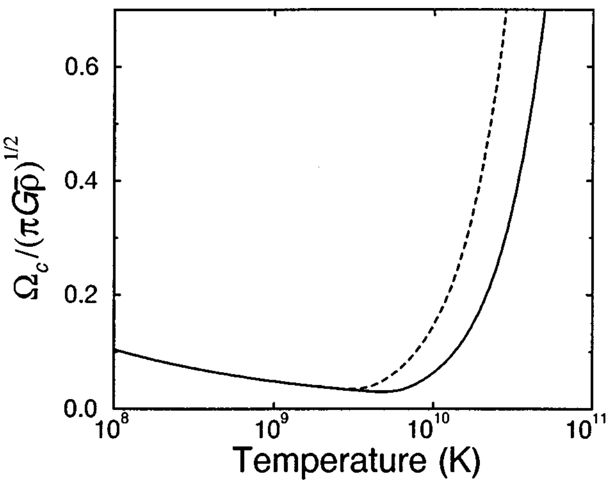

FIG. 5. Critical angular velocities for the r-mode instability as a function of temperature. The dashed curve gives the critical angular velocities based on the approximate bulk viscosity damping time $\tau_{\tilde{B}}$.

Fig. 5 occurs at $\min \Omega_c = 0.0301\sqrt{\pi G \bar{\rho}_0}$ which is 4.51% of the approximate maximum angular velocity $\frac{2}{3}\sqrt{\pi G \bar{\rho}_0}$.

Stars composed of strange quark matter are subject to a different form of bulk viscosity caused by weak interactions that transform d quarks to s quarks. The bulk viscosity coefficient that results from this process is given approximately by [27]

$$\zeta = 3.2\times 10^3 \frac{\rho T^2}{\kappa^2\Omega^2}\left(\frac{m_s}{100\ \ \mathrm{MeV}}\right)^4, \tag{6.12}$$

where all quantities (except m_s, the mass of the s quark) are given in cgs units. We have used this form of the bulk viscosity to estimate the damping time of the r-modes in strange stars. We find that in strange stars this damping time scales as

$$\frac{1}{\tau_B} = \frac{1}{\tilde{\tau}_B}\left(\frac{T}{10^9\ \ \mathrm{K}}\right)^2\left(\frac{\Omega^2}{\pi G \bar{\rho}_0}\right)\left(\frac{m_s}{100\ \ \mathrm{MeV}}\right)^4. \tag{6.13}$$

Using the polytropic stellar model described in Sec. II, we have computed the bulk-viscosity damping time of the r-mode to be $\tilde{\tau}_B = 0.886$ s for strange stars. This value is smaller (by about a factor of 7) than that found by Madsen [17], who used a very rough estimate of the bulk-viscosity damping time [12] for the r-modes. We also note that our expression for τ_B does not scale with angular velocity in the same way as Madsen's. Nevertheless, our calculations confirm Madsen's prediction that bulk viscosity completely suppresses the r-mode instability in hot strange stars. Using our expression for τ_B we estimate that the r-mode instability is suppressed in all strange stars with $T \gtrsim 5\times 10^8$ K.

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APPENDIX: NUMERICAL RELAXATION

The operator D defined in Eq. (5.1) is symmetric, in the sense that

$$\int g^* D(f) d^3x = \left[\int f^* D(g) d^3x\right]^*, \tag{A1}$$

for arbitrary functions f and g in any stellar model where $\rho_0 = 0$ on the surface. Thus the discrete representation of the operator D_{ij} will be a Hermitian matrix, and consequently $D^i{}_j$ will have a complete set of eigenvectors. Let e^i_α denote the eigenvector corresponding to eigenvalue d_α: $D^i{}_j e^j_\alpha = d_\alpha e^i_\alpha$. Since these eigenvectors form a complete set, we can express any vector as a linear combination of them. Thus we take $F^i = \Sigma_\alpha F^\alpha e^i_\alpha$, $u^i_n = \Sigma_\alpha u^\alpha_n e^i_\alpha$, etc. The numerical relaxation scheme indicated in Eq. (5.8) can be re-expressed therefore in the eigenvector basis as

$$u^\alpha_{n+1} = \frac{\Delta\lambda F^\alpha - u^\alpha_n}{\Delta\lambda d_\alpha - 1}. \tag{A2}$$

The role of the projection operator $P^i{}_j$ is merely to remove from Eq. (A2) the component corresponding to the zero eigenvalue. The recurrence relation, Eq. (A2), can be solved analytically:

$$u^\alpha_{n+1} = x_\alpha \Delta\lambda F^\alpha \sum_{k=0}^{n} (-x_\alpha)^k, \tag{A3}$$

where

$$x_\alpha = \frac{1}{\Delta\lambda d_\alpha - 1}. \tag{A4}$$

This series converges as long as $|x_\alpha| < 1$. Since the projection operator has eliminated the one equation where $d_\alpha = 0$, it is easy to choose $\Delta\lambda$ so that $|x_\alpha| < 1$ for all α, e.g., by taking $\Delta\lambda$ sufficiently large. Thus, the sequence u^α_{n+1} converges to

$$\lim_{n\to\infty} u_n^\alpha = \frac{x_\alpha \Delta\lambda F^\alpha}{1+x_\alpha} = \frac{F^\alpha}{d_\alpha}. \tag{A5}$$

Thus the implicit relaxation scheme converges to the desired solution to Eq. (5.2).

In contrast to the implicit relaxation method defined in Eq. (5.8), the analogous explicit relaxation method does not converge at all for this problem. An analysis similar to that carried out above reveals that the criterion for convergence of the explicit scheme is that $|\Delta\lambda d_\alpha + 1| < 1$. Clearly this can only hold for operators D where the eigenvalues all have the same sign (as is the case when D is elliptic) and only when $\Delta\lambda$ has the correct sign. Our limited experience with hyperbolic operators D is that their eigenvalues have both signs. Consequently it is not surprising that our attempts at explicit numerical relaxation fail in this case.

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[23] The value of ϵ is essentially independent of the number of angular grid points N_a because the angular dependence of the function δU_2 is determined by only a few terms when expessed as the series in Eq. (5.11). For such functions the angular differencing method discussed in Ipser and Lindblom [15] is essentially exact.

[24] We note that the normalization of these modes used here [as defined in Eq. (4.1)] differs from that used in Lindblom, Owen, and Morsink [3]. The normalization parameter α used here is related to the one used there (call it α') by

$$(\alpha')^2 = \frac{\alpha^2\pi}{m}(m+1)^3(2m+1)!\ .$$

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