PHYSICAL REVIEW D, VOLUME 60, 061901

D-branes in the WZW model

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(Received 22 December 1998; published 10 August 1999)

It is stated in the literature that D-branes in the Wess-Zumino-Witten (WZW) model associated with the gluing condition $J=-\bar{J}$ along the boundary correspond to branes filling out the whole group volume. We show instead that the end points of open strings are rather bound to stay on ''integer'' conjugacy classes. In the case of the $SU(2)$ level k WZW model we obtain $k-1$ two-dimensional Euclidean D-branes and two D particles sitting at the points e and $-e$. [S0556-2821(99)50116-1]

PACS number(s): 11.25.Hf, 11.25.Sq

String theory on a group manifold can be described by the world-sheet Wess-Zumino-Witten (WZW) action

$$S(g)=\frac{k}{8\pi}\int \mathrm{Tr}\left((\partial_\mu g g^{-1})^2+\frac{2}{3}d^{-1}(dgg^{-1})^3\right). \quad (1)$$

This theory possesses chiral currents (with $\partial_\pm=\partial_t\pm\partial_x$),

$$J=-\partial_+ g g^{-1}, \quad \bar{J}=g^{-1}\partial_- g. \quad (2)$$

Let us perform our analysis of branes in the closed string picture where D-branes are described as special ''initial conditions'' for closed strings rather than by boundary conditions in a theory of open strings. We consider D-branes corresponding to the standard gluing condition $J=-\bar{J}$ at the initial time $t=t_0$. The same gluing condition was used in [1]. For comparison with gluing conditions in the open string picture one needs to include an extra factor of -1 coming from the transformation properties of currents under coordinate transformations of the world sheet [2].

D-branes of this type were studied in the literature. For instance, Kato and Okada [3] suggest that they correspond to Neumann boundary conditions in all directions and, hence, that they fill the whole group manifold G. The same assertion is implicitly contained in [1] where the gluing condition $J=-\bar{J}$ is considered as a generalization of Neumann boundary conditions for a free bosonic string. This is clearly not the case: If we insert the parametrization $g=\exp(X)$ of the group valued field g near the group unit into the gluing conditions we obtain $\partial_x X=0$, i.e. the derivative of X along the boundary vanishes. Hence, one should rather view the relation $J=-\bar{J}$ as a generalization of Dirichlet boundary conditions along the boundary. Using this argument, Stanciu and Tseytlin [4] (in the context of Nappi-Witten backgrounds) see a rather pointlike structure of the associated D-branes. Our findings fit well with the analysis of Klimcik and Severa [8]: they identify D-branes in the WZW model with orbits of dressing transformations. If the ''double'' (used in [8]) is chosen as $G\times G$, the dressing orbits coincide with conjugacy classes (see [5] for details). Note, however, that no gluing conditions are specified in [8].

The analysis below will show that the end points of open strings with gluing conditions $J=-\bar{J}$ (in the closed string picture) are localized on special ''integer'' conjugacy classes ghg^{-1} for some fixed h. In particular, for the $SU(2)$ level k WZW model we obtain two D-particles at the points $\pm e$ and $k-1$ two-dimensional Euclidean D-branes.

In terms of ∂_t, ∂_x, the gluing condition $J=-\bar{J}$ reads

$$g^{-1}\partial_t g-\partial_t g g^{-1}=g^{-1}\partial_x g+\partial_x g g^{-1}. \quad (3)$$

It is convenient to introduce a special notation for the adjoint action of G on its Lie algebra, $\mathrm{Ad}(g)y=gyg^{-1}$. Then, Eq. (3) can be rewritten as

$$(1-\mathrm{Ad}(g))g^{-1}\partial_t g=(1+\mathrm{Ad}(g))g^{-1}\partial_x g. \quad (4)$$

We split the tangent space to the group G at the point g into an orthogonal (with respect to the Killing metric) sum, $T_gG=T_g^\perp G\oplus T_g^\parallel G$, where $T_g^\parallel G$ consists of vectors tangential to the orbit of Ad through g. Observe that on $T_g^\perp G$ the operator $1-\mathrm{Ad}(g)$ vanishes whereas $1+\mathrm{Ad}(g)=2$. Hence, we conclude that

$$(g^{-1}\partial_x g)^\perp=0, \quad (5)$$

and the corresponding D-branes coincide with the conjugacy classes. In the open string picture (which has t and x exchanged), the previous equation is a Dirichlet-type condition for components orthogonal to the conjugacy class.

If we restrict our consideration to some conjugacy class C, the operator $(1-\mathrm{Ad}(g))$ acting on the tangent space $T_g^\parallel G$ becomes invertible, and Eq. (4) can be rewritten in the form

$$g^{-1}\partial_t g=\frac{1+\mathrm{Ad}(g)}{1-\mathrm{Ad}(g)}g^{-1}\partial_x g. \quad (6)$$

Thus, it gives rise to a 2-form (B field) on the conjugacy class

$$\omega=\frac{k}{8\pi}\mathrm{Tr}\left(g^{-1}dg\frac{1+\mathrm{Ad}(g)}{1-\mathrm{Ad}(g)}g^{-1}dg\right), \quad (7)$$

where we have taken into account the normalization of the action (1). The 2-form ω is not closed, instead

$$d\omega = -\frac{k}{12\pi}\mathrm{Tr}(dgg^{-1})^3. \tag{8}$$

According to [8], D-branes in the WZW model are specified by a choice of a submanifold $D\subset G$ such that the restriction of the Wess-Zumino form $\eta = -k/12\pi\,\mathrm{Tr}(dgg^{-1})^3$ to D is exact, together with a 2-form ω on D such that $d\omega=\eta$. Equation (8) shows that conjugacy classes satisfy this condition, and that the form (7) gives a canonical choice of the primitive ω. Conjugacy classes equipped with such 2-forms were considered in [5] as examples of Hamiltonian spaces which admit group-valued moment maps.

Together, the B-field ω and the topological Wess-Zumino term in Eq. (1) impose a further constraint on the choice of conjugacy classes which can be used as D-branes. For simplicity, we analyze it only in the case of $G=SU(2)$. A D-brane in the target space corresponds to a boundary state of the world-sheet theory. In the case of the WZW model such a boundary state can be visualized as wave functional $\Psi(g(x))$ on the space of closed loops $g(x)$ in some conjugacy class C. Typical conjugacy classes in $G=SU(2)$ (other than e and $-e$) are 2-spheres. So, a closed loop on C can be contracted in two different ways giving rise to an ambiguity in the phase of the wave functional

$$\Delta\phi = \int_C \omega + \frac{k}{12\pi}\int_B \mathrm{Tr}(dgg^{-1})^3, \tag{9}$$

where B is one of the 3-balls in $SU(2)=S^3$ bounded by the conjugacy class C. Boundary states correspond to conjugacy classes with $\Delta\phi = 2\pi j$ with integer j. Equation (9) generalizes the standard Bohr-Sommerfeld quantization condition $\int\omega = 2\pi j$. An elementary calculation [see [6], Eq. (C.21)] shows that the conjugacy classes corresponding to $\Delta\phi = 2\pi j$, $j=1,\ldots,k-1$ pass through the points $\mathrm{diag}[\exp(\pi ij/k),\exp(-\pi ij/k)]$. The pointlike conjugacy classes e and $-e$ allow for an unambiguous choice of the wave functional $\Psi(g)=1$ and correspond to D particles.

These findings give a complete list of boundary conditions associated with the gluing condition $J=-\bar{J}$ which is in perfect agreement with the results of Cardy [2]. Other boundary conditions can come only from different choices of the gluing condition. In the simplest case we modify the relation $J=-\bar{J}$ by acting with an inner automorphism of the finite Lie algebra on the left-hand side. In contrast to what was suggested in [3], such a shift with inner automorphisms cannot possibly change the geometry of branes. It is rather associated with symmetries of the target, namely, with the right action of G on itself (see, e.g., [7]). In case of $G=SU(2)$, our discussion exhausts all boundary conditions with a maximal chiral Kac-Moody symmetry. Let us note that the non-Abelian structure of the Kac-Moody algebra places severe constraints for the construction of boundary conditions which preserve a non-Abelian symmetry (see e.g. [3]). In particular, it forbids the reversal of signs in all directions of the Lie algebra (i.e., the gluing condition $J=\bar{J}$) which is used in abelian theories to manufacture, e.g., volume filling branes from D particles etc.

If the Lie algebra of G admits outer automorphisms there can be additional boundary conditions with maximal symmetry which are not covered by our analysis above. The same applies to boundary conditions that do not preserve the full chiral Kac-Moody symmetry of the WZW model. The construction and interpretation of such branes remains an interesting open problem.

We thank J. Fuchs, C. Klimcik, A. Recknagel, C. Schweigert, S. Stanciu, A. Tseytlin, and T. Strobl for their useful comments and critical remarks. The stay of V.S. at the ITP in Uppsala was supported by the DAAD.

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