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Accelerated convergence of perturbative QCD by optimal conformal mapping of the Borel plane

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The technique of conformal mapping is applied to enlarge the convergence domain of the Borel series and to accelerate the convergence of Borel-summed Green functions in perturbative QCD. We use the optimal mapping, which takes into account the location of all the singularities of the Borel transform as well as the present knowledge about its behavior near the first branch points. The determination of $\alpha_s(m_\tau^2)$ from the hadronic decay rate of the τ lepton is discussed as an illustration of the method. [S0556-2821(99)00915-7]

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I. INTRODUCTION

The behavior of the large order terms in perturbative quantum field theory has been a subject of permanent interest [1–9]. Recently, this problem received much attention in the case of QCD [10–20]. As is known, the creation of instanton–anti-instanton pairs and certain classes of Feynman diagrams are responsible for a factorial increase of the large order coefficients of the perturbative expansion of the QCD Green functions, making this series divergent in the mathematical sense. Moreover, the growth of the large order coefficients is so dramatic that, when combined with other difficult circumstances (such as their nonalternating sign and the extraordinarily small analyticity domain of the Green functions in the coupling variable [9]), it leads to the situation that some of the usually efficient summation techniques [21] are not applicable. One of them, the Borel summation, has been very much investigated in recent times. The growth of the large order perturbative coefficients of the QCD Green functions leads, under certain conditions, to Borel transforms with singularities in the Borel plane that make the integral defining the Green function by the Laplace transform ill defined. Of course, as discussed in [19], the Borel technique is not the only mathematical method by which a divergent series can be summed, but an ambiguity emerges in every summation method, once it is discovered in one of them. The real source of nonuniqueness consists in a missing piece of information about the quantities to be calculated, which adopts different forms in different summation methods, but has to be added to eliminate the ambiguity.

In QCD the Borel nonsummability originates from the infrared regions of the Feynman diagrams, where nonperturbative effects also play an important role. Therefore it is natural to assume that the ambiguities of perturbation theory must be compensated by nonperturbative contributions. In fact, it turns out that general concepts like analyticity, renormalization group, or specific properties of the QCD vacuum are unavoidable when discussing the large order behavior of perturbation theory [4,9,19]. As an example, we recall that the argument given by 't Hooft [9] for the Borel nonsummability of QCD relies on nonperturbative properties, mainly the momentum plane analyticity combined with renormalization group invariance. So the perturbative and the nonperturbative regimes of the theory cannot be separated, and their interplay is very clear when attempting to perform the summation of the large orders of perturbation theory. The Borel plane is particularly suitable for discussing this aspect, since the singularities of the Borel transform offer a very intuitive measure of the ambiguities of the perturbation theory and suggest a way to compensate them. The properties of these singularities in some approximations (such as massless QCD in the large β_0 limit, when they are poles [13]) were used recently to provide estimates of the truncation error in the theoretical determinations of some accurately measured quantities.

A natural question is whether it is possible to improve the accuracy of the Borel summation using the first Taylor coefficients of the Borel transform known from explicit low order calculations, supplemented by some (approximate) information about its singularities in the Borel plane. We address this question in the present paper. We use as input the assumption that there are no other singularities in the Borel plane other than those located on the real axis, at a nonvanishing distance from the origin. The precise nature and strength of these singularities are not known in general, except for the nearest ones, which can be characterized by using general principles [20,10]. As discussed in [4], the singularities of the Borel transform require the introduction of higher dimensional operators, which ensure compensation of the ambiguities present in the usual perturbative terms by the ambiguities inherent in their Wilson coefficients. This allows one also to infer a universal behavior of the Borel transform near the first ultraviolet renormalon [20]. On the other hand, as discussed in [10], the location and nature of the first infrared renormalon can be plausibly predicted too, by nonperturbative arguments. In our approach, we take as input this knowledge about the first ultraviolet and infrared renormalons.

Our purpose was to exploit in an optimal way this information, in order to improve the accuracy of the Borel summation in the frame of a specific prescription of handling the singularities of the Borel transform. To this end we use the analytic continuation of the Borel transform outside the circle of convergence of the Taylor expansion, achieved by the technique of conformal mapping. As is known, the conformal mappings are very suitable for accelerating the convergence of power series. The existence of an optimal expansion, with the largest convergence domain and the best asymptotic convergence rate, was proved in [22] a long time ago. The method is applicable if the position of the singularities of the function to be approximated is known or can be reasonably guessed, which is the case in many situations in particle physics.

In the context of the Borel summation in quantum field theory, conformal mappings were first considered in [5–8]. More recently, the method was applied in [23,24] to the Borel transform of QCD Green functions, following a suggestion made in [10]. The purpose was to estimate (and possibly reduce) the influence of the first ultraviolet renormalon and of the associated power corrections on observable quantities, by using a variable in which this singularity is pushed farther away from the integration range of the Borel transform. However, from the point of view of the convergence rate the mapping used in [23,24] is not optimal, since it takes into account only a part of the singularities of the Borel transform, the ultraviolet (UV) renormalons. By using an optimal treatment, which takes into account also the infrared (IR) renormalons and the behavior near the first singularities, an increased convergence rate and consequently a smaller truncation error are to be expected.

The objective of our work is to establish whether the optimal mapping technique is numerically relevant in the Borel plane for situations of physical interest (we mention alternative attempts to enlarge the convergence domain of the Borel transform, based on Padé approximants [25]). To illustrate our discussion we consider, as in [23], the Adler function of the massless QCD vacuum polarization and the determination of the strong coupling constant $\alpha_s(m_\tau^2)$ from the hadronic τ decay rate. In the next section we briefly review some properties of the Adler function and of its Borel transform. In Sec. III we present the technique of optimal conformal mapping and investigate its efficiency in the Borel plane on several mathematical models which simulate the physical situation. We discuss also the determination of the strong coupling constant $\alpha_s(m_\tau^2)$ using the present technique. Some conclusions are formulated in Sec. IV.

II. ADLER FUNCTION AND ITS BOREL TRANSFORM

We consider the correlator

$$i\int \mathrm{d}^4x e^{iq\cdot x}\langle 0|T\{V^\mu(x),V^\nu(0)^\dagger\}|0\rangle=(q^\mu q^\nu-g^{\mu\nu}q^2)\Pi(s), \tag{1}$$

where $s=q^2$ and $V^\mu=\overline{q}\gamma^\mu q$ is the current of a massless quark. From the general principles of causality and unitarity it follows that the amplitude $\Pi(s)$ is an analytic function of real type in the complex plane s, cut along the real positive axis from the threshold $4m_\pi^2$ of hadron production to infinity. It is convenient to define the Adler function

$$D(s)=-\frac{\mathrm{d}}{\mathrm{d}s}\Pi(s), \tag{2}$$

which is ultraviolet finite and is also analytic in the complex s-plane cut above the unitarity threshold. This function was much investigated lately in connection with the determination of the strong coupling constant $\alpha_s(m_\tau^2)$ from the hadronic decay of the τ lepton [26–28,17,29]. The hadronic decay width, normalized to the leptonic one, is defined as

$$R_\tau=\frac{\Gamma(\tau\to\nu_\tau+\text{hadrons})}{\Gamma(\tau\to\nu_\tau e\overline{\nu}_e)}=\int_0^{m_\tau^2}\mathrm{d}s\,\frac{\mathrm{d}R_\tau(s)}{\mathrm{d}s}, \tag{3}$$

where the inclusive hadronic spectrum $\mathrm{d}R_\tau(s)/\mathrm{d}s$ is related to the spectral part of the correlator (1):

$$\frac{\mathrm{d}R_\tau(s)}{\mathrm{d}s}=\frac{3(1+\delta_{\mathrm{EW}})}{\pi m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+\frac{2s}{m_\tau^2}\right)\mathrm{Im}\,\Pi(s+i\epsilon). \tag{4}$$

The factor $\delta_{\mathrm{EW}}\simeq 0.0194$ accounts for electroweak radiative corrections. The decay rate (3) was measured recently with great accuracy [31,32].

Using the analyticity properties of the function $\Pi(s)$ in the momentum plane, relation (3) can be transformed by a Cauchy relation into

$$R_\tau=\frac{3(1+\delta_{\mathrm{EW}})}{2\pi i}\oint\frac{\mathrm{d}s}{s}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)D(s), \tag{5}$$

where the integration runs along a closed contour in the complex plane, taken usually to be the circle $|s|=m_\tau^2$.

Relation (5) is the starting point for the computation of the τ hadronic width in perturbative QCD. At complex values of s the Adler function admits the formal renormalization group-improved expansion

$$D(s)=1+\sum_{n=1}^{\infty}D_n\left(\frac{\alpha_s(-s)}{\pi}\right)^n. \tag{6}$$

The strong coupling $\alpha_s(\mu^2)$ satisfies the renormalization group equation

$$\mu^2\frac{\mathrm{d}\alpha_s(\mu^2)}{\mathrm{d}\mu^2}=-\alpha_s(\mu^2)\sum_{n=0}^{\infty}\beta_n[\alpha_s(\mu^2)]^{n+1}, \tag{7}$$

with the first coefficients β_n defined in terms of the number n_f of quark flavors as

$$\beta_0=\frac{33-2n_f}{12\pi},$$

$$\beta_1=\frac{153-19n_f}{24\pi^2}. \tag{8}$$

The coefficients D_n for $N<3$ in the expansion (6) were computed for $N<3$ [33–37]. In the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme with $n_f=3$ they are

$$D_1=1,$$

$$D_2 = 1.63982,$$

$$D_3 = 6.37101. \tag{9}$$

On the other hand, the large order coefficients, D_n for large n, have the generic factorial behavior

$$D_n \approx C_k n! n^{\delta_k} \left(\frac{\pi \beta_0}{k} \right)^n + \cdots, \tag{10}$$

where the index k takes the values $-1, \pm 2, \pm 3, \ldots$.

In the Borel method of summation one defines the Borel transform of the Adler function as

$$B(u) = \sum_{n=0}^{\infty} b_n u^n, \tag{11}$$

where

$$b_n = \frac{1}{n!} \frac{D_{n+1}}{(\pi \beta_0)^n} = \frac{\tilde{D}_{n+1}}{n!}. \tag{12}$$

Then $D(s)$ can be expressed formally in terms of $B(u)$ by the Laplace transform

$$D(s) = 1 + \frac{1}{\pi \beta_0} \int_0^{\infty} \mathrm{d}u B(u) \exp\left(- \frac{u}{\beta_0 \alpha_s(-s)} \right). \tag{13}$$

We need also the Borel-summed expression of the hadronic decay rate R_τ. It was obtained in [23] by inserting in Eq. (5) the Borel representation (13) of the Adler function and performing the integration along the circle $|s| = m_\tau^2$ using the one-loop expression

$$\alpha_s(-s) = \frac{1}{\beta_0 \ln(-s/\Lambda^2)}, \tag{14}$$

of the running coupling. This gives [23]

$$R_\tau = 3(1+\delta_{\mathrm{EW}}) \left[1 + \frac{1}{\pi \beta_0} \int_0^{\infty} \mathrm{d}u \times \exp\left(- \frac{u}{\beta_0 \alpha_s(m_\tau^2)} \right) B(u) F(u) \right], \tag{15}$$

where

$$F(u) = \frac{-12 \sin(\pi u)}{\pi u (u-1)(u-3)(u-4)}. \tag{16}$$

In Sec. III we shall use Eq. (15) as a starting point for the determination of $\alpha_s(m_\tau^2)$ using the method of conformal mapping.

The growth (10) of the Taylor coefficients D_n leads to the dominant behavior

$$B(u) \approx C_k \Gamma(\delta_k + 1) \left(1 - \frac{u}{k} \right)^{-\delta_k - 1} + \cdots, \tag{17}$$

which shows that the function $B(u)$ becomes singular at the points $u = k$, with $k = -1, \pm 2, \pm 3, \ldots$. The precise values of C_k and δ_k are not known in general. However, from general arguments it can be shown that the nature of the first branch points of the Borel transform is universal [10,20]. More precisely, near the first UV renormalon at $u = -1$ the Borel transform behaves as

$$B(u) \simeq r_1 (1+u)^{-\gamma_1}, \tag{18}$$

where [20]

$$\gamma_1 = 3 - \frac{\beta_1}{\beta_0^2} + \lambda_1. \tag{19}$$

Here λ_1 is a parameter depending on the number of flavors, which reflects the mixing of higher dimensional operators in the renormalization group equations [20]. Similarly, near the first IR renormalon at $u = 2$ the behavior is

$$B(u) \simeq r_2 (2-u)^{-\gamma_2}, \tag{20}$$

where [10]

$$\gamma_2 = 1 + 2 \frac{\beta_1}{\beta_0^2}. \tag{21}$$

Using the first coefficients β_i from Eqs. (8) and the parameter λ_1 given in [20] (equal to 0.379 for $n_f = 3$ and 0.630 for $n_f = 5$) we obtain

$$\gamma_1 = 2.589, \quad \gamma_2 = 2.580 \tag{22}$$

for $n_f = 3$ and

$$\gamma_1 = 2.972, \quad \gamma_2 = 2.316 \tag{23}$$

for $n_f = 5$. We emphasize that only the nature of the first renormalons is known, and nothing can be said about the residues r_1 and r_2 appearing in Eqs. (18) and (20), respectively.

Strictly speaking, the integrals (13) and (15) have nothing to do with the summation of the perturbative series, because one condition of the Borel theorem (the existence of the analytic continuation in the α_s plane from the convergence disk to an infinite strip around the positive real semiaxis) is not satisfied [9]. This can be seen from the singularities of the Borel transform given in Eq. (17): the poles situated on the real positive axis (IR renormalons) make the integrals (13) and (15) ambiguous. In order to compute it a prescription has to be adopted, by suitably choosing the integration contour in order to avoid the singularities. But this is not the Borel summation. Different prescriptions give different results, and a measure of the intrinsic ambiguity of the perturbation expansion is given by the difference between these results, if no *a priori* arguments in favor of a certain choice exist.

A prescription adopted by several authors [8,14,17] is the ''principal value'' (PV), defined as

$$\mathrm{PV}\int_0^\infty \mathrm{d}u\,\exp\left(-\frac{u}{a_s}\right)f(u) \overset{\epsilon\to 0}{=} \frac{1}{2}\left[\int_0^\infty \mathrm{d}u\,\exp\left(-\frac{u}{a_s}\right)f(u+i\epsilon)\mathrm{d}u + \int_0^\infty \mathrm{d}u\,\exp\left(-\frac{u}{a_s}\right)f(u-i\epsilon)\right]. \tag{24}$$

This definition is a generalization to arbitrary singularities of the Cauchy principal value for the case of simple poles, and has the advantage of yielding a real result when a_s is real. Although this prescription is not unique, we adopted it below as a working hypothesis.

We applied the definition (24) for computing the Borel-summed Adler function [given by the Laplace integral (13)] and the hadronic τ decay rate [given by Eq. (15)]. In the first case the parameter a_s is related to the running coupling $\alpha_s(-s)$ (and may be complex if s is complex or in the Minkowskian region $s>0$), while in the second case it is proportional to $\alpha_s(m_\tau^2)$. We use as input the first Taylor coefficients of $B(u)$ [known from Eq. (12) and the calculated values (9)], supplemented by knowledge of the location and nature of singularities of this function. As discussed in the Introduction, our purpose is to improve the accuracy of the calculation by the technique of conformal mapping, which exploits in an optimal way this input information.

III. OPTIMAL CONFORMAL MAPPING OF THE BOREL PLANE

A. Remarks on the theory

The use of conformal mapping for improving the convergence of power series in particle physics was discussed for the first time in Refs. [22]. The problem formulated in [22] was to find the optimal conformal transformation which minimizes the asymptotic truncation error of a power series, taking into account the location of the singularities of the function to be approximated. First we briefly describe the results obtained in [22]. We consider a function $f(u)$ analytic in a domain $\mathcal{D}$ of the complex u plane containing the origin and write its Taylor series truncated at a finite order N as

$$f^{(N)}(u)=\sum_{n=0}^{N} f_n u^n. \tag{25}$$

According to general theory, the series (25) converges, for $N\to\infty$, inside the circle passing through the nearest singularity of the function $f(u)$ in the complex plane, the rate of convergence at a point u situated inside the circle being that of the geometrical series in powers of r/R, where $r=|u|$ and R is the radius of the convergence circle. Therefore, the convergence rate is strongly influenced by the distance of the singularities of $f(u)$ from the origin, and can be improved by using a suitable change of variable, in which the singularities are pushed further away from the region of interest.

Let us consider a conformal mapping $w=w(u)$ of the plane u onto the plane w, such that $w(0)=0$, and write the truncated Taylor expansion of the function $f(u)$ in the variable w:

$$f_w^{(N)}(u)=\sum_{n=0}^{N} c_n w^n. \tag{26}$$

As pointed out in [22], the best asymptotic rate of convergence of this series in a certain region of the complex plane is achieved when w is such a transformation of the u plane that the corresponding ratio r/R is minimal for every point in that region. According to the theorem proved in [22] this is realized by a conformal mapping $w(u)$ which maps the whole analyticity domain $\mathcal{D}$ of the function $f(u)$ in the u plane into the interior of a circle in the plane w. The proof of the theorem is based on the fact that the circle is the natural domain of convergence for power series and on the Schwartz lemma, which implies that the larger the domain mapped inside the circle, the better is the asymptotic rate of convergence (for details see [22]).

In what follows we shall apply this technique to the Borel transform $B(u)$ of the Adler function. The nearest singularities of the function $B(u)$ are situated at $u=-1$ and $u=2$ and the power expansion (11) converges only inside the circle $|u|<1$ passing through the first UV renormalon. It is easy to see that the optimal conformal mapping in the sense explained above is given in our case by

$$w=\frac{\sqrt{1+u}-\sqrt{1-u/2}}{\sqrt{1+u}+\sqrt{1-u/2}}. \tag{27}$$

By this mapping, the complex u plane cut along the real axis for $u>2$ and $u<-1$ is mapped onto the interior of the circle $|w|<1$ in the complex w plane, the origin $u=0$ of the u plane becoming the origin $w=0$ of the w plane, and the upper (lower) lips of the cuts are mapped onto the upper (lower) semicircles in the plane w. Particularly, all the singularities of the Borel transform, the UV and IR renormalons, are now situated on the boundary of the unit disk in the w plane, all at equal distance from the origin. The Taylor expansion of the Borel transform in powers of w,

$$\tilde{B}_w^{(N)}(u)=\sum_{n=0}^{N} \tilde{c}_n w^n, \tag{28}$$

will converge for $N\to\infty$ up to points close to the renormalons. This is a considerable improvement with respect to the usual expansion (11), whose convergence is limited by the presence of the first UV renormalon. In this way, the expansion in powers of the function $w(u)$ makes full use of the analyticity property that is universally (but tacitly) assumed in all QCD considerations, namely, that there are no singularities in the Borel plane other than those situated on the real axis, at a nonvanishing distance from the origin. This essential, additional assumption has, to our knowledge, never been explicitly used.

For comparison we give the conformal mapping used in [23,24],

$$z=\frac{\sqrt{1+u}-1}{\sqrt{1+u}+1}, \tag{29}$$

which maps the u plane cut along $u<-1$ onto the interior of the unit circle in the z plane. In the z plane the UV renormalons are situated along the boundary of the unit circle $|z|=1$, but the IR renormalons are situated inside this circle. As noticed in [24], pushing away the ultraviolet renormalons by (29) has a price in moving the first infrared renormalon (and actually, the whole positive real semiaxis) closer to the origin. This is why the convergence domain of the power series in z is limited by the first IR renormalon and, as a consequence, the convergence rate of the series in powers z will be worse than that obtained with the optimal variable (27). The use of the optimal conformal mapping (27) is therefore highly desirable, because it does not suffer from this shortcoming, placing *all* renormalons onto the circumference of the unit disk.

As was pointed out in [38], a further improvement of the convergence rate can be reached if some information about the nature of the singularities of the expanded function $f(u)$ is available. The idea is that the power variable $w(u)$, taken as a function of u, should resemble $f(u)$ as much as our knowledge of $f(u)$ allows it. (As it was put in [22], if we were to know f exactly, the most rapidly convergent expansion would be that in powers of f itself, in which case it would reduce to the identity $f\equiv f$.)

In practice, however, our knowledge of the expanded function is only approximative. For instance, as discussed above, we know that near the branch points $u=-1$ and $u=2$ the function $B(u)$ behaves like $(u+1)^{-\gamma_1}$ and $(2-u)^{-\gamma_2}$, respectively, with γ_i real positive numbers. Then it is convenient to expand the product $(u+1)^{\gamma_1}(2-u)^{\gamma_2}B(u)$ in powers of the optimal variable w defined in Eq. (27). The expansion of the function $B(u)$ will be therefore

$$\hat{B}_w^{(N)}(u)=\frac{1}{(u+1)^{\gamma_1}(2-u)^{\gamma_2}}\sum_{n=0}^{N}\hat{c}_n w^n. \tag{30}$$

The singularities themselves may survive as positive powers $(u+1)^{\gamma_1}$ and $(2-u)^{\gamma_2}$, the bonus nevertheless being that the positive exponents γ_i keep the values of the function

$$(u+1)^{\gamma_1}(2-u)^{\gamma_2}B(u) \tag{31}$$

finite near $u=u_1$ and $u=u_2$, which softens their numerical impact. This step will imply no large order improvement of the convergence rate (because the rate is given by the *position* of the nearest singularities), unless some of the two singularities is fully removed by it. But it may represent a considerable improvement at low orders, even if the nature of the nearest singularities is known only approximately. A nice example of efficiency of this approach in practice was presented by Soper and Surguladze in [24].

B. Discussion of mathematical models

(1) As a first application, let us consider the use of the optimal conformal mapping to estimate the first unknown term of a truncated power expansion.

We tested the practical efficiency of the conformal mapping (27) for a number of functions having logarithmic or power branch points at $u=-1$ and $u=2$. We took in particular functions of the form

$$B(u)=\frac{r_1}{(1+u)^{\gamma_1}}+\frac{r_2}{(2-u)^{\gamma_2}}+\sum_{n=3}^{N_{IR}}\frac{r_n}{(n-u)^{\gamma_n}}+\sum_{n=2}^{N_{UV}}\frac{r_{-n}}{(n+u)^{\gamma_{-n}}}, \tag{32}$$

which simulate the contribution of a number of ultraviolet and infrared renormalons to the Borel transform. We started from the ''perturbative'' expansion

$$B^{(N)}(u)=\sum_{n=0}^{N}b_n u^n, \tag{33}$$

and obtained the expansion (28) in terms of the variable w by replacing u in Eq. (33) with the expansion

$$u_w^{(N)}=\sum_{n=1}^{N}C_n w^n, \tag{34}$$

which follows from the inverse of Eq. (27), and keeping only terms up to the order N [i.e., $(u_w^{(N)})^N=C_1^N w^N$, etc.]. The numerical values of the first coefficients C_n are

$$C_1=\frac{8}{3},\quad C_2=\frac{16}{9},\quad C_3=-\frac{40}{27},$$
$$C_4=-\frac{224}{81},\quad C_5=-\frac{88}{243}\dots. \tag{35}$$

For comparison, the expansion in powers of the variable z given in Eq. (29) is obtained using

$$u_z^{(N)}=\sum_{n=1}^{N}\bar{C}_n z^n \tag{36}$$

with the numerical values [23]

$$\bar{C}_1=4,\quad \bar{C}_2=8,\quad \bar{C}_3=12,\quad \bar{C}_4=16,\quad \bar{C}_5=20\dots. \tag{37}$$

We computed the model functions and the truncated power expansions in the variables u, z, and w at points u inside the analyticity region, near the origin of the Borel plane. In many cases the expansions in powers of the optimal variable w approximated the exact functions better than the standard expansion (33) or the series in terms of the variable z. This feature was visible even with a few terms in the expansion.

A nice illustration of the accelerated asymptotic convergence achieved by the expansion (28) is provided by the following test: assume that the first N Taylor coefficients in

TABLE I. The next coefficient of the expansion (33), obtained by reexpressing in powers of u the expansions in powers of w and z truncated at N terms: b_{N+1} is the true Taylor coefficient; $b_{N+1}^{(w)}$ and $b_{N+1}^{(z)}$ denote the values obtained from the expansion in powers of w and z, respectively.

N	b_{N+1}	$b_{N+1}^{(w)}$	$b_{N+1}^{(z)}$
2	0.327	−0.45	−1.208
3	0.778	−0.099	−1.198
4	−0.2100	−0.606	−2.33
6	−0.609	−0.76	−3.20
7	0.897	0.736	−1.896
9	1.144	1.093	−1.964
14	−1.951	−1.952	−5.522
19	2.919	2.919	−0.729

Eq. (33) are known. Using the above procedure, we obtain the expansion (28) in powers of w, containing N terms with known coefficients. Let us now expand this expression back in powers of the variable u, using the expression (27) of w as a function of u. In doing this we of course recover the first N Taylor coefficients of Eq. (33), used as input. But the expansion contains also higher order powers of u, beyond the truncation order N. Since the expansion in powers of w has better convergence than the original expansion (33), i.e., since it contains more information on the exact function [note that $w(u)$ has the same location of singularities as $B(u)$], we expect that the Taylor coefficients of the higher order terms are close to those of the exact function, at least for large N. In particular, the next coefficient b_{N+1} of the Taylor expansion of $B(u)$ is expected to be correctly reproduced by this procedure, using as input only the first N terms of the expansion (33). The implications for the calculation of perturbation terms would be very interesting: one could predict the higher order term with no calculations, just using knowledge of the first terms and the region of analyticity.[1] We tested this conjecture for model functions $B(u)$ of the form (32), for several choices of the parameters. In Table I we give some results, corresponding to a function with two infrared and two ultraviolet renormalons, with the residua $r_1=1$, $r_2=100$, $r_{-2}=4$, $r_3=50$, $\gamma_1=\gamma_2=2.5$, $\gamma_3=\gamma_{-2}=2$ [with this choice the expression (32), normalized to one at $u=0$, has an expansion similar to the physical Borel transform discussed in the previous section]. As seen from Table I, the values of $N\leqslant 3$ of interest for the present situation in perturbative QCD are too low to allow application of the procedure: in other words, for $N\leqslant 3$, there is no hope of obtaining correctly, without an explicit calculation, the next coefficient of perturbation theory, using as input the low order terms. However, starting from $N=7$ the next-order coefficient b_{N+1} is correctly reproduced by the first N terms. The optimal expansion predicts correctly the results of next orders of perturbation theory. Similar, surprisingly good as well, predictions are obtained also for a few next coefficients b_{N+2}, b_{N+3}, ...

[1]We mention that this virtue of conformal mapping was remarked on some time ago in the context of partial wave expansions [39].

On the other hand, the expansion in powers of z [which is not optimal; see Eq. (29)] systematically fails to reproduce even the next-order coefficient, as is seen from the last column of Table I.

As discussed in the previous section, for the physical applications we are interested in the calculation of Laplace integrals such as Eq. (13) or (15). It is therefore necessary to test the usefulness of the optimal conformal mapping for the evaluation of this integral. The principal value (24) of the Laplace integral was computed numerically. The accuracy of the calculations was tested using some specific examples for which analytical formulas exist. We used the relation [40]

$$\lim_{\epsilon\to 0}\int_0^\infty \mathrm{d}u\,\frac{\exp(-u/a_s)}{(u+b+i\epsilon)^\nu}=a_s^{1-\nu}\mathrm{e}^{b/a_s}\Gamma(-\nu+1,b/a_s),\qquad \mathrm{Re}\,a_s>0, \tag{38}$$

where $\epsilon>0$ and $\Gamma(\nu,z)$ is the incomplete gamma function [40], analytically continued from the region $\mathrm{Re}\,z>0$ to the whole complex plane z cut along the negative real axis. For integer ν this can be expressed equivalently as [41]

$$\lim_{\epsilon\to 0}\int_0^\infty \mathrm{d}u\,\frac{\exp(-u/a_s)}{(u+b+i\epsilon)^n}=\frac{\mathrm{e}^{b/a_s}}{b^{n-1}}E_n(b/a_s),\qquad \mathrm{Re}\,a_s>0, \tag{39}$$

in terms of the exponential integral functions $E_n(z)$. Actually, as seen from Eq. (32), in the physical case the denominators must be defined so as to give the correct cut structure of the Borel transform. This case is obtained from Eq. (39) as

$$\lim_{\epsilon\to 0}\int_0^\infty \mathrm{d}u\,\frac{\exp(-u/a_s)}{(|b|-u-i\epsilon)^n}=\frac{\mathrm{e}^{b/a_s}}{-|b|^{n-1}}E_n(b/a_s), \tag{40}$$

where $b=-|b|$.

As a side remark, we mention that the above relations are useful for defining the principal value prescription for arbitrary values of a_s. First, by means of repeated integration by parts in Eq. (39) we can express the left hand side as [40]

$$\int_0^\infty \mathrm{d}u\,\frac{\exp(-u/a_s)}{(u+b+i\epsilon)^n}=\sum_{m=1}^{n-1}\frac{(m-1)!}{(n-1)!}\,\frac{(-a_s)^{1-n+m}}{b^m}+\frac{(-a_s)^{1-n}}{(n-1)!}\int_0^\infty \mathrm{d}u\,\frac{\exp(-u/a_s)}{(u+b+i\epsilon)},\qquad \mathrm{Re}\,a_s>0. \tag{41}$$

We now apply the definition (24) of the principal value and use the known relation

$$\frac{1}{(u+b\pm i\epsilon)}=\mathrm{PV}\frac{1}{(u+b)}\mp i\pi\delta(u+b) \tag{42}$$

in the last term in Eq. (41). We obtain thus the following expression of the principal value:

$$\mathrm{PV}\int_0^\infty \mathrm{d}u\,\frac{\exp(-u/a_s)}{(u+b)^n}=\frac{\mathrm{e}^{b/a_s}}{b^{n-1}}E_n(b/a_s)+i\pi\frac{(-a_s)^{1-n}}{(n-1)!}\mathrm{e}^{b/a_s},\quad \mathrm{Re}\,a_s>0. \tag{43}$$

For real values of a_s the last term in the above relation is purely imaginary and the definition (43) amounts to taking the real part of the right hand side of Eq. (39). For complex a_s the last term in Eq. (43) contains both real and imaginary parts. As discussed below Eq. (24), complex values of a_s appear in the Borel summation of the Green functions in the complex momentum plane or in the timelike region. For some Minkowskian quantities a definition of the principal value, based on physical arguments, was proposed in [14]. The expression (43) is general and covers all these cases. In the present work we used the above relations for testing the numerical evaluation of the principal value (24) of the Laplace integral, for model functions of the form (32).

(2) As a next application, we consider the test proposed by Altarelli, Nason, and Ridolfi. In Ref. [23], by analyzing the conformal mapping (29) in a specific model, these authors concluded that this mapping is useful only if the ultraviolet renormalon is already dominant at the low orders of truncation. An interesting question is whether similar requirements hold for the optimal conformal mapping (27). In order to answer this question we first perform an analysis similar to that presented in [23]. We assume the case when the Borel function is exactly given by

$$B_{true}(u)=1+\tilde{D}_2u+\frac{\tilde{D}_3}{2}u^2+\left[\hat{B}(u)-\sum_{n=0}^{2}\hat{b}_nu^n\right]. \tag{44}$$

In this expression the parameters $\tilde{D}_2$ and $\tilde{D}_3$ are for the moment arbitrary and the numbers $\hat{b}_n$ are the Taylor coefficients of the expansion of $\hat{B}$ around the origin:

$$\hat{B}(u)=\sum_{n=0}^{\infty}\hat{b}_nu^n. \tag{45}$$

The perturbative expression of this model is

$$B_{pert}(u)=1+\tilde{D}_2u+\frac{\tilde{D}_3}{2}u^2. \tag{46}$$

The expansion to the same order in terms of the optimal conformal variable w can be obtained easily using Eq. (34):

$$B_{pert,w}(u)=1+\tilde{D}_2C_1w+\left(\tilde{D}_2C_2+\frac{\tilde{D}_3}{2}C_1^2\right)w^2. \tag{47}$$

We consider also, for comparison, the expansion in terms of the variable (29) used in [23]:

$$B_{pert,z}(u)=1+\tilde{D}_2\overline{C}_1z+\left(\tilde{D}_2\overline{C}_2+\frac{\tilde{D}_3}{2}\overline{C}_1^2\right)z^2, \tag{48}$$

with $\overline{C}_N$ defined in Eq. (36). We introduce now the expressions B_{true}, B_{pert}, $B_{pert,w}$, and $B_{pert,z}$ in the Laplace integral (13) and define the corresponding quantities D_{true}, D_{pert}, $D_{pert,w}$ and $D_{pert,z}$.

Following [23] we consider the ratio

$$H_w=\frac{D_{true}-D_{pert,w}}{D_{true}-D_{pert}} \tag{49}$$

and the similar quantity H_z. Clearly, the inequalities $|H_w|<1$ (or $|H_z|<1$) are the conditions for the accelerated methods based on the conformal mappings w (or z) to be successful. As in [23] we look for the domain in the plane $\tilde{D}_2,\tilde{D}_3$ for which the accelerated methods give better results than the usual perturbation theory. It is easy to see that this domain is a band bounded by the parallel straight lines

$$\tilde{D}_3=s\tilde{D}_2,\quad \tilde{D}_3=s\tilde{D}_2+I. \tag{50}$$

When the denominator in Eq. (49) is positive the slope and the intercept are given by

$$s=-2\frac{\int_0^\infty \mathrm{d}u\,\mathrm{e}^{-u/a_s}[u-C_1w-C_2w^2]}{\int_0^\infty \mathrm{d}u\,\mathrm{e}^{-u/a_s}[u^2-C_1^2w^2]} \tag{51}$$

and

$$I=-4\frac{\int_0^\infty \mathrm{d}u\,\mathrm{e}^{-u/a_s}\left[\hat{B}(u)-\sum_0^2\hat{b}_nu^n\right]}{\int_0^\infty \mathrm{d}u\,\mathrm{e}^{-u/a_s}[u^2-C_1^2w^2]}, \tag{52}$$

respectively. All the integrals are evaluated with the principal value prescription (24). The above relations correspond to the conformal mapping (27), with the coefficients C_n from Eq. (35). Similar relations define the slope and the intercept for the conformal mapping (29).

As seen from Eq. (51) the slope depends uniquely on the conformal mapping and not on the details of the model function. On the other hand, the intercept of the second line in Eq. (50) depends on the choice $\hat{B}(u)$. To illustrate this model we chose for $\hat{B}(u)$ the large β_0 expression [14]

$$\hat{B}(u)=\mathrm{e}^{5u/3}B_V(u), \tag{53}$$

with

$$B_V(u)=\frac{B_0(2)}{(1-u/2)}+\sum_{l=1}^{N_{UV}}\frac{A_0(l)+A_1(l)u}{(1+u/l)^2}+\sum_{l=3}^{N_{IR}}\frac{B_0(l)+B_1(l)u}{(1-u/l)^2}. \tag{54}$$

The exponential factor in Eq. (53) is required to pass from the V scheme to the $\overline{\mathrm{MS}}$ with $\mu=Q$ [14]. The coefficients entering Eq. (54) are [14]

$$A_0(l)=\frac{8}{3}\,\frac{(-1)^{l+1}(3l^2+6l+2)}{l^2(l+1)(l+2)^2},$$

$$A_1(l)=\frac{16}{3}\,\frac{(-1)^{l+1}(l+3/2)}{l^2(l+1)(l+2)^2},$$

$$B_0(2)=1,$$

$$B_0(l)=-A_0(-l),\quad l\geqslant 3,$$

$$B_1(l)=-A_1(-l),\quad l\geqslant 3. \tag{55}$$

With the choice (53), the intercept was rather stable when increasing the number of terms N_{UV} and N_{IR} in the expansion. For $N_{UV}=4$ and $N_{IR}=6$ the domain of interest is defined by the lines

$$\tilde{D}_3=0.96\tilde{D}_2,\quad \tilde{D}_3=0.96\tilde{D}_2-40.9, \tag{56}$$

in the case of the optimal mapping (27), and

$$\tilde{D}_3=-0.30\tilde{D}_2,\quad \tilde{D}_3=-0.30\tilde{D}_2-14.1, \tag{57}$$

for the conformal mapping z used in [23].

As already remarked in [23], the conformal mapping (29) brings no improvement when the low order coefficients $\tilde{D}_2$ and $\tilde{D}_3$ are both positive, as is the case of physical interest [see Eq. (9)]. On the other hand, we note that pairs of positive $(\tilde{D}_2,\tilde{D}_3)$ exist in the band defined in Eq. (56) [the point of coordinates $\tilde{D}_2=0.724$, $\tilde{D}_3=1.23$ obtained from Eqs. (6) and (11) is close to the upper boundary of this domain]. We recall that we consider now only the effect of the conformal mappings, without additional information about the nature of the first singularities of the Borel transform. The mapping used in [23] is useful only when the low order terms are dominated by the ultraviolet renormalons. On the other hand, the optimal mapping, which takes into account the presence of both types of leading renormalons, can be useful also when the low order terms are dominated by the infrared renormalons, as seems to be the physical case.

(3) While the optimal conformal mapping leads to the fastest *asymptotic* convergence rate, it may be interesting to confront it with other methods by looking at their *finite-order* properties. (We saw in the previous analysis that the improvement might be very small at the lowest orders.) We therefore tested the efficiency of the different conformal mappings in approximating the Laplace integral, for various values of the coupling constant a_s and various truncation orders N. We must say that no analytic proof exists ensuring that the optimal variable gives also the best approximation of the Laplace integral, for Borel transforms with singularities along the real axis which make the integral ill defined. One nevertheless expects that the optimal conformal mapping will provide better results, at least for moderate N, since the

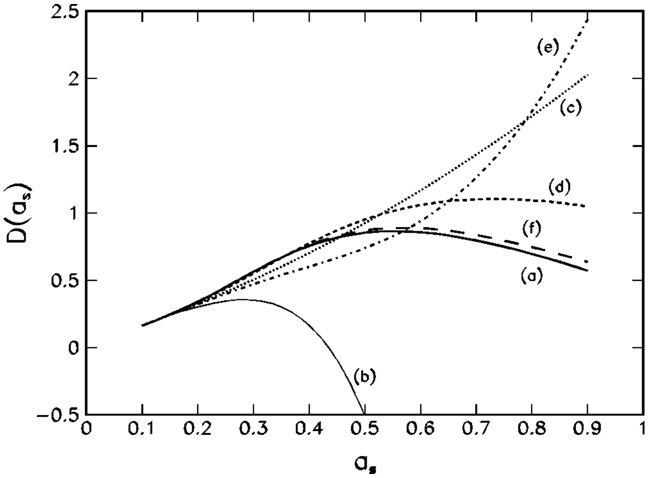

FIG. 1. Laplace integral for the model (32) as a function of a_s: exact values (a), perturbation expansion (b), expansion in powers of z (c), improved expansion in powers of z (d), expansion in powers of w (e), improved expansion in powers of w (f). The series are truncated at $N=3$.

corresponding expansion converges up to points very close to the boundary. Of course, for very large truncation orders the presence of the boundary singularities will manifest in a more dramatic way, and the expansion in powers of w will fail. But it is reasonable to expect that this critical truncation order will be much higher than the order corresponding to the usual variable u. In the present paper we performed a numerical analysis which confirmed these expectations. We evaluated the Laplace integral, with the generalized principal value prescription defined in Eq. (24), for a number of model functions, in particular of the form (32). In Fig. 1 we give for illustration the results of our analysis for the model function (32) with the parameters: $r_1=1$, $r_2=4$, $r_n=0$, $n\geqslant 3$, $\gamma_1=2.5$, $\gamma_2=2.5$, and a_s in the range (0.1–0.9). The Laplace integral of the exact function (32) is indicated together with the results given by its ordinary perturbative expansion (truncated at $N=3$), and the expansions accelerated by the conformal mappings z and w, both in the simple versions and with the improvement explained in Eq. (30). In the present case we assumed that the nature of the first singularities is not exactly known and used in the improved version (30) the product of $B(u)$ with the factors $(u+1)^{2.0}(2-u)^{2.0}$, which do not compensate exactly the singularities of the model function (32).

As seen from Fig. 1, the combined technique of conformal mapping and the explicit treatment of the branch points, supposing that some (approximate) information about the behavior of the function near the first singularities is available, improves the accuracy of the Borel integral, especially for large a_s. The values $a_s\approx 0.2$–0.3 (of interest in the hadronic τ decay) are on the boundary of the region for which the improvement is significant at this order, $N=3$. We notice that a major part of the improvement is brought by the separate treatment of the branch points, according to Eq. (30), especially at low N. Even the standard expansion in powers of the Borel variable u gives good results if the nature of the lowest singularities, assumed to be exactly known, is treated

TABLE II. Approximations of the Laplace integral for $a_s=0.3$ for different truncation orders: (a) expansion in powers of u, (b) ''improved'' expansion in powers of u, (c) expansion in powers of z, (d) ''improved'' expansion in powers of z, (e) expansion in powers of w, (f) ''improved'' expansion in powers of w. The exact value is $D(0.3)=0.563683$. The results close to the exact value are indicated with a greater number of digits.

N	(a)	(b)	(c)	(d)	(e)	(f)
3	0.35	0.530	0.505	0.546	0.471	0.5596
4	0.90	0.610	0.540	0.5625	0.642	0.5701
5	−0.10	0.5732	0.547	0.5722	0.5457	0.5701
6	2.2	0.516	0.5631	0.5783	0.613	0.56518
7	−3.6	0.82	0.5743	0.582	0.634	0.56518
8	13	−0.12	0.587	0.584	0.515	0.563102
9	−40	2.3	0.599	0.584	0.638	0.563100
10	143	−3.4	0.613	0.583	0.503	0.563470
11	−541	9.5	0.628	0.581	0.540	0.563467
12	2×10^3	−19	0.645	0.5776	0.590	0.563783
15	-2×10^5	199	0.72	0.56400	0.5783	0.563713
20	2×10^9	-8×10^3	1.1	0.530	0.582	0.563689
25	-4×10^{13}	3×10^5	2.7	0.484	0.5687	0.563681
30	2×10^{18}	-5×10^7	13	0.428	0.559754	0.563682

explicitly as above. However, when the behavior near the first singularities is known only approximately, the expansion in the optimal variable gives in general the best approximation, especially when the order N of the truncation is increased. We illustrate this fact in Table II, where we indicate the Laplace integral, for $a_s=0.3$, of the model (32) specified above, as a function of the truncation order N, for different types of expansions. The ''improved'' expansions were obtained now by expanding in powers the product of $B(u)$ with the factors $(u+1)^{1.5}(2-u)^{1.5}$, close but not identical to the actual behavior of Eq. (32). For larger values of a_s the improved accuracy obtained by using the optimal mapping is even more impressive. Some results are presented in Table III for $a_s=0.5$.

Similar results are obtained for model functions with subleading renormalons, which simulate closer the physical situation. A number of such functions were analyzed, and we present in Tables IV and V some results for the function considered in the test discussed in Table I. The ''improved''

TABLE III. The same as in Table II, for $a_s=0.5$. The exact value is $D(0.5)=0.853427$.

N	(a)	(b)	(c)	(d)	(e)	(f)
3	−0.507	1.04	0.925	1.01	0.739	0.967
5	−15	1.9	1.22	1.05	1.7	0.87689
10	4×10^4	−75	2.4	0.8412	2.2	0.857092
12	2×10^6	349	3.6	0.678	1.8	0.853068
15	-9×10^8	3×10^3	8.2	0.371	1.56	0.852614
20	8×10^{13}	-1×10^5	56	−0.27	2×10^{-3}	0.853263
25	-2×10^{19}	1×10^8	597	−1.06	0.590	0.853463
30	1×10^{25}	-3×10^{13}	8×10^3	−2.0	1.51	0.853438

TABLE IV. The same as in Table II, for $a_s=0.3$ and a different model function (see the text). In the improved expansion the actual behavior near the branch points is included. The exact value is $D(0.3)=0.4809$.

N	(a)	(b)	(c)	(d)	(e)	(f)
2	0.4358	0.4812	0.4065	0.4575	0.4312	0.4712
5	0.4787	0.4817	0.4646	0.4814	0.5084	0.4838
10	6.4378	0.4806	0.5291	0.4813	0.4633	0.4811
20	6.37×10^7	−0.280	0.9654	0.4807	0.4783	0.4809

expansions are now defined by multiplying with factors which exactly compensate singularities near the first branch points. As expected, in this case the benefic role of the ''improvement'' is more pronounced for all the expansions, and the deviation of the standard series from the true result begins at a higher N.

A closer look at the Tables II–V reveals that there are essentially three circumstances affecting the convergence properties: (i) the use of a convenient (including the optimal) conformal mapping, (ii) explicit (even approximative) account of the branch point singularities, and (iii) exponential damping of the integrand by $\exp(-u/a_s)$.

The effect of the factor (i) can be seen from the fact that, in each of the tables, column (c) possesses better convergence properties than column (a), and column (e) has better properties than column (c). The effect of (ii) is seen from the fact that, again in all the tables, columns (f), (d), and (b) have better convergence properties than columns (e), (c), and (a), respectively. As concerns point (iii), the salutary effect of the optimal conformal mapping is more important when the damping of the exponential function $\exp(-u/a_s)$ is weaker, i.e., when a_s is larger. The results obtained with a_s larger than 0.5 confirm this effect in a spectacular way (we do not give these results here, since they are not of physical relevance in our case).

In the case of a stronger damping (Tables II and IV), the role of the optimal conformal mapping combined with the treatment of the branch point singularities is again important, but good results are obtained also by the other methods, the success varying with the perturbation order N used. The asymptotic superiority of the optimal mapping [columns (e) and (f)] emerges at very high values of N; this mapping supersedes the other methods and turns out to be the best at least starting from $N=20$. It is not excluded, on the other hand, that even the best series, column (f), will exhibit numerical indications of divergence at still higher orders, due

TABLE V. The same as in Table IV, for $a_s=0.5$. The exact value is $D(0.5)=0.8038$.

N	(a)	(b)	(c)	(d)	(e)	(f)
2	0.9816	0.7938	0.8106	0.7911	0.9597	0.7607
5	1.2945	0.8056	1.1759	0.805	1.251	0.7960
10	1736.28	0.7829	2.3563	0.8018	0.6631	0.8106
20	1.34×10^{12}	−14.29	55.419	0.810	0.9880	0.8083

to the singularities of the Borel function along the integration path. Consequently, as we already mentioned, since the Borel integral path runs along the cut, which in the w plane is mapped onto the *boundary* circle of the convergence disk, no convergence is warranted even in column (f).

Concluding this discussion we mention that, as is seen from Fig. 1 and Tables II–V, for realistic values of a_s between 0.2 and 0.3 and $N=3$ the numerical difference between the two conformal mappings is small, and so is the difference between whether or not the known singularities are explicitly taken into account. The reason is that, at high values of u, where the difference is important, the integrand of Eq. (13) is, for small values of a_s, strongly suppressed by the exponential factor. The difference becomes visible when a_s and/or N are big enough. Tables II–V show how this effect develops with increasing N.

C. Determination of $\alpha_s(m_\tau^2)$ from τ decay

As a final application of the method we discuss the determination of the strong coupling constant $\alpha_s(m_\tau^2)$ from the hadronic τ decay width. It is known that the theoretical error is at present the dominant ambiguity in this determination, and the main source of this error arises from higher orders in perturbation theory. This makes the hadronic τ decay a very suitable place to apply the technique of conformal mapping, which accelerates the convergence of the perturbative expansion and reduces the truncation error. As we mentioned, this problem was studied previously in [23], where the conformal mappings were used to reduce only the effect of the UV renormalons. It is of interest to use also the optimal conformal mapping, whose properties were demonstrated on mathematical models. We do not attempt to make here a complete analysis of $\alpha_s(m_\tau^2)$ determination, but only point out the effect of the combined technique of optimal conformal mapping and the implementation of the correct behavior of the Borel transform near the first singularities.

We used as starting point the Borel sum (15) of R_τ and evaluated this expression using both the standard Taylor expansion (11) of the Borel transform in powers of u and the optimized expression (30) proposed by us. For comparison with previous work, we notice that the ''standard expansion'' in our approach is equivalent to the method of integration along the circle proposed in [28], in the particular case of the one-loop running coupling. The expansions were truncated at $N=2$, with the coefficients b_n determined from Eqs. (9) and (12). In the improved expansion (30) we used the values γ_1 and γ_2 given in Eqs. (22) and the coefficients $\hat{c}_n$ were computed such as to reproduce the first three coefficients b_n from (12).

In Fig. 2 we give the results corresponding to the standard Taylor expansion (11) of the Borel transform [curve (a)] and the improved expansion (30) [curve (b)], for various values of $\alpha_s(m_\tau^2)$. Using the experimental value [31]

$$(R_\tau)_{expt}=3.645\pm 0.024, \tag{58}$$

we obtain

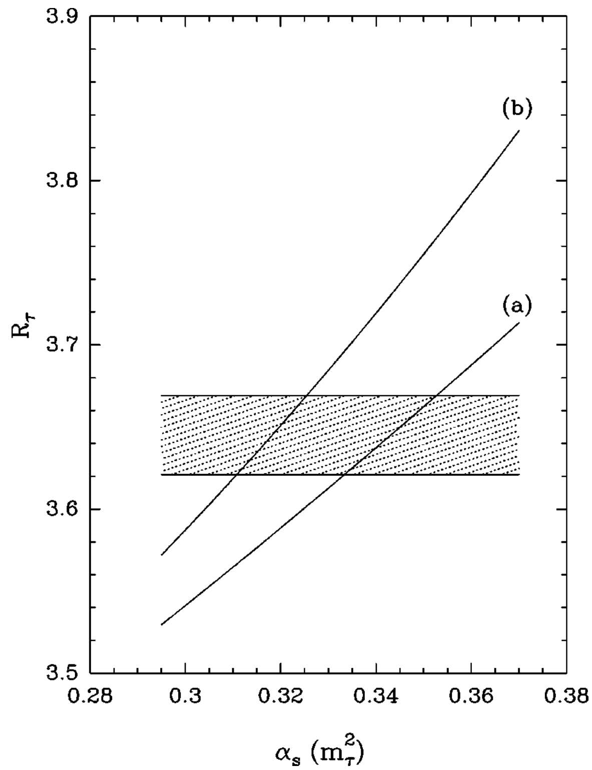

FIG. 2. The Borel summation (15) of R_τ, using the standard Taylor expansion (a) and the improved expansion (30) (b), as functions of $\alpha_s(m_\tau^2)$. The band indicates the experimental values.

$$\alpha_s(m_\tau^2)=0.343\pm(0.009)_{expt}, \tag{59}$$

using the standard Taylor expansion (11), and

$$\alpha_s(m_\tau^2)=0.318\pm(0.007)_{expt}, \tag{60}$$

using the optimized expression (30). The improved expansion leads to a value of the coupling constant $\alpha_s(m_\tau^2)$ lower by about 8% than the result given by the standard Taylor expansion of the Borel transform. Actually, as in the above discussion of the model functions at low N and similar values of α_s, the major contribution in shifting the value of α_s towards smaller values is brought by the explicit treatment of the first singularities of the Borel transform. At the small values of α_s relevant for the present problem, the effect of the conformal mapping is barely seen.

In Eqs. (59) and (60) we indicated only the experimental error, which is very small. On the other hand, it is not easy to ascribe a definite theoretical error to these results. The problem of the theoretical error of $\alpha_s(m_\tau^2)$ was discussed in many papers, in particular in [17,23,28–30], with different conclusions about its magnitude. One can safely neglect the effect of the uncertainties in the QCD parameters (quark masses, gluon condensate, etc.), which is small [27,29] (leaving aside the still open problem of the $1/s$ corrections). The ambiguities related to the prescription chosen for computing the Laplace integral are believed to be small too, due to the conjecture that these ambiguities must be compensated by corresponding ambiguities in the condensates. The most important sources of theoretical error remain therefore those related to the analytic continuation from the Euclidian to the Minkowskian region, and the truncation of the perturbative expansion. A complete discussion of these errors is outside the objective of this paper. Concerning the analytic continuation, we only mention that in the derivation of Eq. (15) the perturbative expansion of the Adler function was assumed to be equally valid in the Euclidian region and in the complex plane near the timelike axis, which is certainly not true. It is not trivial to relax this assumption and see its impact on the determination of $\alpha_s(m_\tau^2)$. As concerns the truncation error, the estimate $\delta\alpha_s(m_\tau^2)\simeq 0.05$ was suggested in [23] and [29], by comparing the predictions of different summation procedures. In [30] it was claimed on the other hand that much smaller errors are obtained if the renormalization group invariance of the perturbation series is exploited in an optimal way. The present work points towards a similar conclusion: indeed, as was remarked also in [17], it is rather arbitrary to interpret the spread of the results produced by different conformal mappings as a measure of the theoretical error, as suggested in [23]. Our investigation of mathematical models shows that the truncation error depends on the choice of the conformal mapping, being smaller if more information on the analyticity of the function is taken into account. The expansion proposed in our work exploits in an optimal way the (renormalization group invariant) information on the first renormalons of the Borel transform, and we therefore expect that the truncation error of the result (60) is smaller than the estimate given above.

IV. CONCLUSIONS

The technique of the optimal conformal mapping of the Borel plane, discussed in this paper, can be seen as an alternative resummation of higher order effects in perturbative QCD. This resummation method has a physical content in the sense that the requirement of convergence in powers of the optimal variable $w(u)$ amounts to a statement on *analyticity in the whole double-cut Borel plane*. Indeed, the theorem [22] on the asymptotic rate of convergence of power series, on which it is based, is dependent upon the condition that the function $f(u)$ (which is expanded) and the function $w(u)$ [in powers of which $f(u)$ is expanded; see Eq. (26)] should have the same location of singularities. The method of the optimal conformal mapping allows us to make full use of this analyticity property.

This remarkable feature is lost if the function is expanded in powers of some other variable, be it u or a conformal mapping of u such that only a part of the analyticity domain is mapped inside the convergence circle. In such cases, the convergence domain is smaller than the region of analyticity, and the requirement of convergence has to be supplemented with the analyticity condition. Only in the case of the optimal mapping the two regions are identical.

As renormalons express the properties of the Feynman diagrams of the process, a statement about their location implies a statement about the physics of the process considered.

If the power expansion is truncated at a definite order, as is the case in practice, the roles of u and $w(u)$ are modified. While a polynomial in u is holomorphic in u and has no singularites in the Borel plane, a polynomial in w has the same analyticity region as the expanded function, having the cuts equally located. Since singularities have physical interpretation, every polynomial in $w(u)$ carries this piece of information.

We demonstrated the practical role of the optimized expansion numerically on a large number of model functions. We found that the expansion in the optimized variable allows one to predict, for a sufficiently large truncation order, the $N+1$ coefficient of the perturbation series, using as input the first N coefficients. For functions resembling the physical Borel transform this procedure works starting from $N\geqslant 7$. We investigated the usefulness of the optimized expansion for the evaluation of the Laplace integral and found that it is noticeable if the coupling constant is large and the exponential damping of the integrand is weak. In these cases knowledge (even approximate) of the behavior near the first renormalons, combined with the expansion in the optimal variable, leads to very accurate results, while the expansion in the Borel variable, though partially improved by the treatment of the branch points, fails dramatically. On the other hand, at low orders of perturbation expansion and for values of the coupling constant of physical interest the effect of the optimal conformal mapping is not very visible and the predominant effect is given by the explicit treatment of the nearest branch points. This was actually the case with the determination of the strong coupling constant $\alpha_s(m_\tau^2)$ from the hadronic τ decay width: the combined technique of conformal mapping and the explicit treatment of the first branch points of the Borel transform reduces by about 8% the value given by the usual Taylor expansion in the Borel variable. The major contribution to this result is brought by the theoretical information [10,20] about the nature of the first renormalons.

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