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Free energy of bubbles and droplets in the quark-hadron phase transition

Gregers Neergaard and Jes Madsen
Institute of Physics and Astronomy, University of Aarhus, DK-8000 Århus C, Denmark
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Using the MIT bag model, we calculate the free energy of droplets of a quark-gluon plasma in a bulk hadronic medium, and of hadronic bubbles in a bulk quark-gluon plasma, under the assumption of vanishing chemical potentials. We investigate the validity of the multiple reflection expansion approximation, and we devise a novel procedure for calculating finite-size corrections to the free energy of hadronic bubbles in a bulk quark-gluon plasma. While our results agree largely with earlier calculations, we show that the usual multiple reflection expansion should be used with caution, and we propose a modification of the multiple reflection expansion, which makes this approximation agree nicely with direct numerical calculations. The results should be of relevance in connection with the cosmological quark-hadron transition as well as for ultrarelativistic heavy ion collisions. [S0556-2821(99)03815-1]

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I. INTRODUCTION

The quark-hadron phase transition is of significant interest in connection with ultrarelativistic heavy ion collision experiments, the interior of neutron stars, and the evolution of the early Universe. A calculation from first principles using QCD is at present impossible, but lattice-QCD studies have shed some light on the transition, for instance, demonstrating that the transition is apparently first order for pure glue, whereas the order for physical QCD is still a matter of investigation.

Awaiting more definite answers to come from such investigations, numerous studies have been performed using phenomenological models in order to gain insight into the physics of the transition. Many such studies have used the MIT bag model, which in a relatively simple manner incorporates confinement in terms of a set of boundary conditions for quarks and gluons.

A very interesting result of a detailed study within the MIT bag model was presented by Mardor and Svetitsky [1], who considered the zero chemical potential case of relevance for the cosmological quark-hadron transition. For a droplet of quark-gluon plasma within a bulk medium of pions, a direct numerical calculation of the partition sum using quark and gluon energy levels led to a behavior of free energy as a function of radius $F(R)$ as expected for a first order transition, namely, a minimum of F for $R=0$ when T is below the transition temperature T_0 and an energy barrier for R of order a few fm separating a local minimum at $R=0$ from the true minimum (diverging negative energy) for $R\rightarrow\infty$.

To treat the ''inverse'' problem of a vacuum (hadron) bubble within a bulk phase of a quark-gluon plasma, the authors employed a phase shift formula to calculate the changes in quark and gluon density of states stemming from the presence of the hadron bubble; again calculating the contribution to the free energy by a direct numerical integration. In this case, a peculiar feature was observed, namely, that $F(R)$ had a negative minimum for radii of 1–2 fm, even for $T>T_0$, apparently indicating an instability of the quark-gluon plasma above T_0, since there was no energy barrier to prevent formation of hadron bubbles.

An interpretation of the result was put forward in terms of an expansion of the free energy in terms of volume, surface, and curvature contributions:

$$F(R)=\Delta P\frac{4}{3}\pi R^3+\sigma 4\pi R^2+\alpha 8\pi R+\cdots. \tag{1}$$

Here R is the radius of the droplet or bubble, ΔP is the pressure difference between quark and hadron phases (with $\Delta P=0$ defining the transition temperature T_0), σ is the surface tension, and α the curvature coefficient, where volume, surface, and curvature terms can be calculated from the smoothed quark and gluon densities of state within the MIT bag model (see below). The results of Ref. [1] were apparently well reproduced under the assumption that a vacuum bubble behaves as a plasma droplet turned inside out, so that the radius changes sign. This leaves the area term unchanged, but volume and curvature terms change sign. Or, if R is defined to be positive in the expression above, then σ is unchanged, but pressure difference as well as curvature coefficient changes sign when going from the case of a plasma droplet to that of a vacuum bubble. The fact that the curvature contributions from massless quarks and gluons are very large compared to the surface contributions coming only from the massive s quarks, could explain that an energy barrier for the plasma droplet could turn into an energy minimum in the reverse case of a hadron bubble.

The authors of Ref. [1] were careful to point out a number of reasons to be cautious about the result. First of all that the MIT bag model is clearly just a phenomenological model, and also that the radii of relevance for the interesting bubbles and droplets were perhaps too small to justify the ideal gas approximations. But if the result was of a physical nature, it did have important implications for the understanding of the quark-hadron transition [1–4]. And the procedure of $R\rightarrow -R$ gave a simple recipe for treating other situations, such as quark-hadron mixed phases in neutron stars. Some consequences, however, were rather strange. For instance, it apparently pays energetically to fill a strangelet with vacuum bubbles, so that it looks more similar to a Swiss cheese than a uniform mixture of quarks [5].

The aim of the present paper is to compare several different calculations of the free energy of a vacuum bubble embedded in quark-gluon plasma as well as a quark-gluon plasma droplet within a bulk phase of hadrons. For the plasma droplet we focus on a direct sum over states compared with a multiple reflection expansion, showing that terms beyond volume, surface, and curvature in the free energy are necessary to avoid unphysical behavior for small radii. We demonstrate how the next important contributions to F [proportional to $T\ln(RT)$ and T] arise naturally if the density of states is truncated below some value of kR (where k denotes momentum) instead of integrating over unphysical, negative values for the density of states all the way from $k=0$. For the vacuum bubble we confirm the results of the phase shift approach of Ref. [1] by comparing to a more direct sum over states approach introduced below, which we refer to as the *concentric spheres method*. Again we show how an improvement of the multiple reflection expansion leads to correction terms in F, such terms arising naturally from a truncation of the density of states.

The general framework and basic equations are described in Sec. II. In Sec. III we present our numerical results, largely confirming the calculations of Ref. [1]. Our results show how and why the usual version of the multiple reflection expansion is not always accurate. In Sec. IV we show how further terms in the analytical expansion of the free energy proposed in the literature improves the agreement with the numerical results, and we show how a physically motivated truncation of the density of states from the multiple reflection expansion resolves most of the problems encountered in Sec. III. Section V contains our conclusions.

II. THEORETICAL FRAMEWORK

In this section, we give the basic equations needed for an analysis of the quark-hadron phase transition within the MIT bag model. We consider the case of zero chemical potential which is of particular relevance to the cosmological quark-hadron transition, but also of interest for ultrarelativistic heavy ion collisions. The quark-gluon plasma is taken to consist of three quark flavors (u, d, and s), the corresponding antiquarks, and eight noninteracting gluons, these particles being described by the MIT bag model presented below. The hadron phase is considered a mixture of the three pions $\pi^0,\pi^\pm$, since all other (much heavier) hadrons contribute only insignificantly to the free energy. Further, we shall assume that the pions contribute volume terms only (see Sec. II B). We have taken $m_u=m_d=0$, $m_s=150$ MeV, and $m_\pi=138$ MeV.

A. The MIT bag model

The MIT bag model [6,7] is defined by the Lagrangian

$$L=\int_\Omega d^3x(\mathcal{L}_{\rm QCD}-B). \tag{2}$$

$\mathcal{L}_{\rm QCD}$ is the usual QCD Lagrangian density, and $L=0$ outside the bag. Ω is the bag volume. $B>0$ is a phenomenological parameter, the bag constant, which models the difference in energy density between the perturbative vacuum inside the bag and the nonperturbative QCD vacuum outside the bag. Requiring the action $W=\int_{t_1}^{t_2}dtL$ to be stationary with respect to variations of the fields yields the equations of motion.

At the surface of the bag, the fields are taken to satisfy boundary conditions which correspond to the fields being confined inside the bag volume. We neglect gluon-exchange interactions.

The equations of motion for the fields become the Dirac equation for the quark fields and the source-free Maxwell equations for the gluon fields. The complete set of equations governing the behavior of the fields, including the boundary conditions, is

$$(i\gamma^\mu\partial_\mu-m)\Psi(x)=0,\quad \vec{x}\in\Omega, \tag{3}$$

$$\partial_\mu F^{\mu\nu}(x)=0,\quad \vec{x}\in\Omega, \tag{4}$$

$$in_\mu\gamma^\mu\Psi(x)=\Psi(x),\quad \vec{x}\in\partial\Omega, \tag{5}$$

$$n_\mu F^{\mu\nu}(x)=0,\quad \vec{x}\in\partial\Omega, \tag{6}$$

in the following notation: $x=(x^0,\vec{x})$ is a space-time four-vector, $\partial\Omega$ is the surface of the bag volume Ω, for $\vec{x}\in\partial\Omega$ we define $n^\mu(x)=(0,-\vec{x}/|\vec{x}|)$ as an inward-directed unit-normal three-vector to the surface of the bag, $\Psi(x)$ is the quark-spinor [there will be one for each quark flavor (u, d, s, . . .) and one for each of the three color states of a quark], and $F^{\mu\nu}(x)=\partial^\mu A^\nu(x)-\partial^\nu A^\mu(x)$ is the (noninteracting) gluon field (there are eight copies of this field).

We fix the bag constant by demanding bulk pressure balance at the transition temperature. Somewhat symbolically the bag constant is thus determined by the equation

$$B=\lim_{V\to\infty}\left\{-\frac{\partial F_{\rm quarks}}{\partial V}-\frac{\partial F_{\rm gluons}}{\partial V}+\frac{\partial F_{\rm pions}}{\partial V}\right\}_{T=T_0}, \tag{7}$$

F being the free energy. In the following, we shall set the transition temperature to $T_0=150$ MeV, thus fixing the bag constant $B=312.6\ \ {\rm MeV/fm^3}=(221.4\ \ {\rm MeV})^4$.

We can immediately write down the expression for the gluon field, since this is just the solution to the source-free Maxwell equations. Expressing the gluon field in terms of color-electric and color-magnetic fields, writing $A^\mu(x)=[V(x),\vec{A}(x)]$, $\nabla\times\vec{A}=\vec{B}(x)$, and $-\nabla V(x)-\partial\vec{A}(x)/\partial x^0=\vec{E}(x)$, there are two sets of solutions to Eq. (4), labeled TM and TE (the $l=0$ fields are absent since for $l=0$, the only solution to the source-free Maxwell equations is $\vec{B}_{00}=\vec{E}_{00}=0$ [8]):

$$\{\vec{B}_{lm}^{TM}e^{-x^\mu k_\mu},\vec{E}_{lm}^{TM}e^{-x^\mu k_\mu}\}_{m=-l,-l+1,\ldots,l}^{l=1,2,\ldots}, \tag{8}$$

$$\{\vec{B}_{lm}^{TE}e^{-x^\mu k_\mu},\vec{E}_{lm}^{TE}e^{-x^\mu k_\mu}\}_{m=-l,-l+1,\ldots,l}^{l=1,2,\ldots}, \tag{9}$$

where

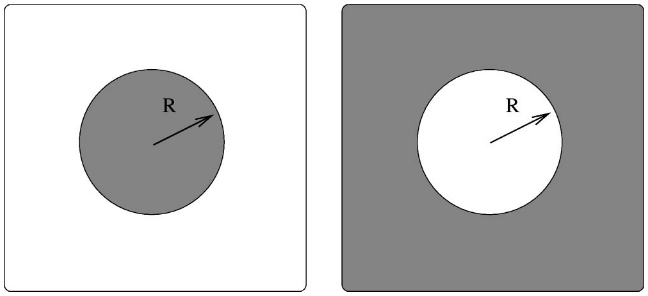

FIG. 1. Left: A plasma droplet with nonperturbative vacuum outside; this is essentially an MIT bag. Right: A vacuum bubble surrounded by plasma, the boundary conditions being those of the MIT bag, but corresponding to no flux of plasma *into* the bubble. The phase outside a droplet or bubble extends to infinity.

$$\vec{B}_{lm}^{TM} = -i f_l(kr)\vec{x}\times[\nabla Y_{lm}(\theta,\phi)] \tag{10}$$

$$-i\nabla\times[\vec{x} f_l(kr) Y_{lm}(\theta,\phi)], \tag{11}$$

$$\vec{E}_{lm}^{TM} = \frac{i}{k}\nabla\times\vec{B}_{lm}^{TM}(x) \tag{12}$$

$$= \frac{1}{k}\nabla\times\nabla\times[\vec{x} f_l(kr) Y_{lm}(\theta,\phi)], \tag{13}$$

$$\vec{E}_{lm}^{TE} = -i f_l(kr)\vec{x}\times[\nabla Y_{lm}(\theta,\phi)] \tag{14}$$

$$= -i\nabla\times[\vec{x} f_l(kr) Y_{lm}(\theta,\phi)], \tag{15}$$

$$\vec{B}_{lm}^{TE} = -\frac{i}{k}\nabla\times\vec{E}_{lm}^{TE}(x) \tag{16}$$

$$= -\frac{1}{k}\nabla\times\nabla\times[\vec{x} f_l(kr) Y_{lm}(\theta,\phi)], \tag{17}$$

$Y_{lm}(\theta,\phi)$ are the usual spherical harmonics, $f_l(z) = a j_l(z) + b n_l(z)$, the spherical Bessel-functions $j_l(z)$ and $n_l(z)$ being the two linearly independent solutions of the equation

$$z^2 g''(z) + 2z g'(z) + [z^2 - l(l+1)]g(z) = 0,$$

$r = |\vec{x}|$, and $k^\mu = (|\vec{k}|,\vec{k})$. The constants a and b appearing in the function f_l, and the possible values of k, must be fixed from the boundary conditions (6). Expressed in terms of the fields $\vec{E}$ and $\vec{B}$, these boundary conditions read

$$\vec{x}\cdot\vec{E} = \vec{x}\times\vec{B} = 0, \quad \vec{x}\in\partial\Omega. \tag{18}$$

Extension of the MIT bag model. The MIT bag is a finite region of space(-time) to which quarks and gluons are confined by boundary conditions (5),(6) corresponding to no flux of plasma out of the droplet. We shall refer to this configuration as a plasma droplet. However, in the following we shall also use the MIT bag model in a slightly different way, namely, the case where quarks and gluons are kept *outside* a finite region of space, see Fig. 1. The equations describing the quarks and gluons in the second configuration, the ''vacuum bubble,'' are still Eqs. (3)–(6), but now using $n^\mu(x) = (0,\vec{x}/|\vec{x}|)$ in the boundary conditions.

B. Thermodynamical relations

For a system of noninteracting fermions (upper sign) or bosons (lower sign) we can calculate the free energy for each particle degree of freedom as

$$F(T,V) = \mp T\sum_{i=1}^{\infty} \ln(1 \pm e^{-E_i(V)/T}), \tag{19}$$

where $E_i(V) = \sqrt{m^2 + k_i^2(V)}$. In the continuum case we have

$$F(T,V) = \mp T\int d^3k\,\tilde{\rho}(\vec{k},V)\ln(1 \pm e^{-\sqrt{m^2+k^2}/T}), \tag{20}$$

$\tilde{\rho}(\vec{k},V)$ being the density of states, defined such that $\tilde{\rho}(\vec{k},V)d^3k$ is the number of states in the volume V with momentum in d^3k around $\vec{k}$. The importance of the free energy stems from the fact that the configuration realized in nature is characterized by a minimum in this free energy.

In the following we shall speak of the volume part resp. the surface part of the free energy. In the case of noninteracting Dirac particles and noninteracting gluons (these are just Maxwell fields) which are the particle species relevant to us, the density of states $\tilde{\rho}$ in any sufficiently large volume V contains a term proportional to the volume. In fact,[1]

$$\tilde{\rho}(V\to\infty) \simeq g_i \frac{V}{8\pi^3} \tag{21}$$

independent of which particle species we consider. Therefore, also the free energy will contain a term proportional to the volume of the system. We name this term the *volume free energy*. The total free energy being F_{tot}, we can write $F_{\mathrm{tot}} = f_{\mathrm{vol}}V + F_{\mathrm{sur}}$, where $F_{\mathrm{sur}}/V\to 0$ as $V\to\infty$, and f_{vol} does not depend on the volume. We shall call F_{sur} the surface part of the free energy, or simply the *surface free energy*.

C. The multiple reflection expansion, MRE

The multiple reflection expansion (MRE) is an approximation for the density of states, also commonly referred to as the asymptotic expansion of the density of states. Since we only consider systems with spherical symmetry, we define the spherically symmetric density of states $\rho(k,V) \equiv 4\pi k^2\tilde{\rho}(\vec{k},V)$. Consider a spherical volume $V = (4\pi/3)R^3$ of quarks and gluons, described by the bag model. The MRE for this system, as a sum of volume, area, and curvature contributions, valid for sufficiently large volumes, is

[1] g_i accounts for spin (helicity) degeneracy.

$$\rho_i(k,V)=\frac{Vk^2}{2\pi^2}+f_{A,i}(k/m)k4\pi R^2 + f_{C,i}(k/m)8\pi R+\cdots, \quad i=q,g, \tag{22}$$

where

$$f_{A,q}(k/m)=-\frac{1}{8\pi}\left(1-\frac{2}{\pi}\arctan(k/m)\right), \tag{23}$$

$$f_{C,q}(k/m)=\frac{1}{12\pi^2}\left[1-\frac{3k}{2m}\left(\frac{\pi}{2}-\arctan(k/m)\right)\right], \tag{24}$$

$$f_{A,g}=0, \tag{25}$$

$$f_{C,g}=-\frac{1}{6\pi^2}. \tag{26}$$

Here, A stands for area, C for curvature, and indices q and g denote quarks and gluons. Note that $\lim_{m\to 0}f_{C,q}(k/m)=-1/24\pi^2$, and that $\lim_{m\to 0}f_{A,q}(k/m)=0$.

The MRE was developed by Balian and Bloch [9], and the above expressions for the area and curvature terms have appeared in the literature. The area term for quarks is given (though not derived) in Ref. [10]. The curvature term for massless quarks seems to appear explicitly for the first time in Ref. [11], whereas the full expression (24) for massive quarks is introduced in Ref. [12]. The gluon expressions, valid for noninteracting gluons, is calculated in Ref. [13].

As indicated by the dots in Eq. (22), the expression for $\rho_i(k,V)$ should in principle contain terms proportional to $1/R$, $1/R^2$, etc., but as these terms become small in the limit of large R, and since the MRE is an approximation valid for large systems, these terms are usually neglected. However, we shall see in the following that the MRE as it stands in Eq. (22) is not only inaccurate, but also unphysical at small radii, having negative density of states. Further, we shall argue that, when used in calculations of the free energy, the MRE (22) containing only area and curvature terms leads to errors even at larger radii, where the MRE itself *is* a good approximation to the density of states. We also suggest a solution to these problems.

Everywhere in the following, unless explicitly stated, reference to the MRE means the approximation (22) to the density of states *without* further correction terms such as $1/R$, $1/R^2$.

D. The inverse multiple reflection expansion, MRE($-R$)

Because of the ''symmetry'' between the two situations (i) quark-gluon plasma confined by MIT bag boundary conditions within a sphere of radius R (a ''plasma droplet'') and (ii) quark-gluon plasma kept outside a sphere of radius R by MIT bag boundary conditions (a ''vacuum bubble''), it has been argued [1] that there should exist a simple relation between the density of states in the two cases, i.e., that the density of states of quarks and gluons in the case of a vacuum bubble can be found from the expressions (22)–(26) for a plasma bubble by simply inverting the sign of R. We shall refer to this hypothesis as the MRE($-R$). Since the MRE($-R$) is derived from the MRE, we expect the MRE($-R$) to have problems related to those of the MRE mentioned in the previous section.

E. The phase shift approach

In this section we briefly describe the phase shift approach to calculating the free energy of a vacuum bubble. The phase shift formula (27) was introduced in this context by Mardor and Svetitsky [1]. The phase shift approach is based on a relation between the density of states and the scattering phase shifts:[2]

$$\Delta\rho_l(k)=\frac{1}{\pi}\frac{d\delta_l(k)}{dk}. \tag{27}$$

For a derivation of this relation in the nonrelativistic case, see, e.g., Ref. [14]. Here, $\Delta\rho_l$ is the change in the density of states (at a given angular momentum) induced by the scatterer.

In order to use this phase shift approach to calculate the free energy, we need the scattering phase shifts for quarks and gluons. These are derived from the defining equations (3) and (5) for the quarks, and from Eqs. (8), (9), and (18) for the gluons. The phase shift for the j component of the quark field ($j=1/2,3/2,\ldots,$ is total angular momentum) is the sum of two components

$$\delta_j(k)=\delta_j^{l=j-1/2}(k)+\delta_j^{l=j+1/2}(k), \tag{28}$$

where, for a surface of radius R,

$$\delta_j^{l=j\mp 1/2}(k)=\arctan\left(\frac{j_l(kR)\pm[k/(E+m_q)]j_{l\pm 1}(kR)}{n_l(kR)\pm[k/(E+m_q)]n_{l\pm 1}(kR)}\right), \tag{29}$$

and $E=\sqrt{m_q^2+k^2}$, $q=u,d,s$. The phase shift for the gluon field also consists of two parts, $\delta_l^{TM}(k)$ and $\delta_l^{TE}(k)$, where

$$\delta_l^{TE}(k)=\arctan\left(\frac{(d/dr)[rj_l(kr)]}{(d/dr)[rn_l(kr)]}\right)_{r=R} \tag{30}$$

$$=\arctan\left(\frac{j_l(kR)(1+l)-kRj_{l+1}(kR)}{n_l(kR)(1+l)-kRn_{l+1}(kR)}\right), \tag{31}$$

and

$$\delta_l^{TM}(k)=\arctan\left(\frac{j_l(kR)}{n_l(kR)}\right), \tag{32}$$

[2]In the case of spherical symmetry, the phase shifts $\delta_l(k)$ are defined such that the effect of the scatterer is to change the spatial part of the wave function far away from the scatterer for a given angular momentum l from $\propto(1/kr)\sin(kr-l\pi/2)$ to $\propto(1/kr)\sin[kr-l\pi/2+\delta_l(k)]$.

again for a surface of radius R. $l=1,2,\ldots,$ labels orbital angular momentum, and here, as opposed to the quark situation, it is a good quantum number.

Knowing the phase shifts, the contribution to the free energy from the quarks and gluons outside a vacuum bubble of radius R is calculated using Eq. (20), so that

$$F_i(T,R)=\mp g_i\frac{T}{\pi}\int_0^\infty dk\frac{d\delta_i(k,R)}{dk}\ln(1\pm e^{-E(k)/T}). \tag{33}$$

The label i stands for different particle types (quarks, gluons) *and* angular momentum. Again, the upper sign applies to the fermions (quarks), lower sign to bosons (gluons). The appropriate degeneracy factors are $g_{\text{quark}}=6$ and $g_{\text{gluon}}=8$.

We make two remarks about the formulas (27) and (33). (i) The free energy (33) includes the contribution from the excluded volume. (ii) When the phase shifts contain functions with multiple branches, such as the arctan function in our case, we choose the branch which makes the phase shifts continuous functions of the energy. (In the case of potential scattering where the potential obeys certain integrability conditions, one can prove that the phase shifts are continuous functions of the energy [15].)

The free energy in the limit $RT\to 0$. By expanding the Bessel-functions appearing in Eqs. (29), (31), and (32) as

$$j_l(x)=\sum_{k=0}^{\infty} a_k(l)x^{l+2k} \tag{34}$$

and

$$n_l(x)=\sum_{k=0}^{\infty} b_k(l)x^{2k-l-1} \tag{35}$$

(valid for $l>0$) and keeping only the lowest order terms, we obtain via Eq. (33) the following analytical expressions valid for $RT\ll 1$ for the surface free energy of (one flavor of) massless quarks (index j and l means that we consider each angular momentum component separately):

$$\frac{F_{S,q}^j(RT)}{T}\simeq -12\frac{(2j+1)(2^{2j+2}-1)\pi^{2j+2}}{(2j+3)}\alpha_q(j)\times(RT)^{2j+2}B_{j+3/2} \tag{36}$$

and for the eight gluons

$$\frac{F_{S,g}^l(RT)}{T}\simeq -8\frac{(2l+1)(2\pi)^{2l+1}}{2l+2}\alpha_g(l)(RT)^{2l+1}B_{l+1}, \tag{37}$$

where

$$\alpha_q(j)=\frac{a_0(j+1/2)b_0(j+1/2)-a_0(j-1/2)b_0(j-1/2)}{b_0^2(j+1/2)}, \tag{38}$$

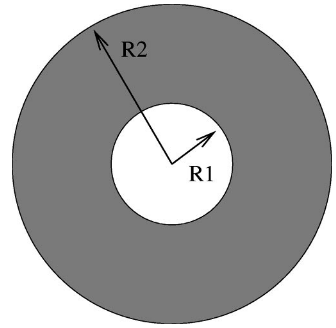

FIG. 2. The concentric spheres configuration: The inner sphere has radius R_1, the outer sphere has radius R_2. There is nonperturbative vacuum inside the inner sphere and outside the outer sphere. Quark-gluon plasma is confined between the two spheres by MIT bag boundary conditions corresponding to no flux of plasma across the spheres. At the outer sphere the boundary conditions are the usual MIT bag conditions (5),(6), but at the inner sphere we use $n^\mu(x)|_{r=R_1}=-n^\mu(x)|_{r=R_2}=(0,\vec{x}/|\vec{x}|)$.

$$\alpha_g(l)=\frac{a_0(l)}{b_0(l)}+\frac{a_0(l)(l+1)}{b_0(l)(l+1)-b_0(l+1)}, \tag{39}$$

and

$$a_k(l)=\frac{(-1)^k}{2^k k!1\times3\times5\cdots(2l+2k+1)}, \tag{40}$$

$$b_k(l)=(-1)^{k+1}\frac{1\times3\times5\cdots(2l-1)}{2^k k!(1-2l)(3-2l)\cdots(2k-1-2l)}. \tag{41}$$

[The factors $(1-2l)(3-2l)\cdots$ in the denominator of Eq. (41) appear only when $2k-1\geqslant 1$.] The B_n appearing in Eqs. (36) and (37) are the Bernoulli numbers, defined by

$$\frac{x}{e^x-1}=1-\frac{x}{2}+B_1\frac{x^2}{2!}-B_2\frac{x^4}{4!}+B_3\frac{x^6}{6!}-\cdots \tag{42}$$

the first few of these being

$$B_1=1/6,\quad B_2=1/30,\quad B_3=1/42,\quad B_4=1/30. \tag{43}$$

On the basis of Eqs. (36) and (37), we conclude that the first energy term of importance for $R\to 0$ is proportional to R^3; no terms proportional to R or R^2 appear in this limit. This is in contrast to the MRE$(-R)$ conjecture, where a curvature term proportional to R dominates for $R\to 0$. The difference is clearly demonstrated in the figures in the next section as a difference between zero and finite slope of the free energy for $R\to 0$.

F. The concentric spheres method

The concentric spheres method is a new way to calculate the surface contribution to the free energy of quarks and gluons outside a vacuum bubble. The idea is to extract the contribution from the inner surface to the total free energy of the concentric spheres configuration in Fig. 2. Assuming that

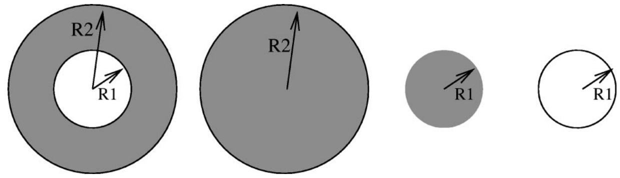

FIG. 3. How to extract the surface free energy in the concentric spheres method: Shaded areas are plasma, white areas are nonperturbative vacuum. Contributions to free energy (left to right): $F_{\rm total}(R_1,R_2)$, $F_{\rm total}(R_2)$, $F_{\rm volume}(R_1)$, and $F_{\rm surface}(R_1)$. $F_{\rm total}(R_1,R_2)$ and $F_{\rm total}(R_2)$ are calculated by summation over energy levels of the particles in the relevant configuration. $F_{\rm volume}(R_1)$ is only the *volume* free energy of particles occupying a volume $(4\pi/3)R_1^3$. The purpose is to calculate the contribution to the free energy from the inner surface, $F_{\rm surface}(R_1)$, and this can be done as in Eq. (44).

the splitting of the free energy in volume and surface contributions is valid, we can extract the free energy contribution from the inner surface (see Fig. 3) from a calculation of the total free energy of particles contained between two concentric spheres as

$$F_{\rm surface}(R_1)=F_{\rm total}(R_1,R_2)+F_{\rm volume}(R_1)-F_{\rm total}(R_2), \tag{44}$$

where $F_{\rm total}(R_1,R_2)$ is the total free energy (including both surface contributions) of the particles contained between the spheres with radii R_1 and R_2, $F_{\rm volume}(R_1)$ is the volume contribution to the free energy of the particles in a sphere of radius R_1, and $F_{\rm total}(R_2)$ is the total free energy (including surface contribution) of the particles in a sphere of radius R_2. It is precisely $F_{\rm surface}(R_1)$, the surface free energy of the particles outside a sphere, that we are interested in. Here, the terms $F_{\rm total}(R_1,R_2)$ and $F_{\rm total}(R_2)$ are calculated directly by summation over energy levels, whereas $F_{\rm volume}(R_1)$ is calculated from the MRE (using a positive radius). Explicitly, the term $F_{\rm total}(R_1,R_2)$ in the case of quarks of mass m_q is

$$F_{\rm quarks}(R_1,R_2)=-6T\sum_{j=1/2,3/2}^{\infty}(2j+1)\sum_{l=j\pm 1/2}\sum_n \ln(1+e^{-E_{jln}/T}),\quad E_{jln}=\sqrt{k_{jln}^2+m_q^2}, \tag{45}$$

where $k_{j,l=j\pm 1/2,n}$ is the nth solution of the equation

$$\begin{aligned}&[\alpha(k)j_{l-1}(kR_1)-j_l(kR_1)][\alpha(k)n_{l-1}(kR_2)+n_l(kR_2)]\\&\quad-[\alpha(k)j_{l-1}(kR_2)+j_l(kR_2)]\\&\quad\times[\alpha(k)n_{l-1}(kR_1)-n_l(kR_1)]=0\end{aligned} \tag{46}$$

for $l=j+1/2$, and

$$\begin{aligned}&[\alpha(k)n_{l+1}(kR_1)+n_l(kR_1)][j_l(kR_2)-\alpha(k)j_{l+1}(kR_2)]\\&\quad+[\alpha(k)n_{l+1}(kR_2)-n_l(kR_2)]\\&\quad\times[\alpha(k)j_{l+1}(kR_1)+j_l(kR_1)]=0\end{aligned} \tag{47}$$

for $l=j-1/2$, with $\alpha(k)=k/\sqrt{k^2+m_q^2}+m_q$. In the case of gluons, the term $F_{\rm total}(R_1,R_2)$ is

$$F_{\rm gluons}(R_1,R_2)=8T\sum_{l=1}^{\infty}(2l+1)\times\sum_{a=TM,TE}\sum_n \ln(1-e^{-k_{l,n}^a/T}), \tag{48}$$

where $k_{l,n}^a$ is the nth solution of the equation

$$j_l(kR_2)n_l(kR_1)-j_l(kR_1)n_l(kR_2)=0 \tag{49}$$

for the TM gluons, and

$$\begin{aligned}&(l+1)^2\{j_l(kR_1)n_l(kR_2)-j_l(kR_2)n_l(kR_1)\}\\&\quad+(l+1)kR_1\{j_l(kR_2)n_{l+1}(kR_1)-j_{l+1}(kR_1)n_l(kR_2)\}\\&\quad+(l+1)kR_2\{j_{l+1}(kR_2)n_l(kR_1)-j_l(kR_1)n_{l+1}(kR_2)\}\\&\quad+kR_1kR_2\{j_{l+1}(kR_1)n_{l+1}(kR_2)\\&\quad-j_{l+1}(kR_2)n_{l+1}(kR_1)\}=0,\end{aligned} \tag{50}$$

for the TE gluons. The term $F_{\rm total}(R_2)$ is just the free energy of a quark or gluon droplet, so this is calculated using Eqs. (53)–(57). Finally, the volume term $F_{\rm volume}(R_1)$ is

$$F_{\rm volume}(R_1)=\mp g_iT\int_0^\infty dk\frac{Vk^2}{2\pi^2}\ln(1\pm e^{-E(k)/T}),\quad i=q,g, \tag{51}$$

where $V=(4\pi/3)R_1^3$, $g_q=12$ for each flavor and $g_g=16$.

Formally, the splitting in surface and volume terms is appropriate only when $R_2-R_1\to\infty$. However, calculations for 10 fm$<R_2<$20 fm suggest that the concentric spheres method yields the correct free energy contribution from the inner surface up to $R_1\simeq R_2/2$.

III. NUMERICAL RESULTS

We are now going to use the different techniques described in the previous section to calculate the free energy of (i) a plasma droplet in a bulk hadronic medium and (ii) a hadron bubble in a bulk plasma. In both cases we normalize the free energy such that it is zero when there is no droplet, respectively, bubble, i.e., we calculate the free energy relative to an infinite hadron, respectively, plasma phase. In the adopted model, we have the following contributions to the free energy: Quarks (u, d, and s, and their antiquarks), gluons, the bag contribution $BV_{\rm QGP}$, where $V_{\rm QGP}$ is the plasma volume, and the contribution from the three pions [using Eq. (20) with $\rho_\pi(k,V)=3(Vk^2/2\pi^2)$ and $m=m_\pi$, this is

$$F_\pi(T,V_\pi)=\frac{3TV_\pi}{2\pi^2}\int_0^\infty dk k^2\ln(1-e^{-\sqrt{m_\pi^2+k^2}/T}), \tag{52}$$

where V_π is the pion volume]. The bag and pion contributions are thus simple and universal, the interesting part of the free energy is the quark and gluon contributions.

A. Free energy of a plasma droplet in a bulk hadronic medium

The plasma droplet is an MIT bag with pions around it, so the calculation of the free energy is straightforward, i.e., we can calculate the energy levels of quarks and gluons in the bag, and perform the partition sum (19) directly. This, of course, gives the true free energy.

We can also use the MRE (22) to calculate the free energy (20). This way, two approximations are involved: (1) The spectrum is discrete, but the MRE treats the energy levels as continuous and (2) we have discarded terms of the form $1/R$, $1/R^2$, etc., in Eq. (22). Since we are considering the high temperature case, the first approximation is well justified: The low ($E<T$) energy levels, where the level spacing is large, do not contribute significantly to the free energy, whereas the main contribution to the true free energy (19) comes from the higher levels where the spacing is small. Thus, any difference in the free energy between a direct calculation and the MRE approximation is a measure of the importance of the neglected terms in Eq. (22) and/or the choice of truncation discussed in Sec. IV.

1. Direct calculation

In this case, we need to solve the set of equations (3)–(6), and then perform the sum (19) over these levels. We thus obtain the following equations (to be solved numerically) for the quarks ($l=j\pm 1/2$):

$$j_l(kR)=\pm\frac{k}{E+m_q}j_{l\pm 1}(kR),\quad E=\sqrt{k^2+m_q^2}. \tag{53}$$

For the TM gluons

$$j_l(kR)=0 \tag{54}$$

and for the TE gluons

$$j_l(kR)(l+1)=kRj_{l+1}(kR). \tag{55}$$

These equations provide a series of solutions, that we label E_{jln} for the quarks and k^a_{ln}, $a=TM,TE$ for the gluons. The contribution to the free energy from quarks and gluons are then

$$F_q(T,R)=-6T\sum_{j=1/2,3/2,..}^{\infty}(2j+1)\times\sum_{l=j\pm 1/2}\sum_n \ln(1+e^{-E_{jln}/T}) \tag{56}$$

and

$$F_g(T,R)=8T\sum_{l=1}^{\infty}(2l+1)\sum_{a=TM,TE}\sum_n \ln(1-e^{-k^a_{ln}/T}), \tag{57}$$

respectively. Note that when dealing with each angular momentum component separately, the degeneracy factors g_i only account for the degeneracy due to color and particle-antiparticle, so we have $g_q=6$ for each quark flavor and $g_g=8$ for the gluons.

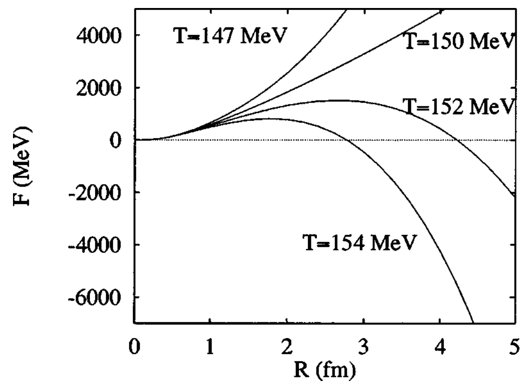

FIG. 4. Total free energy (calculated directly by summation over states) of a quark-gluon plasma droplet of radius R surrounded by pions. The phase transition temperature is set to $T_0=150$ MeV. Results are shown for several temperatures around T_0.

We imagine a spherical plasma droplet of volume $V_{\mathrm{QGP}}=(4\pi/3)R^3$ embedded in a large volume of pions, the total volume of this system being V_∞. The pions therefore inhabit a volume $V_\pi=V_\infty-V_{\mathrm{QGP}}$. Since we calculate the free energy relative to a system with no plasma droplet, and pions in the whole volume V_∞, the effective pion volume in Eq. (52) is $-V_{\mathrm{QGP}}$. Summing all the contributions we obtain Fig. 4. In Fig. 5 we show the different contributions to Fig. 4. We shall comment on these figures later.

2. Using the MRE

Now let us see how the MRE approximation handles the plasma droplet. The difference from the sum over states method lies entirely in the calculation of the quark and gluon contributions. The pion and bag contributions are the same as before. Now, we use Eq. (20) for the quarks and the gluons with the MRE density of states (22). The result is shown in Fig. 6. Figure 7 shows the different contributions for $T=152$ MeV.

3. Comparison

Comparing Figs. 4 and 6 we see that although they agree qualitatively (except for $R\to 0$, where the MRE is dominated

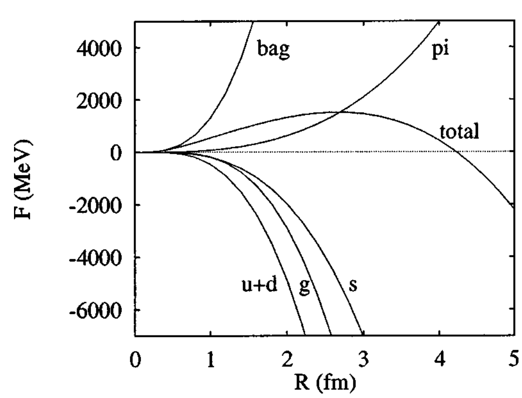

FIG. 5. Different contributions to Fig. 4 for $T=152$ MeV.

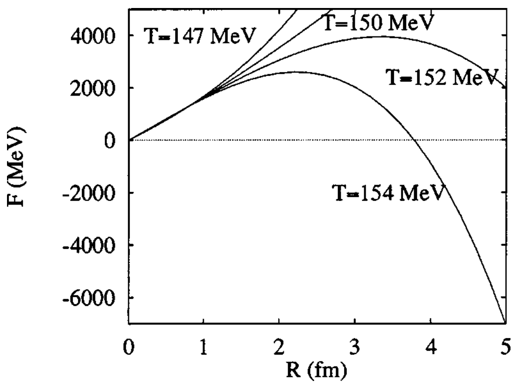

FIG. 6. As Fig. 4, except that the free energies are calculated using the MRE approximation.

by a curvature term proportional to R, whereas the sum over states for massless particles behaves as R^3), there are significant quantitative differences even at large radii. At first sight this is surprising. The MRE should be a good approximation, since each of the terms in Eq. (22) is derived analytically (albeit in the limit $kR \gg 1$). But the MRE is an approximation for the density of states, not for the free energy itself. Because the free energy is an integral over the density of states, it ''picks up'' the wrong behavior of the MRE at low energies, and ''remembers'' this error even at larger radii. This is why the free energy calculated using the MRE is not in quantitative agreement with the correct free energy in Fig. 4, although the MRE is a good approximation for the density of states in the limit $kR \gg 1$.

Looking at Fig. 7, we see that the main difference between the sum over states approach and the MRE approximation is due to the gluons. The gluon free energy being positive for small ($R < 1.5$ fm) radii, corresponds to the density of states being negative. This can also be seen directly from Eqs. (22) and (26). Hence the gluon density of states in the MRE is not only wrong, it is unphysical at small radii. This behavior is due to the way the MRE handles the surface corrections, namely, through Eqs. (23)–(26). Taking more terms ($\propto 1/R$, $\propto 1/R^2$, etc.) into account in Eq. (22) might cure this. We propose another resolution of the problem in Sec. IV.

A few words about the physics implied by Figs. 4 and 6. There are two minima, at $R=0$ and at $R=\infty$, separated by an energy barrier. The picture is therefore the following: When the temperature is $T < T_0$, no stable droplets can form. Even when $T > T_0$ there is an energy barrier to pass before stable droplets of quark-gluon plasma can exist within the pion phase. But once a droplet is created (from fluctuations) with radius $R > 2$ fm, it will expand unimpeded. The energy barrier, of course, is due to the surface terms. The most important quantitative difference between Figs. 4 and 6 is the height of the energy barrier separating the two minima at $R=0$ and $R=\infty$. The height of the barrier is related to the nucleation rate, in this case the plasma formation rate when heating a hadron gas, e.g., in a heavy ion collision. Thus, although generally small, the surface contribution to the free energy has important implications.

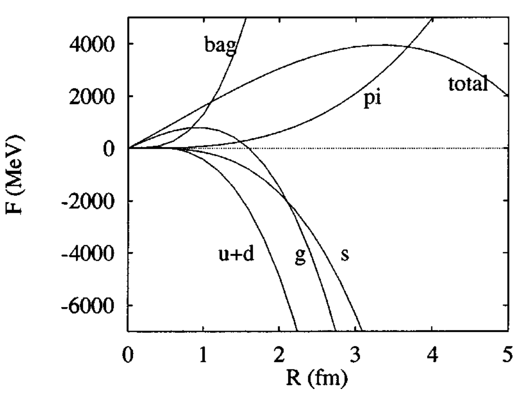

FIG. 7. The different contributions to Fig. 6 for $T=152$ MeV.

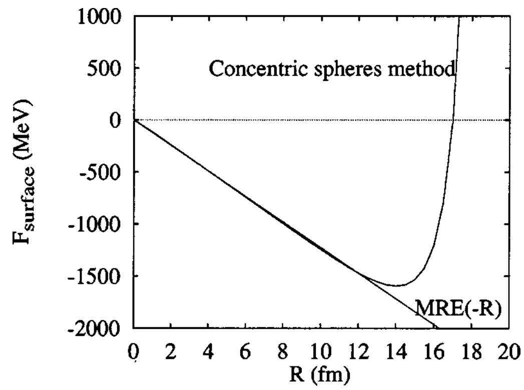

FIG. 8. A comparison between two different methods of calculating the surface free energy of massless quarks outside a sphere (a ''vacuum bubble'') of radius R: (1) Multiple reflection expansion with the sign of R reversed [MRE($-R$)] and (2) the concentric spheres method with an outer radius $R_2 = 20$ fm. When $R < R_2/2$ the two methods yield similar results. This suggests two things: (A) When $R \ll R_2$ the interactions at the outer surface are unimportant and (B) MRE($-R$) describes adequately the way the inner surface alters the density of states in the case of massless quarks. The temperature is $T=152$ MeV.

B. Free energy of a hadron bubble in bulk plasma

This configuration is a vacuum bubble with hadrons (i.e., pions, in our model) inside, surrounded by plasma. In order to emphasize the essential points, we start by focusing on the surface free energy of quarks and gluons, since this is the interesting, non-universal, part of the free energy.

1. Surface free energy of a vacuum bubble

We shall compare results for this surface free energy as calculated by the MRE($-R$) conjecture and by the concentric spheres method. We shall also compare with results obtained by the phase shift approach.

Figures 8, 9, and 10 show the surface free energy of massless quarks, massive quarks, and gluons, respectively. We show the results for just one temperature, $T=152$ MeV, but the picture is qualitatively the same for other temperatures ($T=50,100,160$ MeV). These figures show that the MRE($-R$) works quite well in the case of quarks (massless and massive), but less well for gluons. The *slope* of the gluon

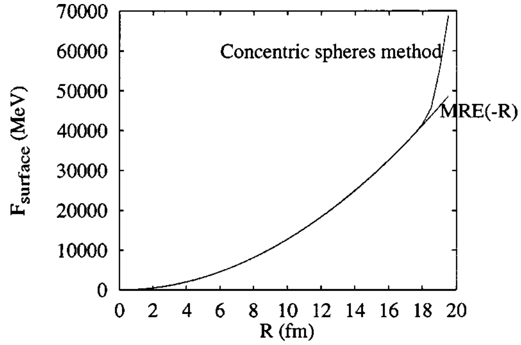

FIG. 9. Same as Fig. 8, but using a quark mass of 150 MeV. In this case, the effect of the outer surface is not visible until R is quite close to the outer surface at $R_2 = 20$ fm. Again, the MRE$(-R)$ seems to be a satisfactory description.

MRE$(-R)$ curve is correct, but there is an offset of about 1000 MeV (see Fig. 10). This is because the MRE$(-R)$ for the density of states, although a good approximation at large values of kR, is wrong at small kR, and since the free energy at a given large radius is an integral over the density of states also at small momenta, the bad behavior of the MRE$(-R)$ at small kR affects the free energy even at large radii (see remarks about the MRE for the plasma droplet). The same remarks apply to the quarks, but here the effect is much less pronounced. To support this picture we show in Fig. 11 a comparison of the quantity $(\pi/R)\rho(kR)$, ρ being the density of states of the eight gluons, calculated by the MRE$(-R)$: $(\pi/R)\rho_{\mathrm{MRE}(-R)}(kR) = -\frac{32}{3}(kR)^2 + \frac{64}{3}$, and by the phase shift approach (33). In contrast to other figures in this section, Fig. 11 includes the volume contribution. Figure 12 compares all three methods of calculating the surface free energy, here shown in the case of gluons. We expect the concentric spheres method to be a correct way of calculating the free energy, as long as we stay in the regime $R \ll R_2$. The agreement of the concentric spheres method and the phase shift approach at $R < 6$ fm in Fig. 12 suggests that both approaches are valid ways to calculate the surface free energy and/or the density of states of quarks and gluons outside a vacuum bubble, for the phase shift approach presumably at any radius. Referring back to Fig. 11 it is therefore clear that the MRE$(-R)$ approach is inadequate at small kR. This was

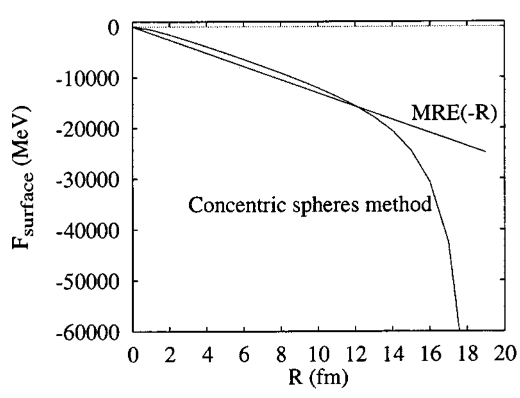

FIG. 10. Same as Fig. 8, but for gluons. The MRE$(-R)$ is not as good a description as in the case of the quarks.

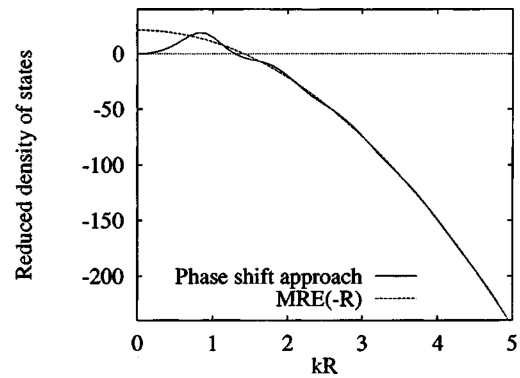

FIG. 11. The "reduced density of states" $(\pi/R)\rho(kR)$ of gluons outside a vacuum bubble, calculated by the phase shift approach, and using the MRE$(-R)$. The phase shift method yields the more correct result (cf. Fig. 12). Note that this is the density of states relative to a situation with no vacuum bubble, so $\rho < 0$ in this case is not unphysical.

to be expected. First, the original MRE (for positive radii) is derived in the limit of large kR, and second, we have seen that the MRE is unphysical at small radii, so it is not surprising that also the MRE$(-R)$ has problems in this regime.

We thus conclude, that as an approximation of the density of states, the MRE$(-R)$ for gluons works well at large values of kR, but is incorrect at small kR. We have only shown the gluon data, but a similar conclusion is valid for the quarks, although the error at small radii is less important than in the case of gluons. Further, we have argued that as far as the free energy is concerned, we should be even more careful when applying the MRE$(-R)$, as the bad behavior of the MRE$(-R)$ at small radii manifests itself as an error in the free energy even at large radii. Finally, our results obtained with the concentric spheres method are consistent with the

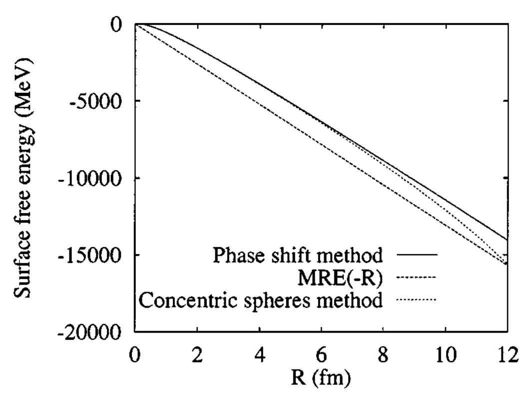

FIG. 12. Surface free energy of gluons outside a vacuum bubble of radius R calculated in the three different ways described in the text. The outer radius used in the concentric spheres method is $R_2 = 20$ fm. The phase shift approach agrees with the concentric spheres method for $R < 6$ fm, whereas the deviation between these two methods at $R > 6$ fm is due to the influence of the outer surface on the result of concentric spheres method. To emphasize the differences between the three methods, we have subtracted the volume free energy from the phase shift results, so that only the surface contributions are shown in this figure. The temperature is $T = 152$ MeV.

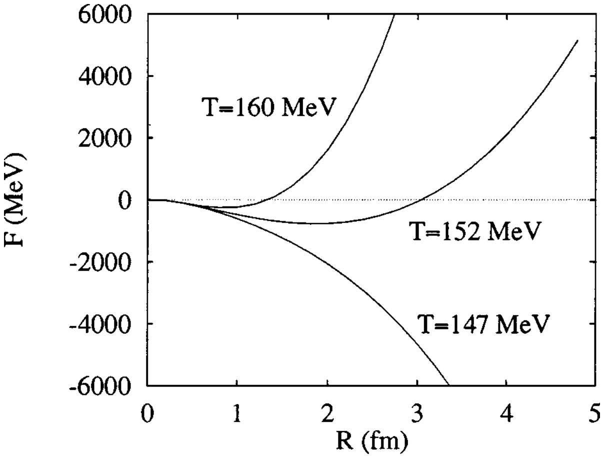

FIG. 13. Free energy of a pion bubble surrounded by quark-gluon plasma, normalized so that the free energy of a pure plasma without pions is zero. Curves are shown for temperatures above and below the transition temperature $T_0 = 150$ MeV. The surface contributions from the quarks and gluons are calculated by the concentric spheres method with an outer surface of $R_2 = 20$ fm. The minimum at $R \simeq 1-2$ fm shows that in this model, bubbles of pions of this radius will form even for temperatures above T_0. Similar results were obtained by Mardor and Svetitsky [1] using the phase shift method.

phase shift formula (27), which seems to be an accurate way to calculate the density of states of quark-gluon plasma outside a vacuum bubble.

2. The total free energy

Knowing the contribution from the surface to the free energy of quarks and gluons outside the vacuum bubble, we can easily calculate the total free energy of the whole configuration: The volume contribution of the plasma is calculated using Eq. (20) with the smoothed density of states (21) inserted. The pion contribution is given by Eq. (52), and the bag constant contributes a term BV_{QGP} as always. Adding these contributions we obtain the results in Fig. 13. The interesting part of this figure is the minimum of the free energy at $R = 1$–2 fm for temperatures well above the transition temperature. Mardor and Svetitsky [1] found a similar minimum in the free energy using the same model as described in this paper, but calculating the free energy of the plasma using the phase shift approach, whereas here we have applied the concentric spheres method.

IV. CORRECTIONS TO THE MRE

We have seen that the MRE and the MRE$(-R)$ for gluons as it stands in Eqs. (22), (25), and (26) have problems at small values of kR, leading to errors in the free energy even at large radii. Although we have numerical methods to calculate correctly the free energy both in the plasma droplet case and in the vacuum bubble case, we would like to be able to use some MRE approximation to gain physical insight, and for practical computations because the direct methods are numerically demanding. In this section, we investigate how to modify the MRE and the MRE$(-R)$, in order for these approximations to describe more correctly the density of states of the configuration in question.

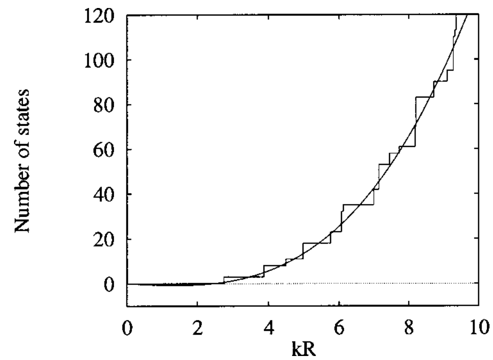

FIG. 14. Number of gluon states with energy less than k in an MIT bag of radius R, calculated (i) directly, solving Eqs. (54) and (55) (discontinuous line, true values) and (ii) by the MRE (continuous line, approximation). Note that the MRE predicts a negative density of states at small values of kR.

A. Gluon droplet

Previously in this paper, we have shown how to calculate exactly the free energy of Abelian gluons in an MIT bag. That was, however, a numerical computation. Analytic calculations of the free energy, not using the MRE, have also appeared in the literature. Using the same model for the gluons as we do, De Francia [16,17] finds for the difference $\Delta F = F_{\mathrm{gluons}} - F_{\mathrm{gluons,MRE}}$ in the limit of large RT

$$\frac{\Delta F}{T} = -0.874 - \frac{5}{8}\ln(RT) + \cdots, \tag{58}$$

where the dots indicate terms of higher order in $(RT)^{-1}$. (De Francia gives such terms explicitly, but they are too small to be relevant in our analysis.) Note that Eq. (58) is calculated for one Abelian gauge field, and should thus be multiplied by 8 in order to describe the gluon free energy.

The main problem with the MRE is that it predicts a negative density of states at small kR where in reality there are no states, see Fig. 14. An error in the density of states at small values of kR is particularly severe, since here the statistical factor in the integrand of the free energy is large. In the following, we will show that using a reduced density of states of the form

$$\frac{\pi}{R}\rho_\Lambda(kR) = \begin{cases} 0, & 0 \leqslant kR < \Lambda, \\ \frac{2}{3}(kR)^2 - \frac{4}{3}, & kR \geqslant \Lambda \end{cases} \tag{59}$$

cures most of the problems of the MRE. We shall refer to this density of states ρ_Λ as the MMRE (modified MRE), since it consists of the usual MRE contributions for $kR \geqslant \Lambda$, but is truncated below $kR = \Lambda$. When we are in a regime where $RT \gg 1$, we can find an approximate analytical expression for the correction $\Delta F = F_{\mathrm{gluons,MMRE}} - F_{\mathrm{gluons,MRE}}$ to the free energy induced by using ρ_Λ instead of ρ as the density of states of gluons:

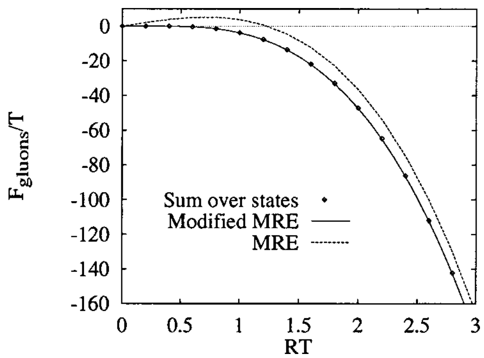

FIG. 15. Gluon contribution to the free energy (in units of the temperature T) of a plasma droplet of radius R, as a function of the dimensionless parameter RT. The unphysical behavior of the usual MRE [Eqs. (22), (25), and (26)] at small radii causes the free energy to deviate from the true free energy (calculated by summing over energy levels) even at large radii. Using the modified MRE, $\rho_{0.832}$, in Eq. (20) makes the free energy agree remarkably well with the true free energy.

$$\frac{\Delta F}{T} = -16\left(\ln(\Lambda)\left(\frac{2}{9\pi}\Lambda^3 - \frac{4}{3\pi}\Lambda\right) - \frac{2}{27\pi}\Lambda^3 + \frac{4}{3\pi}\Lambda + \ln(RT)\left(\frac{4}{3\pi}\Lambda - \frac{2}{9\pi}\Lambda^3\right)\right). \quad (60)$$

We can fix the value of Λ by matching the coefficient of ln(RT) to the analytical result of De Francia, i.e., solving $16[(2/9\pi)\Lambda^3 - (4/3\pi)\Lambda] = -5$, which has $\Lambda = 0.832$ as the relevant solution, and the free energy (60) becomes

$$\frac{\Delta F}{T} = -6.352 - 5\ln(RT). \quad (61)$$

The fact that $6.352 \simeq 8 \times 0.874$ shows the consistency of the procedure, cf. Eq. (58).

In Fig. 15 we compare the proposal (59) with a direct calculation (summing over energy levels) of the free energy. Also shown is the free energy calculated using the usual MRE.

B. Vacuum bubble

Balian and Duplantier [18] have calculated the Casimir energy of a perfectly conducting spherical shell. They find [in the large RT limit, and again not quoting terms of higher order in $(RT)^{-1}$]

$$\frac{\Delta \tilde{F}}{T} = -\frac{0.769}{4} - \frac{\ln(RT)}{4}. \quad (62)$$

In our language, $8\Delta\tilde{F}$ is the sum of (i) the surface free energy of gluons inside an MIT bag and (ii) the surface free energy of gluons outside a vacuum bubble. Using this and De Francia's calculation (58), we can deduce the corrections to the MRE$(-R)$. We obtain, for the difference $\Delta F_{\text{vac}} = F_{\text{gluons,corrected}} - F_{\text{gluons,MRE}(-\text{R})}$,

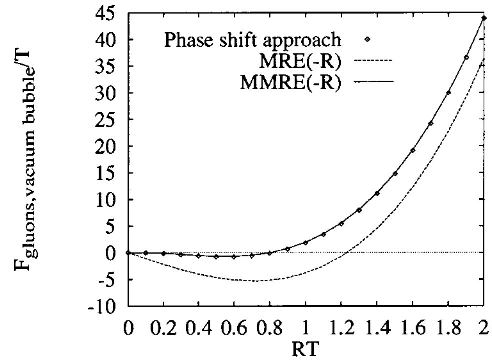

FIG. 16. The free energy of gluons outside a vacuum bubble of radius R (normalized to the temperature T) calculated in three different ways: (1) By the phase shift approach, which we consider an accurate procedure, (2) by the MRE$(-R)$, and (3) using the MMRE$(-R)$.

$$\Delta F_{\text{vac}} = T[5.454 + 3\ln(RT)]. \quad (63)$$

As in the gluon droplet case, we can advise a modification of the MRE$(-R)$, which works very well. Specifically, we propose the following: The reduced density of states of gluons outside a vacuum bubble, is

$$\frac{\pi}{R}\rho_{\text{vac}}(kR) = \begin{cases} 0, & 0 \leqslant kR < 0.458, \\ -\frac{2}{3}(kR)^2 + \frac{4}{3}, & kR \geqslant 0.458, \end{cases} \quad (64)$$

where $R > 0$ is the radius of the vacuum bubble. We refer to this density of states as the MMRE$(-R)$. The value of $kR = 0.458$ where we cut the MRE$(-R)$ is fixed by the same procedure as in the gluon droplet case. The difference $\Delta F = F_{\text{gluons,MMRE}(-\text{R})} - F_{\text{gluons,MRE}(-\text{R})}$ is then

$$\Delta F = T[5.417 + 3\ln(RT)], \quad (65)$$

showing the consistency of the procedure [cf. Eq. (63)]. The fact that we should cut the density of states at $kR = 0.458$ and not $kR = 0.832$ as in the gluon droplet case, reflects the asymmetry between the gluon droplet and the vacuum bubble configurations.

In Fig. 16 we compare the different methods of calculating the free energy of gluons outside a vacuum bubble. The simple MMRE$(-R)$ suggestion is in nice agreement with the phase shift approach, which (based on our calculations in the previous sections) we consider the most accurate way of calculating the free energy.

V. CONCLUSION

This paper had a twofold purpose. First we introduced the concentric spheres method as a way to calculate the free energy of quark-gluon plasma outside a pion bubble, confirming the peculiar results of Mardor and Svetitsky [1] that, within the MIT bag model, this free energy has a minimum at nonzero radius even well above the transition temperature.

Second, we have shown that terms beyond volume, surface, and curvature are necessary in order to reproduce the free energy of plasma droplets and vacuum bubbles within the multiple reflection expansion, especially for the gluon contributions. We have discussed the reasons for this, and based on previous calculations [16,18], we extracted correction terms to the free energy, which can be understood from a physically motivated truncation of the density of states.

Our calculations were all performed in the limit of vanishing chemical potentials. The results are thus relevant to investigations of the cosmological quark-hadron transition, and possibly to forthcoming ultrarelativistic heavy ion collision experiments at RHIC and LHC. While these are certainly interesting prospects, we plan to extend our analysis to situations of finite chemical potential, such calculations being relevant to a wider range of applications including, e.g., neutron stars.

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