Stress tensors and Casimir energies in the AdS-CFT correspondence

Robert C. Myers*
Department of Physics, McGill University, Montréal, Québec, Canada H3A 2T8
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We discuss various approaches to extracting the full stress-energy tensor of the conformal field theory from the corresponding supergravity solutions, within the framework of the Maldacena conjecture. This provides a more refined probe of the AdS-CFT correspondence. We apply these techniques in considering the Casimir energy of the conformal field theory on a torus. It seems that either generically the corresponding supergravity solutions are singular (i.e., involve regions of large string-scale curvatures), or that they are largely insensitive to the boundary conditions of the CFT on the torus. [S0556-2821(99)05114-0]

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I. INTRODUCTION

The Maldacena conjecture [1] has brought renewed interest in the holographic principle [2], which asserts that a theory of gravity in d dimensions can be described in terms of a nongravitational theory in $d-1$ dimensions. The current activity in string theory is focused on the AdS-CFT correspondence [3], which implements holography with a duality between a gravitational theory in d-dimensional anti–de Sitter space and a conformal field theory living in a $(d-1)$-dimensional ''boundary'' space. This duality is best understood for a specific superstring example with $d=5$ [1,3,4]. In this case, the duality maintains an equivalence between type IIb superstring theory on $AdS_5\times S^5$, and $\mathcal{N}=4$ super-Yang-Mills theory with gauge group $U(N)$ in four dimensions. Further in many interesting cases, it is sufficient to only consider the low energy limit of the superstring theory, namely, supergravity.

A precise formulation of the AdS-CFT correspondence is made in equating the generating function of the connected correlation functions in the CFT with the string or gravity partition function on the AdS space [3,4]. In the approximation of classical (super)gravity:

$$Z_{AdS}(\phi_i)=e^{-I(\phi_i)}=\left\langle e^{\int\phi_{0,i}\mathcal{O}^i}\right\rangle_{CFT} \tag{1}$$

where $I(\phi_i)$ is the classical (super)gravity action as a functional of the supergravity fields, $\phi_{0,i}$ are the asymptotic ''boundary'' values of the bulk fields ϕ_i up to a certain rescaling [3], and $\mathcal{O}^i$ are the dual CFT operators. Treating the ''boundary'' fields $\phi_{0,i}$ as source currents in the CFT, Eq. (1) is used in calculating the correlation functions of the operators $\mathcal{O}^i$. This framework also naturally allows one to evaluate the expectation values of the CFT operators in terms of the asymptotic (super)gravity fields [5,6].

Given that part of the duality is a theory of gravity in AdS space, one of the bulk fields will always be the graviton. So it is natural to ask what the role of the graviton (or metric perturbations) is in the above construction. The appropriate current-source term in the AdS-CFT generating function (1) couples the AdS graviton to the stress-energy tensor of the CFT [7,8]:

$$\int d^{d-1}x\, h^{ab}\, T_{ab} \quad . \tag{2}$$

This coupling has been used to investigate two- and three-point correlation functions of the stress tensor [9–11]. In particular, considering correlations protected by supersymmetry provides a nontrivial consistency test of the duality between IIb supergravity on AdS_5 and four-dimensional super-Yang-Mills theory [8].

Just as in asymptotically flat space, the energy of an asymptotically AdS solution can be determined by the asymptotic behavior of the metric [12,13]. In the context of the AdS-CFT correspondence, this result has the additional interpretation that the asymptotic metric perturbations determine the energy of the corresponding CFT state (or ensemble of states). This element of the correspondence was examined in Ref. [14], where it lead to the conjecture of a new positive energy theorem for general relativity. In the field theory, the energy is given by $E=\int\langle T_{tt}\rangle$, and so as in the general discussion above, one is considering states for which the expectation value of a particular operator, i.e., the stress-energy tensor, is nonvanishing. In fact, the expectation value of all of the individual components of the stress-energy can be determined from the asymptotic metric, and this is the focus of the present paper. Having the entire stress tensor provides a more refined tool with which to investigate the AdS-CFT correspondence, and we will apply it in order to extend the investigation of Casimir energies initiated in Ref. [14].

The remainder of the paper is organized as follows: In Sec. II, we consider in detail various techniques for calculating the expectation value of stress-energy tensor in the CFT from the corresponding supergravity solutions. In Sec. III, we apply these techniques to examine the Casimir energy of the CFT on a toroidal geometry. Finally, we present a discussion of our results in Sec. IV.

While this paper was in preparation, Ref. [15] appeared which discusses calculating the CFT stress-energy using techniques similar to those in Sec. II B.

*Email address: rcm@hep.physics.mcgill.ca

II. STRESS-ENERGY TENSOR

As discussed above, the stress-energy tensor provides an interesting tool with which to study the AdS-CFT correspondence. In the following, we consider three different approaches to extracting the field theory stress tensor from the supergravity solutions via (i) asymptotically flat p-brane geometries, (ii) the quasilocal energy defined by Brown and York [22] and (iii) an expansion of the asymptotic metric with an appropriate choice of coordinates.

A. Asymptotically flat geometries

The Maldacena conjecture [1] originally emerged out of investigations of extended branes in string theory and M theory. Anti–de Sitter space arises as part of the near-horizon geometry of certain branes, e.g., $AdS_{4,5,7}$ for M2-, D3- and M5-branes, respectively [1,16]. So we begin by considering the supergravity solutions describing a near-extremal p-brane in a d-dimensional, asymptotically flat spacetime. The usual formula giving the mass of a point-like object in terms of the asymptotic metric can be extended to give the mass per unit p-volume of such solutions [17]. A simple derivation of this result begins by considering an extended p-dimensional source in the linearized gravity equations. If we assume that the brane directions are also symmetry directions, these results may be further extended to yield the entire stress-energy tensor for the p-brane world-volume:

$$T_{ab}=\frac{1}{16\pi G_d}\oint d\Omega_{d-p-2}r^{d-p-2}\,n^i \times[\eta_{ab}(\partial_i h^c{}_c+\partial_i h^j{}_j-\partial_j h^j{}_i)-\partial_i h_{ab}] \tag{3}$$

where n^i is a radial unit vector in the transverse subspace, while $h_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu}$ is the deviation of the (Einstein frame) metric from that for flat space. Note that $h_{\mu\nu}$ is not a diffeomorphism invariant quantity, and in applying Eq. (3), it must be calculated using asymptotically Cartesian coordinates. Above, the labels $a,b=0,1,\dots,p$ run over the world-volume directions, while $i,j=1,\dots,d-p-1$ denote the transverse directions. For $a=b=0$, Eq. (3) reduces to the standard formula for the mass density of the p-brane [17].

As an application of this formula, let us consider a near-extremal D3-brane for which the 10-dimensional spacetime metric is [18]

$$ds^2=H^{-1/2}(-f^2\,dt^2+dx^2+dy^2+dz^2)+H^{1/2}\left(\frac{dr^2}{f^2}+r^2 d\Omega_5\right)$$

$$\text{with}\quad H=1+\left(\frac{l}{r}\right)^4 \quad\text{and}\quad f^2=1-\left(\frac{\mu}{r}\right)^4 \tag{4}$$

while we fix the constant background dilaton as $e^{\phi}=1$. As mentioned above, to apply Eq. (3), we need to express the metric in isotropic coordinates, at least asymptotically. That is we need to find a new radial coordinate such that $dr/f=F(R)\,dR$ and $r=F(R)\,R$. The final result is that asymptotically

$$r\simeq R\left(1+\frac{\mu^4}{8R^4}\right) \tag{5}$$

so that we may write the metric as

$$ds^2\simeq H(R)^{-1/2}[-f(R)^2\,dt^2+dx^2+dy^2+dz^2]+H(R)^{1/2}F(R)^2(dx^i)^2 \tag{6}$$

with $R^2=\Sigma_{i=1}^6(x^i)^2$. Given this form of the metric, a straightforward calculation of Eq. (3) yields

$$T_{ab}=\frac{\pi^2}{16G_{10}}[-\eta_{ab}(4l^4+\mu^4)+4\mu^4\delta_a^0\delta_b^0]. \tag{7}$$

Now this stress tensor may be regarded as including two contributions: (i) those appearing with the introduction of the extremal D3-brane and (ii) those due to excitations of the D3-brane above extremality, and hence which vanish as $\mu\to0$. The precise nature of these sources can be understood by expanding the Born-Infeld action [7,8,19], or by studying string scattering from D-branes [20]. In the context of the Maldacena conjecture, we are primarily interested in the latter since they represent the contribution to the stress-energy tensor by excitations in the world-volume field theory. To isolate these contributions, we subtract off the extremal contribution but in doing so we must be careful to subtract off that for an extremal D3-brane with the same RR five-form charge as the solution (4) given above. Thus the appropriate extremal stress tensor is found by first setting $\mu=0$ in Eq. (7) and then replacing $l^4\to l^2(l^4+\mu^4)^{1/2}$. It is most interesting to make the subtraction in the limit[1] that $\mu/l\ll1$ which yields

$$\Delta T_{ab}=\frac{\pi^2\mu^4}{16G_{10}}[4\delta_a^0\delta_b^0+\eta_{ab}]=\frac{\pi^2\mu^4}{16G_{10}}\begin{pmatrix}3&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}. \tag{8}$$

This result has a form characteristic of a thermal gas of massless particles. In particular its trace vanishes, i.e., $\Delta T^a{}_a=\eta^{ab}\,\Delta T_{ab}=0$. This is in keeping with the interpretation that this contribution arises from a thermal gas in the super Yang-Mills theory on the world-volume of the D3-brane.

Repeating the calculations for near-extremal M5- and M2-branes yields an analogous ΔT_{ab} which is again isotropic and traceless. Hence, this stress-energy lends itself to the interpretation of being due to a thermal gas of massless particles on the world-volume of these nondilatonic branes as

[1]This corresponds to the decoupling limit [1] in which $l_s\to0$ while holding μ/l_s^2 and l/l_s fixed.

well. In the case of a general Dp-brane, however, the result is isotropic but not traceless. Rather $\Delta T^a{}_a \propto (p-3)^2$ indicating the distinguished position of the D3-brane amongst the Dirichlet branes. Of course, an essential difference is that for the generic Dp-brane the dilaton is no longer constant. In such a situation, there is an intrinsic ambiguity in the definition of the energy (see, e.g., [21]) and so one could with an appropriate conformal transformation find a metric for which Eq. (3) yields a traceless stress tensor.

B. Quasilocal formulation

As discussed above in a more general setting, the AdS-CFT correspondence describes a duality between gravity in AdS spacetimes and a ''boundary'' field theory. For any theory including Einstein gravity coupled to matter fields, the boundary stress-energy tensor may be defined as follows [22]:[2] Consider spacetime manifold $\mathcal{M}$ with time-like boundary $\partial\mathcal{M}$.[3] Denote the spacetime metric as $g_{\mu\nu}$, and n^μ is the outward-pointing normal to $\partial\mathcal{M}$ normalized with $n^\mu n_\mu = 1$. The induced metric on the boundary, $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$, acts as a projection tensor onto $\partial\mathcal{M}$. The extrinsic curvature on $\partial\mathcal{M}$ is given by $\Theta_{\mu\nu} = -\gamma_\mu{}^\rho \nabla_\rho n_\nu$. Now given the standard Einstein action including a boundary term

$$I = \frac{1}{16\pi G_d}\int_{\mathcal{M}} d^d x \sqrt{-g}\,(R-2\Lambda) - \frac{1}{8\pi G_d}\oint_{\partial\mathcal{M}} d^{d-1}x\sqrt{-\gamma}\,\Theta + I_{\rm matter} \tag{9}$$

the boundary stress tensor is given by [22]

$$\tau^{ab} \equiv \frac{2}{\sqrt{-\gamma}}\frac{\delta I}{\delta\gamma_{ab}} = \frac{1}{8\pi G_d}(\Theta^{ab} - \gamma^{ab}\,\Theta^c{}_c) \tag{10}$$

where a,b,c denote directions parallel to the boundary. For a background solving the equations of motion, this stress tensor will satisfy [22]

$$\mathcal{D}_a \tau^{ab} = -T^{nb} \tag{11}$$

where the source on the right-hand side is a projection of the matter stress-energy, $T^{nb} = n_\mu T^{\mu\nu}\gamma_\nu{}^b$, and $\mathcal{D}_a$ is the covariant derivative projected onto $\partial\mathcal{M}$. Equation (11) expresses the local conservation of the boundary stress-energy up to the flow of matter energy-momentum across the boundary into $\mathcal{M}$. Due to the geometric confinement in asymptotically AdS spacetimes, this source term will vanish in the following.

In the case of interest here, the boundary will be an asymptotic surface at some large radius R. A technical problem with the above definitions for the action and the surface stress tensor is that they will both diverge in the limit $R\to\infty$. This problem can be cured [13] for the action by subtracting the same contribution (9) for a reference background geometry for which metric $g^0_{\mu\nu}$ matches $g_{\mu\nu}$ asymptotically, i.e., the boundary $\partial\mathcal{M}$ can be embedded in the reference background such that $\gamma^0_{ab} = \gamma_{ab}$. This background subtraction procedure produces a finite action, $\hat{I} = I(g) - I^0(g^0)$, and further yields a finite surface stress tensor

$$\hat{\tau}^{ab} \equiv \frac{2}{\sqrt{-\gamma}}\frac{\delta\hat{I}}{\delta\gamma_{ab}} = \tau^{ab} - (\tau^0)^{ab}. \tag{12}$$

Now consider the case that the spacetime has a Killing vector ξ^μ which is asymptotically time-like and surface forming, and also that $\partial\mathcal{M}$ is chosen so that the Killing vector remains an isometry of the boundary, i.e., $\mathcal{D}_{(a}\xi_{b)} = 0$. In this situation, one can show [22] that

$$E(\xi) = \oint_B d^{d-2}x\sqrt{-\gamma}\,\xi^a \hat{\tau}_{ab}\xi^b \tag{13}$$

with B, a hypersurface in $\partial\mathcal{M}$ orthogonal to ξ^a, is a conserved charge. If the boundary contains other spacelike Killing vectors, the latter can also be used to define other conserved charges by replacing one of the factors of ξ by a new Killing vector in Eq. (13). Further with the choice that the norm $\xi^a\xi_a = -1$ on the boundary, $E(\xi)$ coincides precisely with the standard definition of the energy [22,13].

As the supergravity energy should match the total energy measured in the field theory, this definition (13) is useful in the last step required in matching the surface stress tensor (12) with the expectation value of the stress energy in the dual CFT. While the charge in Eq. (13) is finite for asymptotically AdS spacetimes, the measure $\sqrt{-\gamma}$ is actually asymptotically divergent. In this situation, Eq. (13) only yields a finite result because the components of $\hat{\tau}_{ab}$ vanish asymptotically. In the AdS-CFT duality, the asymptotic boundary geometry is related to the background geometry on which the dual field theory lives by a conformal transformation which also diverges asymptotically. This conformal transformation can be accounted for by writing the stress tensor expectation value in the field theory as follows:

$$\sqrt{-h}\,h^{ab}\langle T_{bc}\rangle = \lim_{R\to\infty}\sqrt{-\gamma}\,\gamma^{ab}\hat{\tau}_{bc} \tag{14}$$

where h_{ab} is the background metric of the field theory.

At this point, it may be useful to examine an explicit example in which to apply the above analysis. Hence consider the spherically symmetric Schwarzschild-AdS metric in $d = p+2$ dimensions

$$ds^2 = -f(r)^2 dt^2 + f(r)^{-2}dr^2 + r^2 d\Omega_p \quad \text{with} \quad f(r)^2 = \frac{r^2}{l^2} + 1 - \frac{\mu^{p+1}}{l^2\, r^{p-1}}\ . \tag{15}$$

[2]Brown and York's quasilocal stress tensor [22] was first considered in the context of the AdS-CFT correspondence in Ref. [15].

[3]In general in a Minkowski-signature spacetime, one would expect the boundary to include space-like components as well, but in the present context, these components will not play a role.

(We have chosen a slightly unusual normalization for the mass term to facilitate comparisons with the results for the planar black holes below.) The normal vector to the surface $r=R$ is

$$n^\mu = f(R)\,\delta_r^\mu \tag{16}$$

and so the nonvanishing components of the boundary metric are

$$\gamma_{00} = -f(R)^2, \quad \gamma_{\bar{a}\bar{b}} = R^2\,[\bar{\gamma}_{(p)}]_{\bar{a}\bar{b}} \tag{17}$$

where $\bar{a}$ and $\bar{b}$ denote angular directions, and $[\bar{\gamma}_{(p)}]_{\bar{a}\bar{b}}$ is the metric on a unit p-sphere. In this simple situation, the extrinsic curvature reduces to

$$\Theta_{ab} = n_r \Gamma^r{}_{ab} = -\frac{1}{2} n^r \partial_r g_{ab} \tag{18}$$

and with a straightforward calculation, Eq. (10) yields

$$\tau_{tt} = -\frac{p\, f^3(R)}{8\pi G_d R},$$

$$\tau_{\bar{a}\bar{b}} = \frac{R}{8\pi G_d\, f(R)}[\bar{\gamma}_{(p)}]_{\bar{a}\bar{b}} \times\left(p\frac{R^2}{l^2} + p - 1 - \frac{p-1}{2}\frac{\mu^{p+1}}{l^2 R^{p-1}}\right). \tag{19}$$

With this result, we see that the nonvanishing components of the boundary stress tensor are all diverging as R^2 as $R\to\infty$, making clear the necessity of the background subtraction in Eq. (12). In the present case, the natural background geometry is simply Eq. (15) with $\mu=0$, which corresponds to AdS_{p+2}. In matching the boundaries, care must be taken to scale the time coordinate in the background metric by a constant so that at $r=R$, we have $\gamma^0_{tt} = \gamma_{tt}$. Equation (12) then yields

$$\hat{\tau}_{tt} = \frac{p}{16\pi G_d l^3}\frac{\mu^{p+1}}{R^{p-1}} + \cdots,$$

$$\tau_{\bar{a}\bar{b}} = \frac{1}{16\pi G_d l}\frac{\mu^{p+1}}{R^{p-1}}[\bar{\gamma}_{(p)}]_{\bar{a}\bar{b}} + \cdots \tag{20}$$

where the ellipsis denotes terms that vanish more quickly as $R\to\infty$. To apply Eq. (14), we first define the background metric for the field theory by stripping off the divergent conformal factor from the boundary metric (17):

$$h_{ab} = \lim_{R\to\infty}\frac{l^2}{R^2}\gamma_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & l^2[\bar{\gamma}_{(p)}]_{\bar{a}\bar{b}}\end{pmatrix}. \tag{21}$$

The field theory stress-energy then becomes

$$\langle T_{ab}\rangle = \frac{\mu^{p+1}}{16\pi G_d l^{p+2}}[(p+1)\,\delta_a^0\delta_b^0 + h_{ab}]. \tag{22}$$

So again we can recognize the form characteristic of a thermal gas of massless particles. As a check, one can easily verify that the total energy is

$$E = \oint d\Omega_p \sqrt{-h}\,\langle T_{tt}\rangle = \frac{p\,\Omega_p}{16\pi G_d l^2}\mu^{p+1} \tag{23}$$

where $\Omega_p = 2\pi^{(p+1)/2}/\Gamma[(p+1)/2]$ is the area of a unit p-sphere. This result agrees precisely with that calculated previously in Ref. [14]. One can also check that this result agrees with the energy calculated from Eq. (13) with $\xi^a\partial_a = f(R)^{-1/2}\partial_t$.

Another interesting example to consider are the planar black holes described by the metric

$$ds^2 = \frac{r^2}{l^2}\left[-\left(1-\frac{\mu^{p+1}}{r^{p+1}}\right)dt^2 + (dx^{\bar{a}})^2\right] + \left(1-\frac{\mu^{p+1}}{r^{p+1}}\right)^{-1}\frac{l^2}{r^2}dr^2 \tag{24}$$

where $\bar{a}=1,\ldots,p$. For certain values of p, these metrics arise in the near-horizon geometry of near-extremal p-branes (see, e.g., [1]). With $\mu=0$, these metrics correspond to AdS_{p+2} space in horospheric coordinates. Following the calculations as above, one finds that in this case the field theory stress tensor is

$$\langle T_{ab}\rangle = \frac{\mu^{p+1}}{16\pi G_d l^{p+2}}[(p+1)\,\delta_a^0\delta_b^0 + \eta_{ab}] \tag{25}$$

where in this case $h_{ab} = \eta_{ab}$ is simply the flat Minkowski metric in $p+1$ dimensions. For $p=3$, Eq. (24) is precisely the throat geometry of a near-extremal D3-brane, and comparing this result to the previous section, we find precise agreement between Eqs. (8) and (25), when we use the identity $G_{10} = G_5\pi^3 l^5$.

C. ‘‘Nice’’ coordinates

In considering absorption of gravitons by D3-branes, one finds that gravitons with polarizations parallel to the brane couple to the world-volume stress tensor [7,8]

$$I_{int} = \frac{1}{2}\int d^4x\, h^{ab}T_{ab}. \tag{26}$$

As discussed in the introduction, this coupling is actually the current-source coupling for the graviton in the AdS-CFT generating function (1), and has been used to investigate correlation functions of the field theory stress tensor [9–11]. As observed in Ref. [10], it is convenient to perform these calculations in ‘‘radiation gauge’’ for which

$$h_{r\mu} = 0 \tag{27}$$

so that the graviton polarizations are automatically in the boundary directions. As the graviton propagates in a higher dimensional space than the field theory, one should understand the non-covariant coupling (2) as being written with this gauge choice in mind.

These observations extend to situations where one is interested in the expectation value of the stress tensor, rather than correlation functions. We wish to determine the stress tensor of some supergravity solution with a given metric g which is to be regarded as an excitation of a background solution with metric g^0 — that is we are again calculating the stress-energy relative to some reference background. Now a convenient choice of coordinates can be found such that asymptotically for large radius

$$g_{rr}-g^0_{rr}\rightarrow o(1/r^{p+3})$$
$$g_{ra}-g^0_{ra}\rightarrow o(1/r^{p+1}) \tag{28}$$

where $o(1/r^q)$ indicates that these differences are falling off more rapidly than the indicated power of r. In principle, one could consider finding coordinates such that these differences fall off even more rapidly, but the above behavior is sufficient to determine the expectation value of the stress-energy. With the above choice of coordinates, the leading asymptotic perturbations of the metric are all in components parallel to the boundary directions. To leading order, the line element will take the form

$$ds^2=g^0_{\mu\nu}dx^\mu dx^\nu+\frac{\hat{T}_{ab}}{r^{p-1}}dx^a dx^b+\cdots. \tag{29}$$

One can now read off the stress tensor from the components of the metric perturbations: $\langle T_{ab}\rangle \propto \hat{T}_{ab}$. The constant of proportionality (which depends only on the spacetime dimension) can be fixed by calculating the mass of the solution and demanding that $\langle T_{tt}\rangle$ gives the correct mass density.

Let us apply the above procedure to the spherically symmetric Schwarzschild-AdS metric as an example. The radial component of the metric (15) is

$$g_{rr}=f(r)^{-2}=\frac{1}{\frac{r^2}{l^2}+1-\frac{\mu^{p+1}}{l^2\, r^{p-1}}} \tag{30}$$

while using AdS space as the background: $g^0_{rr}=(r^2/l^2+1)^{-1}$. Now by making a transformation $r=\tilde{r}+\alpha/\tilde{r}^p$ in the asymptotic region, one can achieve the desired fall off in Eq. (28). To be precise, the $r^{-(p+3)}$ perturbation in g_{rr} is eliminated with the choice $\alpha=\mu^{p+1}/2(p+1)$. Inserting this coordinate transformation into the metric (15), and comparing with the asymptotic behavior in Eq. (29), one finds

$$\hat{T}_{tt}=\frac{p}{p+1}\,\frac{\mu^{p+1}}{l^2},\qquad \hat{T}_{\overline{ab}}=\frac{\mu^{p+1}}{p+1}[\overline{\gamma}_{(p)}]_{\overline{ab}} \tag{31}$$

while $\hat{T}_{t\overline{a}}=0$. In order to produce the correct energy (as given in the previous section), the proportionality constant is fixed to be

$$\langle T_{ab}\rangle=\frac{p+1}{16\pi G_d l^p}\hat{T}_{ab} \tag{32}$$

which yields a precise agreement with the stress-energy in Eq. (22). The same proportionality constant (and in fact precisely the same transformation of the radial coordinate) in considering the planar black holes (24) again yields the correct stress-energy (25) using this procedure.

In the above example, we saw the traceless form of the stress tensor emerging naturally from this choice of coordinates, as well as precise agreement with the results of the previous section. In fact, one can show that the agreement between the present prescription and that of the previous section is quite general. Consider a metric of the above form (29). For some surface of fixed radius $r=R$ in the asymptotic region, the normal becomes

$$n_\mu dx^\mu=\sqrt{g^0_{rr}(R)}dr \tag{33}$$

where to simplify the calculations we have assumed that $g^0_{ra}=0$ — these metric components can be eliminated with an appropriate choice of coordinates for generic solutions. The complicated part of the construction is to match the asymptotic boundary geometries in general. That is one must find coordinates such that

$$g^0_{ab}|_{\overline{r}=R}=(g^0+\delta g)_{ab}|_{r=R} \tag{34}$$

where we have denoted the metric deviation from the background as $\delta g_{ab}=\hat{T}_{ab}/r^{p+1}+\cdots$. We will assume that we can accomplish this matching by a simple scaling of the coordinates, as in the examples considered above — this is a limiting assumption on the generality of the discussion. In this case, the components of the boundary metric in the background become

$$g^0_{ab}(r)\frac{(g^0+\delta g)_{ab}(R)}{g^0_{ab}(R)} \tag{35}$$

where above the values of a and b are fixed. The extrinsic curvature of the boundary now simplifies as in Eq. (18) to yield

$$\Theta_{ab}=-\frac{1}{2\sqrt{g^0_{rr}}}(g^0_{ab,r}+\delta g_{ab,r}). \tag{36}$$

For the background geometry one has

$$\Theta^0_{ab}=-\frac{1}{2\sqrt{g^0_{rr}}}g^0_{ab,r}\frac{(g^0+\delta g)_{ab}(R)}{g^0_{ab}(R)} \tag{37}$$

where again a and b are fixed in the above formula. Carrying out the remaining calculations and substituting in $\delta g_{ab}=\hat{T}_{ab}/r^{p+1}+\cdots$, one then finds that Eq. (14) precisely reproduces the above result

$$\langle T_{ab}\rangle=\frac{p+1}{16\pi G_d l^p}\hat{T}_{ab}. \tag{38}$$

III. CASIMIR ENERGIES

Now we apply the results of the previous section in a discussion of the Casimir stress-energy of the CFT on a torus. The discussion will focus on $p=3$ for which the field theory is best understood, however, for the most part the analysis can be extended to arbitrary dimensions.[4] We begin with a brief review of the results of Ref. [14], which considered Casimir energies with a single compact direction. The investigation there focussed primarily on the AdS soliton, which is the double analytic continuation of a planar black hole given in Eq. (24). The AdS soliton metric for $p=3$ is

$$ds^2=\frac{r^2}{l^2}\left[-dt^2+dx^2+dy^2+\left(1-\frac{\mu^4}{r^4}\right)dz^2\right]+\left(1-\frac{\mu^4}{r^4}\right)^{-1}\frac{l^2}{r^2}dr^2. \tag{39}$$

Here the radial coordinate is restricted to $r\geqslant\mu$, and geometry is smooth at $r=\mu$ provided that z is identified with period $\beta=\pi l^2/\mu$. One can calculate the energy of this configuration relative to a periodically identified AdS_5 spacetime [14]. Using the relations[5] between the AdS supergravity parameters and those in the CFT, which is $\mathcal{N}=4$ super-Yang-Mills theory with gauge group $U(N)$, one finds that the corresponding energy density is [14]

$$\langle T_{tt}\rangle_{\mathrm{sugra}}=-\frac{\pi^2}{8}\frac{N^2}{\beta^4}. \tag{40}$$

Now this negative energy density can be thought of as the Casimir energy that is generated in the CFT when the fermions are antiperiodic on the circle parametrized by z. These asymptotic boundary conditions arise for the supergravity fermions because the S^1 contracts to a point at $r=\mu$. The Casimir energy density can also be calculated directly in the field theory at weak coupling, with the result being

$$\langle T_{tt}\rangle_{\mathrm{gauge}}=-\frac{\pi^2}{6}\frac{N^2}{\beta^4}. \tag{41}$$

Hence one finds that this result and the negative energy density of the supergravity solutions only differ by an overall factor of 3/4. The weak coupling field theory calculations readily yield not just the energy density but also the entire stress-energy tensor which is

$$\langle T_{ab}\rangle_{\mathrm{gauge}}=\frac{\pi^2}{6}\frac{N^2}{\beta^4}\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-3\end{pmatrix}. \tag{42}$$

Using any of the techniques in the previous section,[6] a stress tensor may also be calculated for the AdS soliton with the result

$$\langle T_{ab}\rangle_{\mathrm{sugra}}=\frac{\pi^2}{8}\frac{N^2}{\beta^4}\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-3\end{pmatrix}. \tag{43}$$

These two stress-energy tensors, Eqs. (42) and (43), have precisely the same form except for a single overall factor of 3/4. This discrepancy reflects the fact that the two results apply in different regimes of the dual gauge theory. The supergravity results, Eqs. (40) and (43), correspond to the field theory for large 't Hooft coupling, i.e., $g_{YM}^2N\gg1$, while the explicit field theory results, Eqs. (41) and (42), are calculated for zero coupling, i.e., $g_{YM}^2N=0$. One can expect that the full stress-energy interpolates smoothly between Eqs. (42) and (43) as the coupling ranges between these two extremes (see, for example, Refs. [23,24]).

Motivated by the AdS-CFT correspondence, the authors in Ref. [14] conjectured that the AdS soliton (39) is actually the minimum energy solution with these asymptotic boundary conditions. As further evidence of this conjecture, the authors showed that the solution (39) is perturbatively stable against quadratic fluctuations of the metric. This perturbative stability actually extends to many finite deformations which continuously vary the metric (39) which the authors explored in their investigations [25].

If more than one of the spatial coordinates were compact, i.e., one might consider the CFT in $R^2\times T^2$ or $R\times T^3$ rather than $R^3\times S^1$ as above, then a natural question one might ask if the Casimir energy is further reduced by introducing antiperiodic boundary conditions for the fermions around more than one of the compact directions. It is straightforward to repeat the weak gauge coupling calculation of the stress energy for such generalized boundary conditions. Recall that the $\mathcal{N}=4$ super-Yang-Mills theory contains a $U(N)$ gauge field, six scalars in the adjoint representation, and their superpartner fermions. The stress-energy tensor for this theory may be found in Ref. [8]. To leading order in a gauge coupling expansion, the Casimir stress tensor may be calculated by point-splitting the fields in the stress tensor with the appropriate free-field Green's function and then removing the vacuum divergence before taking the limit of coincident

[4]In particular, the five-dimensional supergravity solutions considered below are easily generalized to other dimensions [26].

[5]That is [1] $g_{YM}^2=2\pi g$, $l^4=4\pi gNl_s^4$ and $G_5=8\pi^3g^2l_s^8/l^5$ where g and l_s are the string theory coupling and length scale, respectively.

[6]Note for the purposes of Sec. II A, that Eq. (39) can be extended to an asymptotically flat solution by taking a doubly analytically continued near-extremal D3-brane solution (4).

fields [27]. The calculation is simplified by choosing orthogonal coordinates to describe the background geometry, i.e.,

$$h_{ab}dx^a dx^b = -dt^2 + dx^2 + dy^2 + dz^2 . \tag{44}$$

To introduce the identifications producing a torus in the spatial part of this geometry, we must now specify three basis vectors $\vec{v}_i$ and points are then identified according to

$$(x,y,z) = (x,y,z) + n_1\vec{v}_1 + n_2\vec{v}_2 + n_3\vec{v}_3 \tag{45}$$

where the n_i are any integers. To simplify the following discussion, we will only consider the case of $R^2\times T^2$, in which case we will drop the vector $\vec{v}_3$. A convenient choice for the remaining two vectors is

$$\begin{aligned}\vec{v}_1 &= (0,0,\beta_z)\\ \vec{v}_2 &= (0,\beta_y\cos\theta,\beta_y\sin\theta) \equiv (0,\beta_y c,\beta_y s)\end{aligned} \tag{46}$$

where we will assume $\cos\theta>0$.

The desired Green's functions may now be determined by the method of images [27]. The point-splitting calculation then yields as the nonvanishing components of the stress tensor:

$$\begin{aligned}\langle T_{tt}\rangle &= -\langle T_{xx}\rangle = -\frac{4N^2}{\pi^2}\sum_{m,n=-\infty}^{\infty}{}' \frac{[1-(-1)^q]}{[(n\beta_z)^2+(m\beta_y)^2+2nms\beta_z\beta_y]^2}\\ \langle T_{yy}\rangle &= \frac{4N^2}{\pi^2}\sum_{m,n=-\infty}^{\infty}{}' \frac{[1-(-1)^q][(mc\beta_y)^2-3(n\beta_z+ms\beta_y)^2]}{[(n\beta_z)^2+(m\beta_y)^2+2nms\beta_z\beta_y]^3}\\ \langle T_{zz}\rangle &= \frac{4N^2}{\pi^2}\sum_{m,n=-\infty}^{\infty}{}' \frac{[1-(-1)^q][(n\beta_z+ms\beta_y)^2-3(mc\beta_y)^2]}{[(n\beta_z)^2+(m\beta_y)^2+2nms\beta_z\beta_y]^3}\\ \langle T_{yz}\rangle &= -\frac{16N^2}{\pi^2}\sum_{m,n=-\infty}^{\infty}{}' \frac{[1-(-1)^q](mc\beta_y)(n\beta_z+ms\beta_y)}{[(n\beta_z)^2+(m\beta_y)^2+2nms\beta_z\beta_y]^3}\end{aligned} \tag{47}$$

where the prime on the summations indicates that summation does not include $(m,n)=(0,0)$. The choice of the exponent q depends on the fermion boundary conditions around the $\vec{v}_1$ and $\vec{v}_2$ cycles:

$$\begin{aligned} q &= 0 & (\vec{v}_1,\vec{v}_2) &= (+,+)\\ &= n & &= (+,-)\\ &= m & &= (-,+)\\ &= n+m & &= (-,-).\end{aligned} \tag{48}$$

Of course, $\langle T_{ab}\rangle=0$ for the $(+,+)$ boundary conditions for which supersymmetry remains unbroken. In the remaining cases, one sees that as expected the result is traceless, i.e., $\eta^{ab}\langle T_{ab}\rangle=0$. Also for generic angles, one has an off-diagonal contribution in $\langle T_{yz}\rangle$. However, it is straightforward to show that this term vanishes for the special case that $\beta_y\sin\theta=k\beta_z$ for some integer k. In this case, one can reorganize the calculation in terms of new orthogonal basis vectors, $(\vec{v}_1,\vec{v}_2')$ where $\vec{v}_1\cdot\vec{v}_2' = 0$. So one may assume that $\beta_y|\sin\theta|<\beta_z$ without loss of generality.

We are particularly interested in the energy density, which may be rewritten as

$$\langle T_{tt}\rangle = -\frac{\pi^2}{6}N^2\left(\frac{\delta_{1,-}}{\beta_z^4}+\frac{\delta_{2,-}}{\beta_y^4}\right) - \frac{16N^2}{\pi^2}\sum_{n,m=1}^{\infty}\frac{[1-(-1)^q][(n^2\beta_z^2+m^2\beta_y^2)^2+4(nms\beta_z\beta_y)^2]}{[(n^2\beta_z^2+m^2\beta_y^2)^2-4(nms\beta_z\beta_y)^2]^2} \tag{49}$$

where $\delta_{i,-}=0$ for periodic boundary conditions around the i cycle, and $\delta_{i,-}=1$ for antiperiodic boundary conditions. With antiperiodic boundary conditions around the first cycle, we recover the previous result (41) by taking the limit $\beta_y\to\infty$. For finite β_y, it is clear that the extra contributions make the Casimir energy density even more negative.[7] In particular, even when

[7]Even without performing the final summation, it is clear that individual terms in the sum are either zero or negative, and that the total sum is finite.

the second cycle has periodic boundary conditions, even though the second term above vanishes the infinite sum is making a negative contribution to lower the Casimir energy below that in the previous result.

One may now ask which dual supergravity solutions describe these field theory configurations. From the discussion in Sec. II C, one can infer the asymptotic form of the solutions. However, finding the full solutions of the nonlinear supergravity equations is a difficult problem. Fortunately, solutions which appear to describe the case where $\sin\theta=0$, i.e., $\vec{v}_1\cdot\vec{v}_2=0$, are already available in the literature [26]. The five-dimensional metric may be written as

$$ds^2=\frac{l^2\,dr^2}{r^2\left(1-\frac{\mu^4}{r^4}\right)}+\frac{r^2}{l^2}\left[\left(1-\frac{\mu^4}{r^4}\right)^{(1/2)(1-\alpha_1-\alpha_2)}\times(-dt^2+dx^2)+\left(1-\frac{\mu^4}{r^4}\right)^{\alpha_1}dy^2+\left(1-\frac{\mu^4}{r^4}\right)^{\alpha_2}dz^2\right] \tag{50}$$

while the dilaton remains constant. The exponents in Eq. (50) lie on the ellipse given by

$$3(\alpha_1^2+\alpha_2^2)+2(\alpha_1\alpha_2-\alpha_1-\alpha_2)=1. \tag{51}$$

It is straightforward to solve this quadratic constraint to eliminate α_1 with

$$\alpha_{1\pm}=\frac{1}{3}[1-\alpha_2\pm2(1+\alpha_2-2\alpha_2^2)^{1/2}]\;. \tag{52}$$

One may note here that if one allows μ^4 to take negative values, the solutions characterized by $(\mu^4,\alpha_1,\alpha_2)$ and $(\tilde{\mu}^4,\tilde{\alpha}_1,\tilde{\alpha}_2)=(-\mu^4,\frac{1}{2}-\alpha_1,\frac{1}{2}-\alpha_2)$ are identical up to a diffeomorphism. Since $\tilde{\alpha}_{1\mp}=\frac{1}{2}-\alpha_{1\pm}$, one need only consider the positive branch α_{1+} in Eq. (52) by including negative values of μ^4.

Equation (50) becomes the AdS soliton (39) for $(\alpha_1,\alpha_2)=(1,0)$ or $(0,1)$, and in the limit $\mu\to0$ the solution reduces to AdS space in horospheric coordinates. Apart from these special cases, the geometry is singular at $r=\mu$. For example, $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\sim\mu^2/l^4/(r-\mu)^2$ as r approaches μ.

To determine the corresponding field theory stress-energy, we use the prescription of Sec. II C. We consider the background to be AdS space, and so asymptotically the radial part of the metric can be put in the AdS form with the coordinate transformation

$$r=R\left(1+\frac{\mu^4}{8R^4}\right). \tag{53}$$

Asymptotically the metric (50) becomes

$$ds^2\simeq\frac{l^2}{R^2}dR^2+\frac{R^2}{l^2}\left[\left(1+\frac{(2\alpha_1+2\alpha_2-1)\mu^4}{4R^4}\right)(-dt^2+dx^2)+\left(1+\frac{(1-4\alpha_1)\mu^4}{4R^4}\right)dy^2+\left(1+\frac{(1-4\alpha_2)\mu^4}{4R^4}\right)dz^2\right]. \tag{54}$$

From this asymptotic metric, one can read off the metric perturbation and then applying Eq. (32) yields

$$\langle T_{ab}\rangle=\frac{\mu^4}{16\pi G_5 l^5}\begin{pmatrix}-(2\alpha_1+2\alpha_2-1)&0&0&0\\0&2\alpha_1+2\alpha_2-1&0&0\\0&0&1-4\alpha_1&0\\0&0&0&1-4\alpha_2\end{pmatrix} \tag{55}$$

which has a form reminiscent of the field theory result (47), i.e., the stress tensor is traceless and generically $-\langle T_{tt}\rangle=\langle T_{xx}\rangle\neq\langle T_{yy}\rangle\neq\langle T_{zz}\rangle$. Since this stress tensor is diagonal, however, it seems that this solution can only describe the situation with $\sin\theta=0$, i.e., the cycles on the torus are orthogonal.

Up to this point, no consideration has been made of identification of the y and z coordinates in the supergravity background (50). To parallel the field theory calculation, we should identify $y\sim y+\beta_y$ and $z\sim z+\beta_z$. In the special case of the AdS soliton, demanding that the geometry be free of singularities relates the periodicity of one of the coordinates to the parameter μ [as described below Eq. (39)]. However, in the present case with generic exponents, one cannot avoid a curvature singularity at $r=\mu$. Therefore without understanding the stringy physics that underlies this region of strong curvature, there is no natural way to relate the periods to the parameter μ. Given the results for the near-extremal D-branes and the AdS soliton, one might expect that the ratios of the various components of the stress-energy are the same in the strong coupling supergravity regime as arise in the weak coupling calculations. With this assumption for a given pair of periodicities (and $\sin\theta=0$), one could calculate the relative size of the components of the weak coupling stress tensor (47), and then match these ratios in the strong coupling result (55) with a choice of exponents. Given the infinite double sums in Eq. (47), we have no analytical results to offer. However, one can examine the field theory stress tensor numerically, and it is clear (for $\sin\theta=0$) that one can always choose the exponents in Eq. (55) to match the overall form of $\langle T_{ab}\rangle$ in the two calculations. For example, with $(-,-)$ boundary conditions, as β_z/β_y varies from 1 to ∞, $\langle T_{ab}\rangle$ is matched by choosing α_{1+} in Eq. (52) and choosing α_2 between $(1+\sqrt{3})/4$ and 1. There remains the question of the overall normalization of the strong coupling stress-energy, but this seems to require a knowledge of physics at string scale curvatures.

IV. DISCUSSION

The field theory stress-energy tensor provides an interesting tool with which to study the AdS-CFT correspondence. In Sec. II, we have provided a number of different approaches to calculating $\langle T_{ab}\rangle$ for a given supergravity solution. Note that each of the calculations presented there requires a background solution, which essentially defines the zero for the stress tensor. That is the calculations yield the stress-energy of the given solution relative to the reference background.

The presence of a background solution is useful in making contact with earlier discussions of expectation values of CFT operators [5]. This discussion originally relied on considering solutions of the linearized equations of motion around AdS space, however, it was actually extended to solutions of the full nonlinear supergravity equations in considering D-instantons [5]. One can (at least roughly) classify the solutions of the linearized equations as modes which are singular at the boundary of AdS, and those which are singular at the interior. The modes that are singular at the boundary are the ones associated with the source currents for the CFT in calculating correlation functions [3], while those that are singular at the interior are associated with expectation values [5]. However the fact that the latter modes are singular (i.e., reach large values) in the interior of AdS means that one must go beyond the linearized equations of motion to consider expectation values in general.[8] The black hole solutions in Eqs. (15) and (24) are a good example in that they are solutions of the full nonlinear (super)gravity equations of motion. However, asymptotically these solutions approach AdS space, and one can regard the deviations of the metric from the AdS solution as solutions of the linearized gravity equations. Closer examination shows that these linearized solutions correspond to modes which become singular in the interior of AdS, which now simply means that the full solutions enter a nonlinear regime. With the choice of coordinates in Sec. II C, one can further match the expectation value of the stress-energy to the general calculations discussed in Ref. [5].

On the other hand, matching the asymptotic geometry of the (super)gravity solutions to that of a reference background is a technical nuisance. In fact there are solutions for which there is no natural choice of a background solution, e.g., Taub-NUT-AdS — as discussed in [28]. From this point of view, Balasubramanian and Kraus [15] recently presented a superior technique. While the basis of their calculations is the quasilocal stress-energy [22] discussed in Sec. II B, they avoid the background subtraction by introducing a ''counterterm subtraction,'' which only relies on the intrinsic boundary geometry of the solution of interest. This new technique provides a definition of the total energy which then represents a remarkable departure from previous investigations of gravitational energy, which appears to be unique to asymptotically AdS spacetimes. While relying on a background subtraction was sufficient for our investigation of the Casimir energies on toroidal geometries in Sec. III, it seems that the counterterm subtraction technique will be essential in calculating the Casimir stress-energy for more complicated geometries. For example, one can determine the Casimir energy for the super-Yang-Mills theory on $R\times S^3$ from a supergravity calculation on AdS_5 alone [15].

By considering the expectation value of the stress-energy tensor in the dual CFT, we seem to have found an interesting interpretation of the supergravity solutions (50) found in Ref. [26]. These solutions appear to be dual to the CFT on a two-torus with nonsupersymmetric boundary conditions imposed on the fermions around the cycles of the torus. As mentioned previously, the solutions provided by Ref. [26] are general enough that the discussion here and in Sec. III can be extended to the CFT on T^3 or T^4, as well as the AdS-CFT correspondence in higher dimensions. However, these solutions appear to be limited to the case where the cycles on the torus are orthogonal, e.g., $\sin\theta=0$ in Eq. (46). It may be interesting to extend this family of solutions to include tori with non-orthogonal cycles, for which generically the stress tensor acquires off-diagonal terms. From the discussion of Sec. II C then, one sees that the new supergravity solutions will include nontrivial off-diagonal metric components. Of course, the coordinates of Sec. II C may not be the optimal choice for actually determining the full nonlinear solutions. It may be that considering the transformation properties of the stress-energy under the action of $SL(d,Z)$ on T^d (see, e.g., [29]) may be useful in trying to construct the extended family of solutions. In any event, given such a set of solutions, it would be interesting to understand the action of the $SL(d,Z)$ symmetry on the supergravity spacetime. It could be that this symmetry would be a useful tool in determining the overall normalization of the strong coupling stress-energy without recourse to a complete understanding of string theory in regions of strong curvature.

We should remark that we have only found an interesting interpretation for a subset of the solutions in Eq. (50). Fixing α_2 (and hence α_{1+}) determines the ratio of the components of the stress-energy (55), and so for a certain set of boundary conditions in the CFT, this then fixes the ratio β_z/β_y from Eq. (47). Fixing μ^4 then sets the overall scale, and so through some unknown stringy physics, this determines β_z. However, any identifications in y and z are left implicit in the supergravity solution (50), and so for fixed α_2 and μ^4, these solutions exist for arbitrary values of β_y and β_z. Therefore our interpretation in terms of antiperiodic fermion boundary conditions only applies to a set of measure zero in the full space of solutions implicitly given by Eq. (50). It may be that more creative boundary conditions may allow one to provide a CFT interpretation for some (discrete number) of the other solutions, but it seems unlikely that a reasonable interpreta-

[8]The linearized modes are still useful in considering the expectation values associated with test probes moving in the supergravity spacetime [5,6].

tion can be found for a generic set of parameters in this family of solutions.

This then raises the question of which singularities or which singular solutions are physically interesting in the context of the AdS-CFT correspondence. One response would be that such singularities simply represent regions of strong curvature where the low energy (super)gravity theory breaks down, but that the singularity would be ''resolved'' in a physically sensible way by the full theory of quantum gravity (e.g., superstring or M theory). It was argued in Ref. [30] that this point of view cannot be correct, by considering negative mass Schwarzschild solution of the Einstein equations. The argument there applies equally well in present context with a negative cosmological constant. If the timelike singularities at the center of negative mass ''black hole''-AdS solutions were resolved within the full quantum gravity theory, no stable ground state would exist. Hence it is clear that certain singularities must be unphysical and so not all singular solutions (or solutions with regions of strong curvature) are physically relevant.

In certain special cases, singular solutions may be distinguished by being supersymmetric, and so there may be greater merit in considering such singularities. We remind the reader that such a solution implicitly played a role in Sec. III. The background solution in that case was AdS space with periodic identifications in the horospheric coordinates. This solution has a conical singularity at the null surface $r=0$, which is a horizon in the absence of any identifications. The fact that the background is supersymmetric would appear to add weight to our assumption that string theory is able to resolve this singularity. In the AdS-CFT context, charged black hole solutions have an interesting interpretation [31], but it turns out that the corresponding supersymmetric solutions [32] are actually naked (null) singularities whose role remains to be determined.

In the case of the metric (50), the singularity is again a null singularity as in the two preceding examples, but the solution is not supersymmetric. We might add that if one evaluates the supergravity action (making a background subtraction with AdS) the result is finite [26]. This comes about because the metric is a solution of the supergravity equations, and so the curvature singularity at $r=0$ does not manifest itself in the Ricci scalar. Divergences are present in the curvature scalars which appear as the higher order α' corrections to the supergravity action — see, for example, [23]. Of course, to properly evaluate the action including such corrections, one would have to construct a solution of the higher order equations of motion. However, if the higher order terms play more than a perturbative role (as they would near the singularity), one must (determine and) solve the full superstring equations of motion to produce a consistent solution.

Thus the AdS-CFT correspondence seems to provide another situation in which curvature singularities seem to play an interesting role in string theory (see, for example, [33]). Given the discussion up to this point, it appears remarkable that the case with the CFT on $R^3\times S^1$ corresponds to a dual supergravity solution which is everywhere smooth. However, we would now like to present an alternative point of view which argues that this behavior should not be exceptional. The motivation for these arguments will be cosmic censorship [34].

Let us begin by considering the AdS_5 spacetime in horospheric coordinates, i.e., Eq. (39) with $\mu^4=0$, in which one of the spatial coordinates is periodically identified, say $y=y+\beta_y$. Further we expect the following conclusion about the supergravity to be independent of any boundary conditions on the supergravity fermions in this direction. Now introducing a planar configuration of matter into this spacetime with sufficient density, we expect that the system will collapse to form a black hole which settles to a metric of the form in Eq. (24) with $p=3$. This expectation is prejudiced by our experience with cosmic censorship and black hole uniqueness theorems [35] in other settings. Distinguishing the metric component g_{yy} from the other spatial directions would be like adding ''hair'' to the black hole and produce a singular configuration similar to the solutions (50). Cosmic censorship dictates that gravity avoid this solution dynamically by radiating away any such hair during the collapse that leads to the formation of the black hole. Thus the black hole solution (24) would appear to be the physically relevant solution *independent of the period* β_y. Given this conclusion, one would expect that a double analytic continutation will yield the supergravity solution dual to the CFT on $R^2\times T^2$. Certainly the calculations for the thermal ensemble on S^1 and the Casimir stress-energy on T^2 have a common Euclidean framework, as discussed in Ref. [14]. One would thus conclude that the AdS soliton (39) is still the relevant dual supergravity solution independent of any additional compactifications, as well as of the boundary conditions imposed in those directions.

Hence independent of the period β_y, the expectation value of the stress tensor would be that of the AdS soliton as given in Eq. (43). In particular then one has $3\langle T_{tt}\rangle=-3\langle T_{xx}\rangle=-3\langle T_{yy}\rangle=\langle T_{zz}\rangle$, independent of β_y or the boundary conditions imposed on the y-cycle. Certainly this result does not match the form of the weak coupling result (even with $\sin\theta=0$) given in Eq. (47). Of course, the form of the latter result does go over to that of the supergravity calculation in the limit that $\beta_y\to\infty$. There is no inconsistency here, of course. One would simply conclude that the various components of the stress tensor evolve independently as the 't Hooft parameter increases from zero in the explicit field theory calculation to large values where the supergravity result applies. Thus the effects of the additional identifications seem to be erased by the supergravity dynamics in the large N limit.

While this scenario may be reasonable in a situation where the periodicity of one direction is much smaller than that of any others, it should appear problematic for periodicities of roughly the same size. Analogous to the discussion in Ref. [3], a possible resolution is that the supergravity partition function (1) includes contributions from different AdS solitons in each of which one of the compact directions with antiperiodic fermions shrinks to zero on the interior. For a given set of periodicities, the relative size of the contributions of the different supergravity solutions would be determined by their Euclidean action. In the case where one of the periodicities is much smaller than the others, the corresponding AdS soliton dominates the partition function. When the periodicities are roughly the same size, no single solution would be dominant. The expectation $\langle T_{ab}\rangle$ would presumably then be given by a weighted sum of that calculated for the individual solutions.[9]

Thus we have presented two very different pictures of the supergravity description of the dual CFT on a torus. The first involves singular solutions, but relies on the assumption that the ratios of the various components of the stress tensor are preserved as the 't Hooft coupling varies. The only evidence given to support of this assumption was that it was observed to be true for the CFT on $R^3\times S^1$. However, this is not a very strong argument since in that case, the form of the stress-energy is really determined by the symmetries of the background geometry and the traceless property of the stress tensor. In the second scenario, the supergravity dynamics erases the complicated details which the field theory boundary conditions produce at weak coupling. This picture is motivated by cosmic censorship and black hole uniqueness which, however, have not been well studied in the context of AdS space. Another weakness in these arguments about the dynamics is that while it is claimed that the post-collapse black hole should be nonsingular, the pre-collapse configuration will almost certainly contain a singularity at the interior due to the periodic identification of z. Even if the second scenario is correct, it seems that the gravitational abhorence of singularities must only be a large N effect in the CFT. It would be interesting to understand the physics of these configurations at finite N where it is claimed that the super-Yang-Mills theory should resolve the singularities associated with gravitational collapse [37].

Finally we would like to comment on the implications of the present results for the positive energy theorem of Ref. [14]. We have found that there are singular supergravity solutions which have a lower energy than the AdS soliton, which was conjectured to be a minimum energy solution. This should not come as a surprise. One can easily find singular solutions with arbitrarily negative energies, such as negative mass "black holes." The positive energy theorem at least in the form of conjectures 2 and 3 in Ref. [14] is that the AdS soliton should be the minimum energy solution within the space of smooth solutions. The important point that was considered here was that some singular solutions may be physically relevant for Type IIb supergravity, since the singularities are resolved in the full string theory. If this is the case, it would certainly affect conjecture 1 which was phrased in terms of the ten-dimensional type IIb supergravity. As we have discussed above though, string theory may still choose to ignore these singular solutions and describe the dual CFT in terms of asymptotically AdS geometries which are everywhere smooth.

[9]It appears that with this approach that there can be a "phase" transition in the thermal ensemble on a torus from the high entropy black hole to the zero-entropy AdS soliton at low temperatures [38] — see Refs. [3,31,36] for discussions of similar phase transitions.

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