Mode regularization of the configuration space path integral in curved space

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The path integral representation of the transition amplitude for a particle moving in curved space has presented unexpected challenges since the introduction of path integrals by Feynman fifty years ago. In this paper we discuss and review mode regularization of the configuration space path integral, and present a three loop computation of the transition amplitude to test with success the consistency of such a regularization. The key features of the method are the use of the ''Lee-Yang'' ghost fields, which guarantee a consistent treatment of the nontrivial path integral measure at higher loops, and an effective potential specific to mode regularization which arises at the two loop order. We also perform the computation of the transition amplitude using the regularization of the path integral by time discretization, which also makes use of Lee-Yang ghost fields and needs its own specific effective potential. This computation is shown to reproduce the same final result as the one performed in mode regularization. [S0556-2821(99)02216-X]

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I. INTRODUCTION

The Schrödinger equation for a particle moving in curved space with the metric $g_{\mu\nu}(x)$ has many applications ranging from nonrelativistic diffusion problems (described by a Wick rotated version of the Schrödinger equation) to the relativistic description of particles moving in curved space-time. However, it cannot be solved exactly for an arbitrary background metric $g_{\mu\nu}(x)$, and one has to resort to some kind of perturbation theory. A very useful perturbative solution can be obtained by employing the well-known ansatz introduced by De Witt [1], also known as the heat kernel ansatz. This ansatz makes use of a power series expansion in the time of propagation of the particle. The coefficients of the power series are then determined iteratively by requiring that the Schrödinger equation be satisfied perturbatively.

Equivalently, the solution of the Schrödinger equation can be represented by a path integral, as shown by Feynman fifty years ago [2]. One can formally write down the path integral for the particle moving in curved space and check that the standard loop expansion reproduces the structure of the heat kernel ansatz of De Witt. However, the proper definition of the path integral in curved space is not straightforward. In fact it has presented many challenges due to complications arising from (i) the nontrivial path integral measure [3] and (ii) the proper discretization of the action necessary to regulate the path integral. A quite extensive literature has been produced over the years addressing especially the latter point [4].

In this paper we short cut most of the literature and discuss a method of defining the path integral by employing mode regularization as it is by now standard in many calculations done in quantum field theory. The methods extends the one employed by Feynman and Hibbs in discussing mode regularization of the path integral in flat space [5]. It has been introduced and successively refined in Refs. [6–8] where quantum mechanics was used to compute one loop trace anomalies of certain quantum field theories. The key feature is to employ ghost fields to treat the nontrivial path integral measure as part of the action, in the spirit of Lee and Yang [3]. These ghost fields have been named ''Lee-Yang'' ghosts and allow us to take care of the nontrivial path integral measure at higher loops in a consistent manner. The path integral is then defined by expanding all fields, including the ghosts, in a sine expansion about the classical trajectories and integrating over the corresponding Fourier coefficients. The necessary regularization is obtained by integrating all Fourier coefficients up to a fixed mode M, which is eventually taken to infinity. A drawback of mode regularization is that it does not respect general coordinate invariance in target space: a particular noncovariant counterterm has to be used in order to restore that symmetry [8]. General arguments based on power counting (quantum mechanics can be thought as a superrenormalizable quantum field theory) plus the fact that the correct trace anomalies are obtained by the use of this path integral suggest that the mode regularization described above is consistent to any loop order without any additional input.

As usual when dealing with formal constructions, it is a good practice to check with explicit calculations the proposed scheme. It is the purpose of this paper to present a full three loop computation of the transition amplitude. The result is found to be correct since it solves the correct Schrödinger equation at the required loop order. This gives a powerful check on the method of mode regularization for quantum mechanical path integrals on curved space. In addi-

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tion, we present our computation in such a way that it can be easily extended and compared to the time discretization method developed in Ref. [9], which is also based on the use of the Lee-Yang ghosts. This method requires its own specific counterterm (also called effective potential) to restore general coordinate invariance. As expected both schemes give the same answer.

The paper is structured as follows. In Sec. II we review the method of mode regularization and discuss the effective potential specific to this regularization. In Sec. III we present a three loop computation of the transition amplitude. Here we make use of general coordinate invariance to select Riemann normal coordinates to simplify an otherwise gigantic computation. We check that the result satisfies the Schrödinger equation at the correct loop order. Moreover, from this result one can extract the leading terms of the Seeley–De Witt coefficients a_0, a_1, a_2 for noncoinciding points, as defined in the forthcoming Eq. (6). In Sec. IV we extend our computation to the time discretization scheme. This is found to compare successfully with the results previously obtained in Sec. III. Finally, in Sec. V we present our conclusions and perspectives. In the Appendix we present a technical section with a list of loop integrals employed in the text. In particular, we discuss how to compute them in mode regularization as well as in time discretization regularization.

II. MODE REGULARIZATION

The Schrödinger equation for a particle of mass m moving in a D-dimensional curved space with metric $g_{\mu\nu}(x)$ and coupled to a scalar potential $V(x)$ is given by

$$i\hbar \frac{\partial}{\partial t}\Psi = H\Psi, \tag{1}$$

where

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(x) \tag{2}$$

with ∇^2 the covariant Laplacian acting on scalars. It can be obtained by canonical quantization of the model described by the classical action

$$S_{\rm cl}[x] = \int dt \left[\frac{m}{2} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu - V(x)\right] \tag{3}$$

when ordering ambiguities are fixed by requiring general coordinate invariance in target space and requiring in addition that no scalar curvature term be generated by the orderings in the quantum potential.[1] For convenience we will Wick rotate the time variable $t \to -it$ and set $m=\hbar=1$ to obtain the following heat equation:

$$-\frac{\partial}{\partial t}\Psi = \left[-\frac{1}{2}\nabla^2 + V(x)\right]\Psi \tag{4}$$

and corresponding euclidean action

$$S[x] = \int dt \left[\frac{1}{2} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + V(x)\right]. \tag{5}$$

As mentioned in the Introduction the heat equation can be solved by the heat kernel ansatz of De Witt [1]:

$$\Psi(x,y,t) = \frac{1}{(2\pi t)^{D/2}} e^{-\sigma(x,y)/t} \sum_{n=0}^{\infty} a_n(x,y) t^n \tag{6}$$

which depends parametrically on the point y^μ that specifies the boundary condition $\Psi(x,y,0) = \delta^D(x-y)/\sqrt{g(x)}$. Here $\sigma(x,y)$ is the so-called Synge world function and corresponds to half the squared geodesic distance. The coefficients $a_n(x,y)$ are sometimes called Seeley–De Witt coefficients[2] and are determined by plugging the ansatz (6) into Eq. (4) and matching powers of t.

Now we want to describe in detail how to get the solution of Eq. (4) by the use of a path integral which employs the classical action in Eq. (5). Following Refs. [6–8] we write the transition amplitude for the particle to propagate from the initial point x_i^μ at time t_i to the final point x_f^μ at time t_f as follows:

$$\langle x_f^\mu, t_f | x_i^\mu, t_i \rangle \equiv \langle x_f^\mu | e^{-\beta H} | x_i^\mu \rangle = \int_{x(-1)=x_i}^{x(0)=x_f} \tilde{\mathcal{D}}x \exp\left[-\frac{1}{\beta} S\right], \tag{7}$$

where

$$\begin{aligned} S &\equiv S[x,a,b,c] \\ &= \int_{-1}^{0} d\tau \left[\frac{1}{2} g_{\mu\nu}(x)(\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu) \right. \\ &\quad \left. + \beta^2 [V(x) + V_{\rm MR}(x)]\right], \end{aligned} \tag{8}$$

$$V_{\rm MR} = \frac{1}{8} R - \frac{1}{24} g^{\mu\nu} g^{\alpha\beta} g_{\gamma\delta} \Gamma_{\mu\alpha}{}^{\gamma} \Gamma_{\nu\beta}{}^{\delta}, \tag{9}$$

$$\tilde{\mathcal{D}}x = \mathcal{D}x \mathcal{D}a \mathcal{D}c \mathcal{D}c. \tag{10}$$

For commodity we have shifted and rescaled the time parameter in the action, $t = t_f + \beta\tau$ with $\beta = t_f - t_i$, so that $-1 \leqslant \tau \leqslant 0$. Note that the total time of propagation β plays the role of the Planck constant $\hbar$ (which we have already set to 1) and counts the number of loops. In the loop expansion generated by β the potentials V and $V_{\rm MR}$ start contributing

[1] One could be more general by coupling the particle also to a vector potential $A_\mu(x)$. It is simple to do so, since mode regularization will respect the corresponding gauge invariance [7]. For simplicity we set $A_\mu(x)=0$ in this paper.

[2] It is also customary to redefine the $a_n(x,y)$ by extracting a common factor $\Delta^{1/2}(x,y)$, where $\Delta(x,y)$ is a scalar version of the so-called Van Vleck–Morette determinant.

only at two loops.[3] The full action S includes terms proportional to the Lee-Yang ghosts, namely the commuting ghosts a^μ and the anticommuting ghosts b^μ and c^μ. Their effect is to reproduce a formally covariant measure: integrating them out produces $\tilde{\mathcal{D}}x=\Pi\{\det g_{\mu\nu}[x(\tau)]\}^{1/2}d^D x(\tau)$. As we will discuss, mode regularization destroys this formal covariance. Nevertheless general covariance is recovered thanks to the effects of the potential $V_{\rm MR}$ directly included in the action (8). With precisely this counterterm the mode regulated path integral in Eq. (7) solves the equation in Eq. (4) in both sets of variables (x_f^μ,t_f) and (x_i^μ,t_i) and with the boundary condition $\langle x_f^\mu,t|x_i^\mu,t\rangle=\delta^D(x_f^\mu-x_i^\mu)/\sqrt{g(x)}$.

For an arbitrary metric $g_{\mu\nu}(x)$ one is able to calculate the path integral only in a perturbative expansion in β and in the coordinate displacements ξ^μ about the final point x_f^μ: $\xi^\mu \equiv x_i^\mu - x_f^\mu$. The actual computation starts by parametrizing

$$x^\mu(\tau)=x_{bg}^\mu(\tau)+q^\mu(\tau), \tag{11}$$

where $x_{bg}^\mu(\tau)$ is a background trajectory and $q^\mu(\tau)$ the quantum fluctuations. The background trajectory is taken to satisfy the free equations of motion and is a function linear in τ connecting x_i^μ to x_f^μ in the chosen coordinate system, thus enforcing the proper boundary conditions

$$x_{bg}^\mu(\tau)=x_f^\mu-\xi^\mu\tau. \tag{12}$$

Note that by free equations of motion we mean the ones arising from (8) by neglecting the potentials $V+V_{\rm MR}$ and by keeping the constant leading term in the expansion of the metric $g_{\mu\nu}(x)$ around the final point x_f^μ, thus making the space flat. Obviously, one could also use the exact solution of the classical equations of motion as background trajectory, but this would not change the result of the computation. It would correspond just to a different parametrization of the space of paths.

The quantum fields $q^\mu(\tau)$ in Eq. (11) should vanish at the time boundaries since the boundary conditions are already included in $x_{bg}^\mu(\tau)$. Therefore they can be expanded in a sine series. For the Lee-Yang ghosts we use the same Fourier expansion since the classical solutions of their field equations are $a^\mu=b^\mu=c^\mu=0$. Hence

$$\phi^\mu(\tau)=\sum_{m=1}^{\infty}\phi_m^\mu\sin(\pi m\tau), \tag{13}$$

where ϕ stands for all the quantum fields q^μ,a^μ,b^μ,c^μ. The measure $\tilde{\mathcal{D}}x$ in Eq. (10) is now properly defined in terms of integration over the Fourier coefficients ϕ_m^μ as follows:

$$\tilde{\mathcal{D}}x=\mathcal{D}q\mathcal{D}a\mathcal{D}b\mathcal{D}c=\lim_{M\to\infty}A\prod_{m=1}^{M}\prod_{\mu=1}^{D}m\,dq_m^\mu da_m^\mu db_m^\mu dc_m^\mu, \tag{14}$$

where A is a constant. Note that this fixes the path integral for a free particle to

$$\int\tilde{\mathcal{D}}x\exp\left[-\frac{1}{\beta}S_{\rm free}\right]=Ae^{-\xi^2/2\beta}, \tag{15}$$

where

$$S_{\rm free}=\int_{-1}^{0}d\tau\frac{1}{2}\delta_{\mu\nu}(\dot{x}^\mu\dot{x}^\nu+a^\mu a^\nu+b^\mu c^\nu). \tag{16}$$

It is well-known that $A=(2\pi\beta)^{-D/2}$, however, this value can also be deduced later on from a consistency requirement.

The way to implement mode regularization is now quite clear: limiting the integration over the number of modes for each field to a finite mode number M gives the natural regularization of the path integral. This regularization resolves the ambiguities that show up in the continuum limit.

The perturbative expansion is generated by splitting the action into a quadratic part S_2, which defines the propagators, and an interacting part $S_{\rm int}$, which gives the vertices. We do this splitting by expanding the action about the final point x_f^μ and obtain

$$S=S_2+S_{\rm int}=S_2+S_3+S_4+\cdots, \tag{17}$$

where

$$S_2=\int_{-1}^{0}d\tau\,\frac{1}{2}\,g_{\mu\nu}(\xi^\mu\xi^\nu+\dot{q}^\mu\dot{q}^\nu+a^\mu a^\nu+b^\mu c^\nu), \tag{18}$$

$$S_3=\int_{-1}^{0}d\tau\,\frac{1}{2}\,\partial_\alpha g_{\mu\nu}(q^\alpha-\xi^\alpha\tau)(\xi^\mu\xi^\nu+\dot{q}^\mu\dot{q}^\nu+a^\mu a^\nu+b^\mu c^\nu-2\dot{q}^\mu\xi^\nu), \tag{19}$$

$$S_4=\int_{-1}^{0}d\tau\bigg[\frac{1}{4}\partial_\alpha\partial_\beta g_{\mu\nu}(q^\alpha q^\beta+\xi^\alpha\xi^\beta\tau^2-2q^\alpha\xi^\beta\tau)\times(\xi^\mu\xi^\nu+\dot{q}^\mu\dot{q}^\nu+a^\mu a^\nu+b^\mu c^\nu-2\dot{q}^\mu\xi^\nu)+\beta^2(V+V_{\rm MR})\bigg]. \tag{20}$$

In this expansion all geometrical quantities, such as $g_{\mu\nu}$ and $\partial_\alpha g_{\mu\nu}$, as well as V and $V_{\rm MR}$, are evaluated at the final point x_f^μ, but for notational simplicity we do not exhibit this dependence. S_2 is taken as the free part and defines the propagators which are easily obtained from the path integral

$$\begin{aligned}\langle q^\mu(\tau)q^\nu(\sigma)\rangle&=-\beta\,g^{\mu\nu}(x_f)\Delta(\tau,\sigma),\\ \langle a^\mu(\tau)a^\nu(\sigma)\rangle&=\beta\,g^{\mu\nu}(x_f)\,{}^{\bullet\bullet}\Delta(\tau,\sigma),\\ \langle b^\mu(\tau)c^\nu(\sigma)\rangle&=-2\beta\,g^{\mu\nu}(x_f)\,{}^{\bullet\bullet}\Delta(\tau,\sigma),\end{aligned} \tag{21}$$

where Δ is regulated by the mode cutoff

$$\Delta(\tau,\sigma)=\sum_{m=1}^{M}\left[-\frac{2}{\pi^2m^2}\sin(\pi m\tau)\sin(\pi m\sigma)\right] \tag{22}$$

and has the following limiting value for $M\to\infty$:

$$\Delta(\tau,\sigma)=\tau(\sigma+1)\theta(\tau-\sigma)+\sigma(\tau+1)\theta(\sigma-\tau). \tag{23}$$

[3]Reintroducing $\hbar$ one can see that the classical potential V must be of order $\hbar^0$ while the counterterm $V_{\rm MR}$ is a truly two loop effect of order $\hbar^2$.

Note that we indicate ${}^{\bullet}\Delta(\tau,\sigma)=(\partial/\partial\tau)\Delta(\tau,\sigma)$, $\Delta^{\bullet}(\tau,\sigma)=(\partial/\partial\sigma)\Delta(\tau,\sigma)$ and so on. Details on the properties of these functions are given in the Appendix.

Now, the quantum perturbative expansion reads

$$\begin{aligned}\langle x_f^\mu,t_f|x_i^\mu,t_i\rangle &= \int \tilde{\mathcal{D}}x \exp\left[-\frac{1}{\beta}S\right]\\ &= A e^{-(1/2\beta)g_{\mu\nu}\xi^\mu\xi^\nu}\langle e^{-(1/\beta)S_{\rm int}}\rangle\\ &= A e^{-(1/2\beta)g_{\mu\nu}\xi^\mu\xi^\nu}\left(\left\langle 1-\frac{1}{\beta}S_3-\frac{1}{\beta}S_4 \right.\right.\\ &\quad \left.\left. +\frac{1}{2\beta^2}S_3^2\right\rangle + O(\beta^{3/2})\right). \end{aligned} \tag{24}$$

where the angle brackets denote the averaging with the free action S_2, and amount to use the propagators given in Eq. (21) in the perturbative expansion. Note that in the last line of the above equation we have kept only those terms contributing up to two loops, i.e., up to $O(\beta)$, by taking into account that $\xi^\mu \sim O(\beta^{1/2})$, as follows from the exponential appearing in the last line of Eq. (24) after one averages over ξ^μ. Note also that having extracted the coefficient A together with the exponential of the quadratic action S_2 evaluated on the background trajectory implies that the normalization of the left over path integral is such that $\langle 1\rangle=1$.

Using standard Wick contractions and going through a lengthy calculation one gets [8]

$$\left\langle -\frac{1}{\beta}S_3\right\rangle = -\frac{1}{\beta}\frac{1}{4}\partial_\alpha g_{\mu\nu}\xi^\alpha\xi^\mu\xi^\nu, \tag{25}$$

$$\left\langle -\frac{1}{\beta}S_4\right\rangle = \partial_\alpha\partial_\beta g_{\mu\nu}\left[\frac{\beta}{24}(g^{\mu\nu}g^{\alpha\beta}-g^{\mu\alpha}g^{\nu\beta})-\frac{1}{24}(g^{\mu\nu}\xi^\alpha\xi^\beta+g^{\alpha\beta}\xi^\mu\xi^\nu-2g^{\mu\alpha}\xi^\nu\xi^\beta)-\frac{1}{\beta}\frac{1}{12}\xi^\mu\xi^\nu\xi^\alpha\xi^\beta\right]+\beta(V+V_{\rm MR}), \tag{26}$$

$$\begin{aligned}\left\langle \frac{1}{2\beta^2}S_3^2\right\rangle &= \partial_\alpha g_{\mu\nu}\partial_\beta g_{\lambda\rho}\left[\frac{\beta}{96}(g^{\alpha\beta}g^{\mu\nu}g^{\lambda\rho}-4g^{\alpha\rho}g^{\mu\nu}g^{\beta\lambda}-6g^{\alpha\beta}g^{\mu\lambda}g^{\nu\rho}+4g^{\alpha\rho}g^{\beta\mu}g^{\nu\lambda}+4g^{\alpha\mu}g^{\beta\lambda}g^{\nu\rho})+\frac{1}{48}[g^{\mu\lambda}g^{\nu\rho}\xi^\alpha\xi^\beta\right.\\ &\quad +2(g^{\alpha\beta}g^{\mu\lambda}-g^{\alpha\lambda}g^{\mu\beta})\xi^\nu\xi^\rho+(2g^{\alpha\lambda}g^{\beta\rho}-g^{\alpha\beta}g^{\lambda\rho})\xi^\mu\xi^\nu+(2g^{\mu\beta}g^{\lambda\rho}-4g^{\mu\lambda}g^{\beta\rho})\xi^\alpha\xi^\nu]\\ &\quad \left.+\frac{1}{\beta}\frac{1}{96}(g^{\alpha\beta}\xi^\mu\xi^\nu\xi^\lambda\xi^\rho-4g^{\alpha\lambda}\xi^\mu\xi^\nu\xi^\beta\xi^\rho+4g^{\mu\lambda}\xi^\alpha\xi^\nu\xi^\beta\xi^\rho)+\frac{1}{\beta^2}\frac{1}{32}\xi^\alpha\xi^\mu\xi^\nu\xi^\beta\xi^\lambda\xi^\rho\right]. \end{aligned}\tag{27}$$

This gives the transition amplitude in the two-loop approximation.

To test its consistency one can use it to evolve in time an arbitrary wave function $\Psi(x,t)$

$$\Psi(x_f,t_f)=\int d^Dx_i\sqrt{g(x_i)}\langle x_f^\mu,t_f|x_i^\mu,t_i\rangle\Psi(x_i,t_i) \tag{28}$$

and verify if $\Psi(x_f,t_f)$ solves the correct Schrödinger equation.[4] Since the transition amplitude (24) together with the results (25), (26), and (27) is given in terms of an expansion around the final point (x_f,t_f), one Taylor expands the wave function $\Psi(x_i,t_i)$ and the measure $\sqrt{g(x_i)}$ in Eq. (28) about that point, performs the integration over $d^Dx_i^\mu=d^D\xi^\mu$, and matches the various terms. The leading term fixes A

$$\Psi=A(2\pi\beta)^{D/2}\Psi\rightarrow A=(2\pi\beta)^{-D/2}, \tag{29}$$

and the terms of order β give

$$\beta\left[-\partial_t\Psi+\frac{1}{2}\nabla^2\Psi-V\Psi\right]=0. \tag{30}$$

This last equation means that the wave function Ψ satisfies the correct Schrödinger equation (4) at the final point (x_f^μ,t_f).

It is interesting to note that the counterterm V_{MR} appears only in the last line of Eq. (26). Actually the value of the counterterm reported in Eq. (9) has been deduced in Ref. [8] by imposing that the transition amplitude would solve Eq. (30). General arguments can then be used to show that this

[4]The factor $\sqrt{g(x_i)}$ appearing in Eq. (28) is suggested by the expression of the path integral in Eq. (7) which is formally a scalar for $V_{\rm MR}=0$. However, general coordinate invariance is broken by mode regularization, and recovered thanks to the effects of the counterterm $V_{\rm MR}$. Therefore the measure appearing in Eq. (28) should be considered as an ansatz which is verified, for example, by the calculations presented in the next section.

counterterm should be left unmodified at higher loops. In fact one can consider quantum mechanics on curved spaces as a super renormalizable one-dimensional quantum field theory, and check by power counting that all possible superficial divergences can only appear at loop order 2 or less in β. In the next section we are going to check that it is so indeed, expelling doubts which have sometimes been raised that mode regularization would be inconsistent at higher loops. Thus one can consider mode regularization as a viable way of correctly defining the path integral in curved spaces.

III. THE TRANSITION AMPLITUDE AT THREE LOOPS

In this section we want to check Eq. (28) at the next order in β, which is equivalent to showing that the transition amplitude computed by the path integral satisfies the Schrödinger equation not only at the point (x_f^μ, t_f) but in a small neighborhood of it. This computation can be quite lengthy if done in arbitrary coordinates. To make it feasible we select a useful set of coordinates: the Riemann normal coordinates centred at the point x_f^μ. In such a frame of reference the coordinates of an arbitrary point x^μ contained in the neighborhood of the origin are given by a vector $z^\mu(x)$ belonging to the tangent space at the origin. This vector specifies the unique geodesic connecting the origin to the given point x^μ in a unit time. In such a frame of reference the coordinates of the origin are obviously given by $z^\mu(x_f)=0$. In what follows we will use Riemann normal coordinates which we keep denoting by x^μ since no confusion can arise.

The expansion of the metric around the origin is given by (see, e.g., Ref. [7] for a derivation)

$$
\begin{aligned}
g_{\mu\nu}(x)=g_{\mu\nu}(0)&+\frac{1}{3}R_{\alpha\mu\nu\beta}(0)x^\alpha x^\beta\\
&+\frac{1}{6}\nabla_\gamma R_{\alpha\mu\nu\beta}(0)x^\alpha x^\beta x^\gamma\\
&+\left(\frac{1}{20}\nabla_\gamma\nabla_\delta R_{\alpha\mu\nu\beta}(0)+\frac{2}{45}R_{\alpha\mu\sigma\beta}R_{\gamma\nu}{}^{\sigma}{}_{\delta}(0)\right)\\
&\times x^\alpha x^\beta x^\gamma x^\delta+O(x^5).
\end{aligned}\tag{31}
$$

Note that the coefficients in this expansion are tensors belonging to the tangent space at the origin. This is a property of Riemann normal coordinates.

In general, the terms contributing to the transition amplitude up to three loops are given by

$$
\begin{aligned}
\langle x_f^\mu,t_f|x_i^\mu,t_i\rangle&=Ae^{-(1/2\beta)g_{\mu\nu}\xi^\mu\xi^\nu}\langle e^{-(1/\beta)S_{\rm int}}\rangle\\
&=Ae^{-(1/2\beta)g_{\mu\nu}\xi^\mu\xi^\nu}\bigg(\bigg\langle 1-\frac{1}{\beta}(S_3+S_4+S_5+S_6)\\
&\quad+\frac{1}{2\beta^2}[(S_3+S_4)^2+2S_3S_5]-\frac{1}{6\beta^3}(S_3^3\\
&\quad+3S_3^2S_4)+\frac{1}{24\beta^4}S_3^4\bigg\rangle+O(\beta^{5/2})\bigg).
\end{aligned}\tag{32}
$$

Clearly the computation would be quite complex in arbitrary coordinates. Fortunately, in Riemann normal coordinates many terms are absent

$$S_3=0,\tag{33}$$

$$
\begin{aligned}
S_4=\int_{-1}^0 d\tau\bigg[&\frac{1}{6}R_{\alpha\mu\nu\beta}x^\alpha x^\beta(\dot{x}^\mu\dot{x}^\nu+a^\mu a^\nu+b^\mu c^\nu)\\
&+\beta^2(V+V_{\rm MR})\bigg],
\end{aligned}\tag{34}
$$

$$
\begin{aligned}
S_5=\int_{-1}^0 d\tau\bigg[&\frac{1}{12}\nabla_\gamma R_{\alpha\mu\nu\beta}x^\alpha x^\beta x^\gamma(\dot{x}^\mu\dot{x}^\nu+a^\mu a^\nu+b^\mu c^\nu)\\
&+\beta^2 x^\alpha\partial_\alpha(V+V_{\rm MR})\bigg],
\end{aligned}\tag{35}
$$

$$
\begin{aligned}
S_6=\int_{-1}^0 d\tau\bigg[&\left(\frac{1}{40}\nabla_\gamma\nabla_\delta R_{\alpha\mu\nu\beta}+\frac{1}{45}R_{\alpha\mu\sigma\beta}R_{\gamma\nu}{}^{\sigma}{}_{\delta}\right)\\
&\times x^\alpha x^\beta x^\gamma x^\delta(\dot{x}^\mu\dot{x}^\nu+a^\mu a^\nu+b^\mu c^\nu)\\
&+\frac{\beta^2}{2}x^\alpha x^\beta\partial_\alpha\partial_\beta(V+V_{\rm MR})\bigg].
\end{aligned}\tag{36}
$$

Note that all structures such as $R_{\mu\nu\alpha\beta}$, V, $V_{\rm MR}$, and derivatives thereof are evaluated at the origin of the Riemann coordinate system, but for notational simplicity we do not indicate so explicitly. The computation is still quite lengthy and we get

$$\left\langle -\frac{1}{\beta}S_4\right\rangle=\frac{1}{6}R_{\alpha\beta}\xi^\alpha\xi^\beta I_1-\frac{\beta}{6}R\,I_2-\beta(V+V_{\rm MR}),\tag{37}$$

$$\left\langle -\frac{1}{\beta}S_5\right\rangle=-\frac{1}{12}\nabla_\gamma R_{\alpha\beta}\xi^\alpha\xi^\beta\xi^\gamma I_3+\frac{\beta}{6}\nabla_\alpha R\xi^\alpha I_4-\frac{\beta}{2}\partial_\alpha(V+V_{\rm MR})\xi^\alpha,\tag{38}$$

$$\left\langle -\frac{1}{\beta}S_6\right\rangle=\left(\frac{1}{40}\nabla_\gamma\nabla_\delta R_{\alpha\beta}+\frac{1}{45}R_{\alpha\mu\nu\beta}R_\gamma{}^{\mu\nu}{}_\delta\right)\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta I_5-\frac{\beta}{40}\nabla^2R_{\alpha\beta}\xi^\alpha\xi^\beta I_6-\beta\left(\frac{1}{20}(\nabla_\alpha\nabla_\beta R+\nabla^\mu\nabla_\alpha R_{\beta\mu})+\frac{2}{45}R_{\alpha\mu\nu\beta}R^{\mu\nu}\right)\xi^\alpha\xi^\beta I_7-\beta\left(\frac{1}{20}\nabla^\mu\nabla^\nu R_{\alpha\mu\nu\beta}+\frac{1}{45}R_{\alpha\mu}R_\beta{}^\mu\right)\xi^\alpha\xi^\beta(I_6-I_7)-\frac{\beta}{30}R_{\alpha\mu\nu\lambda}R_\beta{}^{\mu\nu\lambda}\times\xi^\alpha\xi^\beta(I_6+I_7)+\beta^2\left(\frac{1}{20}\nabla^2R+\frac{1}{45}R^2_{\mu\nu}+\frac{1}{30}R^2_{\alpha\mu\nu\beta}\right)I_8-\frac{\beta}{6}\partial_\alpha\partial_\beta(V+V_{\rm MR})\xi^\alpha\xi^\beta+\frac{\beta^2}{2}\partial^\alpha\partial_\alpha(V+V_{\rm MR})I_9, \quad (39)$$

$$\left\langle\frac{1}{2\beta^2}S_4^2\right\rangle=\frac{1}{2}\left\langle-\frac{1}{\beta}S_4\right\rangle^2+\left\langle\frac{1}{2\beta^2}S_4^2\right\rangle_{\rm con}, \quad (40)$$

$$\left\langle\frac{1}{2\beta^2}S_4^2\right\rangle_{\rm con}=\frac{1}{72}R_{\alpha\mu\nu\beta}R_\gamma{}^{\mu\nu}{}_\delta\xi^\alpha\xi^\beta\xi^\delta\xi^\gamma I_{10}-\frac{\beta}{72}R_{\alpha\mu}R_\beta{}^\mu\xi^\alpha\xi^\beta\, 4\, I_{11}-\frac{\beta}{72}R_{\alpha\mu\nu\beta}R^{\mu\nu}\xi^\alpha\xi^\beta\, 4\, I_{12}-\frac{\beta}{72}R_{\alpha\mu\nu\lambda}R_\beta{}^{\mu\nu\lambda}\xi^\alpha\xi^\beta\, 6\, I_{13}+\frac{\beta^2}{72}R^2_{\mu\nu}\, 2\, I_{14}+\frac{\beta^2}{72}R^2_{\alpha\mu\nu\beta}\, 3\, I_{15}, \quad (41)$$

where the integrals I_n are listed and evaluated using mode regularization in the Appendix. Inserting the specific values of the terms arising from the effective potential $V_{\rm MR}$ when evaluated at the origin

$$V_{\rm MR}=\frac{1}{8}R, \quad (42)$$

$$\partial_\alpha V_{\rm MR}=\frac{1}{8}\nabla_\alpha R, \quad (43)$$

$$\partial_\alpha\partial_\beta V_{\rm MR}=\frac{1}{8}\nabla_\alpha\nabla_\beta R-\frac{1}{36}R_{\alpha\mu\nu\lambda}R_\beta{}^{\mu\nu\lambda} \quad (44)$$

leads us to the following expression for the transition amplitude at the third loop order

$$\langle x_f^\mu,t_f|x_i^\mu,t_i\rangle=\frac{1}{(2\pi\beta)^{D/2}}e^{-1/2\beta g_{\mu\nu}\xi^\mu\xi^\nu}\left\{1-\frac{1}{12}\xi^\alpha\xi^\beta R_{\alpha\beta}-\beta\left(\frac{1}{12}R+V\right)-\frac{1}{24}\xi^\alpha\xi^\beta\xi^\gamma\nabla_\gamma R_{\alpha\beta}-\frac{1}{2}\beta\xi^\alpha\nabla_\alpha\left(\frac{1}{12}R+V\right)+\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta\left(\frac{1}{360}R_{\alpha\mu\nu\beta}R_\gamma{}^{\mu\nu}{}_\delta+\frac{1}{288}R_{\alpha\beta}R_{\gamma\delta}-\frac{1}{80}\nabla_\gamma\nabla_\delta R_{\alpha\beta}\right)+\beta\xi^\alpha\xi^\beta\left[\frac{1}{360}R_{\alpha\mu\nu\lambda}R_\beta{}^{\mu\nu\lambda}-\frac{1}{720}R_{\alpha\mu\nu\beta}R^{\mu\nu}-\frac{1}{720}R_{\alpha\mu}R_\beta{}^\mu+\frac{1}{12}\left(\frac{1}{12}R+V\right)R_{\alpha\beta}-\frac{1}{240}\nabla^\mu\nabla^\nu R_{\alpha\mu\nu\beta}-\frac{7}{480}\nabla_\alpha\nabla_\beta R-\frac{1}{6}\nabla_\alpha\nabla_\beta V\right]+\beta^2\left[\frac{1}{720}R^2_{\alpha\mu\nu\beta}-\frac{1}{720}R^2_{\alpha\beta}+\frac{1}{2}\left(\frac{1}{12}R+V\right)^2-\frac{1}{120}\nabla^2R-\frac{1}{12}\nabla^2V\right]+O(\beta^{5/2})\right\}. \quad (45)$$

This is the complete expression which should be used to test Eq. (28) at order β^2. A straightforward calculation shows that one indeed obtains an identity after making use of Eq. (30). The mode regulated path integral described in the previous section passes this consistency check. Therefore it can be considered as a well defined way of computing path integrals in curved spaces.

Before closing this section it may be useful to cast the transition amplitude in a more compact form which can be made manifestly symmetric under the exchange of the initial and final point. Keeping on using the Riemann normal coordinates (in which we recall $x_f^\mu=0$ and $\xi^\mu \equiv x_i^\mu - x_f^\mu = x_i^\mu$) and defining symmetrized quantities as

$$\overline{A}=\frac{1}{2}[A(x_i)+A(x_f)] \tag{46}$$

we can write

$$\begin{aligned}\langle x_f^\mu,t_f|x_i^\mu,t_i\rangle=&\frac{1}{(2\pi\beta)^{D/2}}\exp\bigg[-\frac{1}{2\beta}\xi^\mu\xi^\nu\overline{g_{\mu\nu}}-\frac{1}{12}\xi^\alpha\xi^\beta\overline{R_{\alpha\beta}}-\beta\left(\frac{1}{12}\overline{R}+\overline{V}\right)+\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta\left(\frac{1}{360}\overline{R_{\alpha\mu\nu\beta}R_\gamma{}^{\mu\nu}{}_\delta}+\frac{1}{120}\overline{\nabla_\gamma\nabla_\delta R_{\alpha\beta}}\right)\\&+\beta\xi^\alpha\xi^\beta\left(\frac{1}{360}\overline{R_{\alpha\mu\nu\lambda}R_\beta{}^{\mu\nu\lambda}}-\frac{1}{720}\overline{R_{\alpha\mu\nu\beta}R^{\mu\nu}}-\frac{1}{720}\overline{R_{\alpha\mu}R_\beta{}^\mu}-\frac{1}{240}\overline{\nabla^\mu\nabla^\nu R_{\alpha\mu\nu\beta}}+\frac{1}{160}\overline{\nabla_\alpha\nabla_\beta R}+\frac{1}{12}\overline{\nabla_\alpha\nabla_\beta V}\right)\\&+\beta^2\left(\frac{1}{720}\overline{R^2_{\alpha\mu\nu\beta}}-\frac{1}{720}\overline{R^2_{\alpha\beta}}-\frac{1}{120}\overline{\nabla^2 R}-\frac{1}{12}\overline{\nabla^2 V}\right)+O(\beta^{5/2})\bigg].\end{aligned} \tag{47}$$

From this expression one can extract [by reexpanding part of the exponential and comparing with Eq. (6)] the leading terms of the Seeley–De Witt coefficients a_0,a_1,a_2 for non-coinciding points and obtain, in particular, the one loop trace anomalies for the operator $H=-\frac{1}{2}\nabla^2+V(x)$ in two and four dimensions.

IV. TIME DISCRETIZATION

The computation performed in the previous section was cast in such a way that can be easily extended to a different regularization scheme: the time discretization method developed in Ref. [9]. Such a regularization was obtained by deriving directly from operatorial methods a discretized version of the path integral. Taking the continuum limit one recognizes the action with the proper counterterm, and the rules how to compute Feynman graphs. These rules differ in general from the one required by mode regularization. The counterterm V_W arising in time discretization differs from $V_{\rm MR}$, too.

The time discretization method leads to the following path integral expression of the transition amplitude [9]

$$\langle x_f^\mu,t_f|x_i^\mu,t_i\rangle=A\left[\frac{g(x_f)}{g(x_i)}\right]^{1/4}e^{-(1/2\beta)g_{\mu\nu}(x_f)\xi^\mu\xi^\nu}\langle e^{-(1/\beta)S_{\rm int}}\rangle, \tag{48}$$

where

$$S_{\rm int}=\int_{-1}^0 d\tau\bigg[\frac{1}{2}[g_{\mu\nu}(x)-g_{\mu\nu}(x_f)](\dot{x}^\mu\dot{x}^\nu+a^\mu a^\nu+b^\mu c^\nu)+\beta^2[V(x)+V_W(x)]\bigg], \tag{49}$$

$$V_W=\frac{1}{8}R+\frac{1}{8}g^{\mu\nu}\Gamma_{\mu\alpha}{}^\beta\Gamma_{\nu\beta}{}^\alpha, \tag{50}$$

$$A=(2\pi\beta)^{-D/2}. \tag{51}$$

The propagators to be used in the perturbative expansion implied by the brackets on the right hand side of Eq. (49) are the same as in Eq. (21). The only difference is in the prescription how to resolve the ambiguities arising when distributions are multiplied together. The prescription imposed by time discretization consists in integrating the Dirac delta functions coming form the velocities and the ghosts propagators (thanks to the Lee-Yang ghosts they never appear multiplied together) and using consistently the value $\theta(0)=1/2$ for the step function. Note also the presence of the factor $[g(x_f)/g(x_i)]^{1/4}$ appearing in this scheme.

The result of the calculation has the same structure as the one reported in Eqs. (37), (38), (39), (40), (41) with the difference that $V_{\rm MR}$ should be substituted by V_W, leading to

$$V_W=\frac{1}{8}R, \tag{52}$$

$$\partial_\alpha V_W=\frac{1}{8}\nabla_\alpha R, \tag{53}$$

$$\partial_\alpha\partial_\beta V_W=\frac{1}{8}\nabla_\alpha\nabla_\beta R-\frac{1}{24}R_{\alpha\mu\nu\lambda}R_\beta{}^{\mu\nu\lambda}, \tag{54}$$

and with the following different values of the integrals computed in time discretization regularization:

$$I_1=0,\quad I_3=0,\quad I_5=0,\quad I_{10}=0,\quad I_{13}=\frac{1}{12},\quad I_{15}=-\frac{1}{12}. \tag{55}$$

The other integrals are as in mode regularization. Inserting all these values back in Eq. (48) and expanding the coefficient $[g(x_f)/g(x_i)]^{1/4}$ at the required loop order give the same transition amplitude as in Eq. (45) or, equivalently, in Eq. (47). Thus this result constitutes a successful test on the method developed in Ref. [9].

V. CONCLUSIONS

In this paper we have discussed a proper definition of the configuration space path integral for a particle moving in curved spaces. By performing a three loop computation we have tested its consistency and checked that one can equally well obtain the perturbative solution of the Schrödinger equation by path integrals. This fills a conceptual gap, showing that the perturbative description of a quantum particle moving in a curved space obtained by De Witt by solving the Schrödinger equation (i.e., using the canonical formulation of quantum mechanics [1]) can equally well be obtained in the path integral approach introduced by Feynman fifty years ago. This approach may also have practical applications in quantum field theoretical computations when carried out in curved background using the world line formalism [10].

We have mainly described the mode regulated path integral. Its definition was obtained in Refs. [6–8] by using a pragmatic approach to identify its key elements, and needed a strong check to test its foundations. This we have provided in this paper. We find that the method of mode regularization is also quite appealing for aesthetic reasons, since it is close to the spirit of path integrals that are meant to give a global picture of the quantum phenomena.

On the other hand we have also extended our computation to the time discretization method of defining the path integrals [9]. This method is in some sense closer to the local picture given by the differential Schrödinger equation, since one imagines the particle propagating by small time steps. It is nevertheless a consistent way of defining the path integral, maybe superior at this stage, since one obtains its properties directly from canonical methods. As we have seen also this scheme gives the correct result for the transition amplitude.

An annoying property of the two regularization schemes we have been discussing is that they both do not respect general coordinate invariance in target space, and require specific noncovariant counterterms to restore that symmetry. It would be interesting to find a reliable covariant regularization scheme or, at least, a scheme which while breaking covariance (e.g., in the decomposition of the action into free and interacting parts) does not necessitates noncovariant counterterms.

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APPENDIX

The function $\Delta(\tau,\sigma)$ appearing in the propagators is given in mode regularization by

$$\Delta(\tau,\sigma)=\sum_{m=1}^{M}\left[-\frac{2}{\pi^2 n^2}\sin(\pi m\tau)\sin(\pi m\sigma)\right] \tag{A1}$$

and leads to the following limiting values as $M\to\infty$, at least in the bulk (we recall that $-1\leqslant\tau,\sigma\leqslant 0$):

$$\Delta(\tau,\sigma)=\tau(\sigma+1)\,\theta(\tau-\sigma)+\sigma(\tau+1)\,\theta(\sigma-\tau), \tag{A2}$$

$$^{\bullet}\Delta(\tau,\sigma)=\frac{\partial}{\partial\tau}\Delta(\tau,\sigma)=\sigma+\theta(\tau-\sigma), \tag{A3}$$

$$\Delta^{\bullet}(\tau,\sigma)=\frac{\partial}{\partial\sigma}\Delta(\tau,\sigma)=\tau+\theta(\sigma-\tau), \tag{A4}$$

$$^{\bullet}\Delta^{\bullet}(\tau,\sigma)=\frac{\partial}{\partial\tau}\frac{\partial}{\partial\sigma}\Delta(\tau,\sigma)=1-\delta(\tau-\sigma), \tag{A5}$$

$$^{\bullet\bullet}\Delta(\tau,\sigma)=\frac{\partial^2}{\partial\tau^2}\Delta(\tau,\sigma)=\delta(\sigma-\tau). \tag{A6}$$

It is also useful to report the following limiting values for coinciding points:

$$\Delta(\tau,\tau)=\tau^2+\tau, \tag{A7}$$

$$\Delta^{\bullet}(\tau,\tau)={}^{\bullet}\Delta(\tau,\tau)=\tau+\frac{1}{2}. \tag{A8}$$

Note that at the regulated level one can easily obtain the following identities by inspection of Eq. (A1) and its derivatives

$$^{\bullet}\Delta^{\bullet}(\tau,\tau)+{}^{\bullet\bullet}\Delta(\tau,\tau)=\partial_\tau[{}^{\bullet}\Delta(\tau,\tau)], \tag{A9}$$

$$\Delta^{\bullet}(\tau,\tau)=0\quad\text{at}\quad\tau=-1,0, \tag{A10}$$

$$\partial_\tau[\Delta(\tau,\tau)]=2\Delta^{\bullet}(\tau,\tau), \tag{A11}$$

$$^{\bullet\bullet}\Delta(\tau,\sigma)=\Delta^{\bullet\bullet}(\tau,\sigma). \tag{A12}$$

The limiting values given above should be used with care in the perturbative expansion of the path integral. Rather, one should resort to the proper regularized expressions whenever ambiguities arise. In mode regularization we have adopted the following strategy: one should partially integrate as much as possible to reach expressions which are free of ambiguities, and which can be computed directly in the continuum limit. Following this procedure we have obtained the following results needed in the text:

$$I_1=\int_{-1}^{0}d\tau[\tau^2({}^{\bullet}\Delta^{\bullet}+{}^{\bullet\bullet}\Delta)+\Delta-2\tau\,{}^{\bullet}\Delta]|_\tau=-\frac{1}{2}, \tag{A13}$$

$$I_2=\int_{-1}^{0} d\tau[\Delta({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)-{}^\bullet\Delta^2]|_\tau=-\frac{1}{4}, \tag{A14}$$

$$I_3=\int_{-1}^{0} d\tau[\tau^3({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)+\tau\Delta-2\tau^2\,{}^\bullet\Delta]|_\tau=\frac{1}{2}, \tag{A15}$$

$$I_4=\int_{-1}^{0} d\tau[\tau({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)-\tau\,{}^\bullet\Delta^2]|_\tau=\frac{1}{8}, \tag{A16}$$

$$I_5=\int_{-1}^{0} d\tau[\tau^4({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)+\tau^2\Delta-2\tau^3\,{}^\bullet\Delta]|_\tau=-\frac{1}{2}, \tag{A17}$$

$$I_6=\int_{-1}^{0} d\tau[\tau^2\Delta({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)+\Delta^2-2\tau\,{}^\bullet\Delta\,\Delta]|_\tau=0, \tag{A18}$$

$$I_7=\int_{-1}^{0} d\tau[\tau^2\Delta({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)-\tau^2\,{}^\bullet\Delta^2]|_\tau=-\frac{1}{12}, \tag{A19}$$

$$I_8=\int_{-1}^{0} d\tau[\Delta^2({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)-{}^\bullet\Delta^2\,\Delta]|_\tau=\frac{1}{24}, \tag{A20}$$

$$I_9=\int_{-1}^{0} d\tau\,\Delta|_\tau=-\frac{1}{6}, \tag{A21}$$

$$\begin{aligned}I_{10}=\int\int d\tau d\sigma\ &[2\tau^2({}^\bullet\Delta^{\bullet 2}-{}^{\bullet\bullet}\Delta^2)\sigma^2+4\tau^2\,{}^\bullet\Delta^2\\&-8\tau^2\,{}^\bullet\Delta\,{}^\bullet\Delta^\bullet\sigma+2\Delta^2-8\Delta\Delta^\bullet\sigma+4\tau\Delta\,{}^\bullet\Delta^\bullet\sigma\\&+4\tau\,{}^\bullet\Delta\,\Delta^\bullet\sigma]=1,\end{aligned} \tag{A22}$$

$$\begin{aligned}I_{11}=\int\int d\tau d\sigma\ &[\tau({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)|_\tau\Delta({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)|_\sigma\sigma\\&+\tau\Delta^\bullet|_\tau{}^\bullet\Delta^\bullet\Delta^\bullet|_\sigma\sigma-2\tau({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)|_\tau\Delta^\bullet\Delta^\bullet|_\sigma\sigma\\&+\Delta|_\tau{}^\bullet\Delta^\bullet\Delta|_\sigma+\Delta^\bullet|_\tau\Delta\Delta^\bullet|_\sigma-2\Delta|_\tau{}^\bullet\Delta\Delta^\bullet|_\sigma\\&+2\tau({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)|_\tau\Delta^\bullet\Delta|_\sigma+2\tau\Delta^\bullet|_\tau{}^\bullet\Delta\Delta^\bullet|_\sigma\\&-2\tau({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)|_\tau\Delta\Delta^\bullet|_\sigma-2\tau\Delta^\bullet|_\tau{}^\bullet\Delta^\bullet\Delta|_\sigma\sigma]\\=-\frac{1}{12},&\end{aligned} \tag{A23}$$

$$\begin{aligned}I_{12}=\int\int d\tau d\sigma\ &[\tau^2({}^\bullet\Delta^{\bullet 2}-{}^{\bullet\bullet}\Delta^2)\Delta|_\sigma+\Delta^{\bullet 2}\Delta|_\sigma\\&-2\tau\Delta^\bullet\,{}^\bullet\Delta^\bullet\Delta|_\sigma+\tau^2\,{}^\bullet\Delta^2({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)|_\sigma+\Delta^2({}^\bullet\Delta^\bullet\\&+{}^{\bullet\bullet}\Delta)|_\sigma-2\tau\Delta\,{}^\bullet\Delta({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)|_\sigma\\&-2\tau^2\,{}^\bullet\Delta\,{}^\bullet\Delta^\bullet\Delta^\bullet|_\sigma-2\Delta\Delta^\bullet\Delta^\bullet|_\sigma+2\tau\Delta\,{}^\bullet\Delta^\bullet\Delta^\bullet|_\sigma\\&+2\tau\,{}^\bullet\Delta\Delta^\bullet\Delta^\bullet|_\sigma]=\frac{1}{6},\end{aligned} \tag{A24}$$

$$\begin{aligned}I_{13}=\int\int d\tau d\sigma\ &[\tau\Delta({}^\bullet\Delta^{\bullet 2}-{}^{\bullet\bullet}\Delta^2)\sigma-\tau\,{}^\bullet\Delta\Delta^\bullet\,{}^\bullet\Delta^\bullet\sigma\\&+\Delta^2\,{}^\bullet\Delta^\bullet-\Delta\,{}^\bullet\Delta\Delta^\bullet+2\tau\Delta^\bullet\,{}^\bullet\Delta^2-2\tau\Delta\,{}^\bullet\Delta\,{}^\bullet\Delta^\bullet]\\=\frac{1}{18},&\end{aligned} \tag{A25}$$

$$\begin{aligned}I_{14}=\int\int d\tau d\sigma\ &[\Delta|_\tau({}^\bullet\Delta^{\bullet 2}-{}^{\bullet\bullet}\Delta^2)\Delta|_\sigma\\&-4\Delta|_\tau{}^\bullet\Delta^\bullet\,{}^\bullet\Delta\Delta^\bullet|_\sigma+2\Delta|_\tau{}^\bullet\Delta^2({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)|_\sigma\\&+2\Delta^\bullet|_\tau\Delta\,{}^\bullet\Delta^\bullet\Delta^\bullet|_\sigma+2\Delta^\bullet|_\tau{}^\bullet\Delta\Delta^\bullet\Delta^\bullet|_\sigma\\&-4\Delta^\bullet|_\tau\Delta\,{}^\bullet\Delta({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)|_\sigma+({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)|_\tau\Delta^2({}^\bullet\Delta^\bullet\\&+{}^{\bullet\bullet}\Delta)|_\sigma]=-\frac{1}{12},\end{aligned} \tag{A26}$$

$$\begin{aligned}I_{15}=\int\int d\tau d\sigma\ &[\Delta^2({}^\bullet\Delta^{\bullet 2}-{}^{\bullet\bullet}\Delta^2)+{}^\bullet\Delta^2\Delta^{\bullet 2}\\&-2\Delta\,{}^\bullet\Delta\Delta^\bullet\,{}^\bullet\Delta^\bullet]=-\frac{1}{18}.\end{aligned} \tag{A27}$$

On the other hand, the time discretization method needs a different prescription in order to resolve the ambiguities. It consists in integrating the Dirac delta functions whenever they appear (the Lee-Yang ghosts guarantee that they never appear multiplied together) and using consistently the value $\theta(0)=\frac{1}{2}$ for the step function. We present now a list of the elementary integrals needed in the text and whose values differ in the two regularizations. We have reported both values, the one related to time discretization being included in square brackets:

$$\int d\tau\ \ \tau^2({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)|_\tau=-\frac{1}{6}\ \left[\frac{1}{3}\right], \tag{A28}$$

$$\int d\tau\ \ \tau^3({}^\bullet\Delta^\bullet+{}^{\bullet\bullet}\Delta)|_\tau=\frac{1}{4}\ \left[-\frac{1}{4}\right], \tag{A29}$$

$$\int d\tau \;\; \tau^4 ({}^{\bullet}\Delta^{\bullet} + {}^{\bullet\bullet}\Delta)|_{\tau} = -\frac{3}{10} \left[\frac{1}{5}\right], \tag{A30}$$

$$\int \int d\tau d\sigma \;\; \tau^2 ({}^{\bullet}\Delta^{\bullet\,2} - {}^{\bullet\bullet}\Delta^2)\sigma^2 = \frac{19}{90} \left[-\frac{13}{45}\right], \tag{A31}$$

$$\int \int d\tau d\sigma \;\; \tau\, {}^{\bullet}\Delta\, \Delta^{\bullet}\, {}^{\bullet}\Delta^{\bullet}\, \sigma = -\frac{1}{36} \left[-\frac{1}{18}\right], \tag{A32}$$

$$\int \int d\tau d\sigma \;\; \Delta\, {}^{\bullet}\Delta\, \Delta^{\bullet}\, {}^{\bullet}\Delta^{\bullet} = \frac{1}{180} \left[\frac{7}{360}\right]. \tag{A33}$$

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