# Conservation laws, symmetry properties, and the equivalence principle in a class of alternative theories of gravity

D. E. Barraco, E. Domínguez, and R. Guibert
*Fa.M.A.F, Universidad Nacional de Córdoba, Ciudad Universitaria, Cordoba 5000, Argentina*
(Received 3 September 1998; published 20 July 1999)

We consider a subclass of alternative theories of gravitation obtained by a first order formalism from a Lagrangian density $\mathcal{L}_T = f(R)\sqrt{-g} + \mathcal{L}_M$ where the matter field Lagrangian density $\mathcal{L}_M$ does not depend on the connection. For this theory we derive an analogue of the Einstein pseudotensor and the von Freud superpotential. Then we derive, using the arbitrariness that is always present in the choice of pseudotensor and superpotential, a generalization of the Møller superpotential as associated with a double-index differential conservation law. This superpotential allows us to deduce that there are two analogues of the Komar vector of general relativity (GR): one associated with the general connection and the other with the metric connection. Astonishingly both of them satisfy the physical condition that the inertial mass must be equal to the gravitational (active) mass for any class of matter. We also obtain a generalization of Tolman's expression for the energy, and prove that those theories with $f(0)=0$ share with GR the property that the total energy is independent of any two-dimensional surface which encloses the support of the matter distribution. [S0556-2821(99)03614-0]

PACS number(s): 04.50.+h, 04.20.Cv, 04.25.Nx

## I. INTRODUCTION

Attempts to quantize general relativity (GR) or to regularize the stress-energy-momentum tensor of quantum fields propagating in curved space-time have led investigators to consider gravitational actions involving terms quadratic in the Ricci tensor [1,2]. For example, the theory based on the Lagrangian $\mathcal{L} = (\alpha R + \beta R^2 + \gamma R_{ab}R^{ab})\sqrt{-g}$ has better quantum properties than general relativity itself. The particle spectrum of this theory, in the linear approximation, contains a massless graviton, a massive spin-2 ghost, and a physical scalar particle (spin-0), which are, respectively, associated with the first, second, and last terms of the Lagrangian.

In modern cosmology, higher-derivative theories have become standard since the Starobinsky model with curvature-squared terms leads automatically to the desired inflationary period. In the limit of small energies the superstring theories give an action for spacetime in the form of the Einstein-Hilbert action plus terms which are quadratic in the scalar curvature and the Ricci tensor.

Higher-order theories of gravity are the generally covariant extensions of GR when we consider in the Lagrangian density nonlinear terms in the curvature. The field equations derived by second-order variation of this Lagrangian contain metric derivatives of an order higher than the second. The second-order variation or second-order formalism consists of assuming a Riemannian geometry and considering variations of the metric and its first derivatives equal to zero on the boundary of a space-time region $\mathcal{U}$. However, it is possible to modify the Einstein-Hilbert action by adding a boundary term such that, when the variation of this term is taken into account, it cancels the unwanted term which appears when we only impose a null variation of the metric on the boundary and leave its derivatives unrestricted [3,4].

Alternatively, the Palatini approach, or first-order formalism, can be applied to obtain the field equations in GR assuming the metric and the connection as independent variables. This formalism has also been applied to more general Lagrangian densities with quadratic terms [5] or a general function of the scalar curvature [6,7], to study other geometrical theories of gravitation. More recently [8], the latter theories have been extended by including a scalar field in the Lagrangian and a connection allowing torsion [9]. One apparent conceptual advantage of these theories is that quantum fluctuations of the metric and the connection are independent of each other.

The theory behaves in the Newtonian limit [7] as a Newtonian theory with a correction which is proportional to the matter density at the field point. This behavior can be produced by a Yukawa potential with an atomic scale characteristic range $\lambda$ and a coupling constant $\alpha$ proportional to $1/\lambda^2$. This type of potential is not excluded by the present experimental data [10].

In the present work we consider those theories that are obtained from a Lagrangian density $\mathcal{L}_T(R) = f(R)\sqrt{-g} + \mathcal{L}_M$ that depends on the curvature scalar and a matter Lagrangian that does not depend on the connection, and apply Palatini's method to obtain the field equation. Our investigation includes the particular case $f(R) = R + \omega R^2$ whose cosmological solutions has been discussed in some details by Shahid-Saless [11,12].

The theories obtained from this Lagrangian, using the second-order formalism, are, in vacuum, conformally ''equivalent'' to Einstein gravity with a massive scalar field [13]. This conformal equivalence follows from the prescription of a general Legendre transformation [14,15]. Then, in vacuum, we have two frames: the Jordan frame with the original set of variables and the Einstein frame with the transformed set of variables. In the Jordan frame, gravity is entirely described by the metric tensor. In the Einstein frame, the scalar field acts as a source for the transformed metric tensor and formally plays the role of an external matter field for the Einstein field equations, which corresponds to the additional degrees of freedom, due to the higher order of the field equations, in the Jordan frame. We have to decide which frame should be taken as the physical one; i.e., the physical metric should be singled out already in the vacuum theory, and the minimal coupling of the matter field to the Einstein frame's metric or to the Jordan frame's metric depends on this decision. Magnano and Sokolowski [16] have studied this problem.

On the other hand, in the theories considered here it is still an open question whether they can be reformulated as GR plus additional degrees of freedom; we expect to address this point in a future paper.

One of the most intriguing problems to be solved in any theory of gravitation is the definition of energy and, more generally, of conserved quantities associated with the gravitational field itself.

There are various approaches in the existing literature. One of them is directly based on suitable ''covariance requirements'' for the Lagrangian of the theory, together with suitable integration by parts on field equations, to generate families of ''Noether currents'' out of a vector density $\Theta^a$ which is usually the divergence of a skew-symmetric tensor density $U^{ab}$, called a ''superpotential'' for the conserved quantities themselves.

In this work we propose to consider the derivation of conservation laws and the related problem of the symmetry properties of the metric field, applicable to the above theory obtained using a first-order formalism and the general matter Lagrangian.

In the next section, the general structure of the theory is shown. In Sec. III ''double-index'' differential conservation laws (laws involving conserved quantities with two indices) are derived and the analogues of the Einstein pseudotensor [17] and the von Freud superpotential [18] are deduced. Finally we obtain, for this theory, the generalization of the Møller [19] superpotential of GR.

In Sec. IV the ''single-index'' differential conservation laws are discussed and from the generalization of the Møller superpotential we show that we can choose two analogues of the Komar vector of GR to construct a conserved quantity for an isolated asymptotically flat system. One of these generalizations of the Komar vector of GR is obtained using the connection $\Gamma$ of the theory and the other using the metric connection.

The application of the results of Secs. II and III and the integral conservation laws is considered in Sec. V. In a general nonlinear theory the active mass is not necessarily equal to the total mass-energy (or inertial mass), obtained as the conserved quantity associated with time symmetry. However, by choosing any of the two analogues of the Komar vector we show that in this theory (as in general relativity and in some models of higher-order theories [20] and of scalar-tensor theories [21]) the active mass is equal to the inertial mass, and then the weak equivalence principle is satisfied. This property differentiates this theory from other alternative theories, such as, as for example, Brans Dicke theory [21].

## II. GENERAL STRUCTURE OF THE THEORY

Let us consider a Lagrangian density $\mathcal{L}_M = f(R)\sqrt{-g} + \mathcal{L}_M$, where the matter Lagrangian does not depend on the connection, the metric and the connection are independent variables, and the connection is torsionless. Then the field equations [6,7], if we vary with respect to the metric, are

$$f'(R)R_{ab} - \frac{1}{2}f(R)g_{ab} = T_{ab}. \tag{1}$$

The variation with respect to the connection, recalling that this is fixed at the boundary, gives

$$\nabla_c g_{ab} = b_c g_{ab}, \tag{2}$$

$$b_c = -[\ln f'(R)]_{,c}. \tag{3}$$

Thus, we have a Weyl conformal geometry with a Weyl field given by Eq. (3).

The vanishing of the connection in a particular frame, for example in a geodesic frame, however, does not mean that the metric is flat there, because, from Eq. (2), $\partial_c g_{ab} = b_c g_{ab}$. Therefore the strong equivalence principle is in general not satisfied.

From Eq. (1) we obtain

$$f'(R)R - 2f(R) = T, \tag{4}$$

which shows that $b_c$ is determined by $T$ and its derivative except in the case $f(R) = \omega R^2$, for which $Rf' - 2f \equiv 0$, and then we must consistently have $T \equiv 0$.

The connection is

$$\Gamma^a_{bc} = C^a{}_{bc} - \frac{1}{2}(\delta^a_b b_c + \delta^a_c b_b - g_{bc} b^a), \tag{5}$$

where $C^a{}_{bc}$ is the metric connection. The Riemann tensor is defined as usual and then the Ricci tensor and scalar curvature are

$$R_{ab} = R^0_{ab} + \frac{3}{2}D_a b_b - \frac{1}{2}D_b b_a + \frac{1}{2}g_{ab}D\cdot b - \frac{1}{2}b_a b_b + \frac{1}{2}g_{ab}b^2, \tag{6}$$

$$R = R^0 + 3D\cdot b + \frac{3}{2}b^2, \tag{7}$$

where $R^0_{ab}$, $R^0$, and $D_c$ are the Ricci tensor, scalar curvature, and covariant derivative defined from the metric connection, respectively.

From Eq. (6) we obtain

$$R_{[ab]} = \partial_a b_b - \partial_b b_a; \tag{8}$$

then, Eq. (3) gives $R_{(ab)} = R_{ab}$.

Because the matter action must be invariant under diffeomorphisms and the matter field satisfies the matter field equations, $T_{ab}$ is conserved:

$$D^a T_{ab}=0. \qquad (9)$$

Therefore,we may conclude that a test particle will follow the geodesics of the metric connection. Using Eqs. (3) and (4) we have

$$b_c=-\frac{f''\nabla_c T}{f'(Rf''-f')}. \qquad (10)$$

We assume $f'\neq 0$. Except for the case of GR, $f''\equiv 0$, the Weyl field is nonzero wherever the trace of the energy-momentum tensor varies with respect to the coordinates. If $\boldsymbol{T}$ is constant, then $R$ is also constant, $b_c=0$, and Eq. (1) takes the form

$$G_{ab}+\frac{1}{2}\Lambda g_{ab}=\alpha T_{ab}, \qquad (11)$$

where $\Lambda$ and $\alpha$ are two constants depending on $R$. Thus, as we have just proved [6,7] all those cases with a constant trace of the energy-momentum tensor are equivalent to GR for a given cosmological constant. This is the so-called [22] universality of the Einstein equations for matter with constant $T$.

## III. DOUBLE-INDEX DIFFERENTIAL CONSERVATION LAWS

Differential identities analogous to the Bianchi identities of GR are derived [23] by considering the Lagrangian density as an arbitrary function of the coordinates, the field variables, and their first and second derivatives,

$$I(x^i;Y_A(x);Y_{A,j}(x);Y_{A,jk}(x))\equiv I(x;Y_A), \qquad (12)$$

which transforms as a scalar density under the group of general space-time coordinate transformation.

In the first-order formalism the metric and the connection are independent field variables. Under an arbitrary infinitesimal coordinate transformation

$$x'^k=x^k+\xi^k(x), \quad |\xi^k(x)\|\ll 1, \qquad (13)$$

the scalar density will satisfy the equation

$$I(x';Y'_A(x'))d^4x'=I(x;Y_A(x))d^4x. \qquad (14)$$

Then, an infinitesimal coordinate transformation is a symmetry transformation, i.e., a transformation which leaves the form of the equations of motion unaltered. We assume that the description of our physical system is given completely by a system of functions $Y_A(x)$, which includes the metric components, the connection, and the matter fields. In the case of an infinitesimal diffeomorphism the change in the form of the fields is given by

$$\bar{\delta}Y_A=\pounds_\xi Y_A, \qquad (15)$$

where $\xi$ is a vector field called the generator of diffeomorphism. The symmetry transformation under consideration is the infinite group of diffeomorphism, and then we obtain a set of differential identities known as the generalized Bianchi identities [23–25]

$$L^A\gamma_{Aj}+(L^A\gamma_{Aj}{}^{\alpha})_{,\alpha}+(L^A\gamma_{Aj}{}^{\alpha\beta})_{,\alpha\beta}=0, \qquad (16)$$

where $L^A=\{\delta I/\delta g_{ab};\delta I/\delta\Gamma^c{}_{ab}\}$ and

$$\pounds_\xi Y_A=\xi^k\gamma_{Ak}-\xi^k{}_{,l}\gamma_{Ak}{}^l+\xi^k{}_{,lm}\gamma_{Ak}{}^{lm}. \qquad (17)$$

Whenever $Y_A=g_{ab}$ and $Y_A=\Gamma^a{}_{bc}$, we have

$$\pounds_\xi g_{ab}=\xi^k\gamma_{abk}-\xi^k{}_{,l}\gamma_{abk}{}^l, \qquad (18)$$

$$\pounds_\xi\Gamma^c_{ab}=\xi^k\hat{\gamma}^c_{abk}-\xi^k{}_{,l}\gamma_{ab}{}^c{}_k{}^l+\xi^k{}_{,lm}\gamma_{ab}{}^c{}_k{}^{lm}, \qquad (19)$$

where

$$\gamma_{abk}=-g_{ab,k}, \quad \gamma_{abk}{}^l=2\delta^l_{(a}g_{b)k}, \qquad (20)$$

$$\hat{\gamma}_{abk}{}^c=-\Gamma^c{}_{ab,k}, \quad \gamma_{ab}{}^c{}_k{}^l=2\delta^l_{(a}\Gamma^c{}_{b)k}-\delta^c_k\Gamma^l{}_{ab},$$

$$\gamma_{ab}{}^c{}_k{}^{lm}=-\delta^c_k\delta^l_a\delta^m_b. \qquad (21)$$

Then the Bianchi identities (16) can be written

$$-L^{ab}g_{ab,k}+2(L^{ib}g_{bk})_{,i}+L^{ab}{}_c\Gamma^c{}_{ab,k}$$
$$-(L^{ab}{}_k\Gamma^i{}_{ab}-2L^{ib}{}_c\Gamma^c{}_{bk})_{,i}-(L^{ij}{}_k)_{,ij}=0, \qquad (22)$$

with $L^{ab}$ and $L^{ab}{}_c$ given by

$$L^{ab}\equiv\frac{\delta I}{\delta g_{ab}}, \quad L^{ab}{}_c\equiv\frac{\delta I}{\delta\Gamma^c{}_{ab}}. \qquad (23)$$

Now, expanding the variational derivative and making some extra algebra, we can show that

$$L^{ab}g_{ab,k}+L^{ab}{}_c\Gamma^c{}_{ab,k}=\left\{I\delta^i_k-\left[\frac{\partial I}{\partial g_{ab,i}}-\left(\frac{\partial I}{\partial g_{ab,ei}}\right)_{,e}\right]g_{ab,k}-\frac{\partial I}{\partial g_{ab,il}}g_{ab,kl}-\frac{\partial I}{\partial\Gamma^c{}_{ab,i}}\Gamma^c{}_{ab,k}\right\}_{,i}. \qquad (24)$$

From the last result and Eq. (22) we have

$$\left(\frac{\partial I}{\partial g_{bi}}g_{bk}-\frac{1}{2}\left\{I\delta^i_k-\left[\frac{\partial I}{\partial g_{ab,i}}-\left(\frac{\partial I}{\partial g_{ab,ei}}\right)_{,e}\right]g_{ab,k}-\frac{\partial I}{\partial g_{ab,il}}g_{ab,kl}-\frac{\partial I}{\partial\Gamma^c{}_{ab,i}}\Gamma^c{}_{ab,k}+\frac{\partial I}{\partial\Gamma^k{}_{ab}}\Gamma^i{}_{ab}+\left(\frac{\partial I}{\partial\Gamma^k{}_{ji}}\right)_{,j}-2\frac{\partial I}{\partial\Gamma^c{}_{bi}}\Gamma^c{}_{bk}\right\}\right)_{,i}=0. \qquad (25)$$

Defining the analogue of the Einstein pseudotensor of GR as

$$\sqrt{-g}t^i_k \equiv \frac{1}{2}\left\{ I\delta^i_k - \left[ \frac{\partial I}{\partial g_{ab,i}} - \left( \frac{\partial I}{\partial g_{ab,ei}} \right)_{,e} \right] g_{ab,k} - \frac{\partial I}{\partial g_{ab,il}} g_{ab,kl} \frac{\partial I}{\partial \Gamma^c{}_{ab,i}} \Gamma^c{}_{ab,k} + \frac{\partial I}{\partial \Gamma^k{}_{ab}} \Gamma^i{}_{ab} + \left( \frac{\partial I}{\partial \Gamma^k{}_{ji}} \right)_{,j} - 2\frac{\partial I}{\partial \Gamma^c{}_{bi}} \Gamma^c{}_{bk} \right\}, \tag{26}$$

and using the field equation in Eq. (22), we can construct the two-index conserved quantity

$$\Theta^i_k \equiv \sqrt{-g}(T^i_k + t^i_k), \tag{27}$$

which satisfies

$$\Theta^i{}_{k,i} = 0. \tag{28}$$

Furthermore, associated with the infinitesimal symmetry transformation, we can obtain a set of identities known as basic identity [23]:

$$\frac{\delta L}{\delta Y_A}\delta Y_A + t^m{}_{,m} = 0. \tag{29}$$

Then, from Eqs. (18) and (19) we have

$$\frac{\delta I}{\delta g_{ab}}(\xi^k \gamma_{abk} - \xi^k{}_{,l}\gamma_{abk}{}^l) + \frac{\delta I}{\delta \Gamma^c{}_{ab}}(\xi^k \hat{\gamma}_{abk}{}^c - \xi^k{}_{,l}\gamma_{ab}{}^c{}_k{}^l + \xi^k{}_{,lm}\gamma_{ab}{}^c{}_k{}^{lm}) + t^m{}_{,m} = 0, \tag{30}$$

where

$$t^m \equiv A^{abm}(\xi^k \gamma_{abk} - \xi^k{}_{,l}\gamma_{abk}{}^l) + I\delta x^m + B^{abmn}(\xi^k{}_{,n}\gamma_{abk} + \xi^k \gamma_{abk,n} - \xi^k{}_{,ln}\gamma_{abk}{}^l - \xi^k{}_{,l}\gamma_{abk}{}^l{}_{,n}) + A_c{}^{abm}\delta\Gamma^c{}_{ab} \tag{31}$$

and with

$$A^{abm} \equiv \frac{\partial I}{\partial g_{ab,m}} - \left( \frac{\partial I}{\partial g_{ab,mn}} \right)_{,n},$$

$$B^{abmn} \equiv \frac{\partial I}{\partial g_{ab,mn}},$$

$$A_c{}^{abm} \equiv \frac{\partial I}{\partial \Gamma^c{}_{ab,m}}. \tag{32}$$

Now, expanding and recombining the derivative terms, the basic identity (30) can be written as

$$\xi^k X_k + \xi^k{}_{,l}X_k{}^l + \xi^k{}_{,lm}X_k{}^{lm} + \xi^k{}_{,lmn}X_k{}^{lmn} = 0, \tag{33}$$

where we have used the definitions

$$\begin{aligned}
X_k &\equiv U_k - V^a{}_{k,a},\\
X_k{}^l &\equiv -(V^l{}_k + W^{ml}{}_{k,m}),\\
X_k{}^{lm} &\equiv -(W^{ml}{}_k + Z^{nlm}{}_{k,n}),\\
X_k{}^{lmn} &\equiv -Z^{nml}{}_k,\\
U_k &\equiv L^A\gamma_{Aj} + (L^A\gamma_{Aj}{}^{\alpha})_{,\alpha} + (L^A\gamma_{Aj}{}^{\alpha\beta})_{,\alpha\beta} = 0,\\
V^l{}_k &\equiv L^{ab}\gamma_{abk}{}^l + L^{ab}{}_e\gamma_{ab}{}^e{}_k{}^l + (L^{ab}{}_e\gamma_{ab}{}^e{}_k{}^{ld})_{,d} - I\delta^l_k - A^{abl}\gamma_{abk} - A^{abl}{}_e\hat{\gamma}_{abk}{}^e - B^{able}\gamma_{abk,e},\\
W^{li}{}_k &\equiv L^{ab}{}_e\gamma_{ab}{}^e{}_k{}^{li} - A^{abi}\gamma_{abk}{}^l + B^{abli}\gamma_{abk} - B^{abji}\gamma_{abk}{}^l{}_{,j} - A^{abi}{}_c\gamma_{ab}{}^c{}_k{}^l,\\
Z^{iml}{}_k &\equiv B^{abil}\gamma_{abk}{}^m - A^{abi}{}_c\gamma_{ab}{}^c{}_k{}^{ml}.
\end{aligned} \tag{34}$$

Since $\xi^k$ are arbitrary, each coefficient in the expansion (33) must vanish separately. Therefore, we have the equations

$$\begin{aligned}
V^l{}_{k,l} &= 0,\\
V^l{}_k &= -W^{lm}{}_{k,m},\\
W^{(ml)}{}_k &= -Z^{n(lm)}{}_{k,l},\\
Z^{(nml)}{}_k &= 0,
\end{aligned} \tag{35}$$

where we have used the Bianchi identities in the first equation, i.e., $U_k = 0$, in order to obtain a differential conservation law. From the remaining conditions we find that

$$V^l{}_k = U^{li}{}_{k,i}, \tag{36}$$

$$U^{lm}{}_k = W^{lm}{}_k + \frac{1}{3}(Z^{ilm}{}_k - Z^{lim}{}_k)_{,i}. \tag{37}$$

As a result of Eq. (35) we can put

$$W^{(lm)}{}_k = -Z^{j(lm)}{}_{k,j} \tag{38}$$

and

$$Z^{j(lm)}{}_k + Z^{(ml)j}{}_k + Z^{(l|j|m)}{}_k = 0. \tag{39}$$

Therefore, the symmetric part of $U^{lm}_k$ is

$$U^{(lm)}{}_k = \frac{1}{3}(Z^{(ml)j}{}_k - Z^{j(lm)}{}_k)_{,j}. \tag{40}$$

Using the fact that the Lagrangian is an arbitrary function of the curvature $R$ and

$$g^{uv}\delta^a{}_t \frac{\partial \Gamma^t{}_{ua,v}}{\partial \Gamma^j{}_{ik,l}} = g^{l(k}\delta^{i)}{}_j, \tag{41}$$

it is not difficult to prove that

$$\left( \frac{\partial I}{\partial \Gamma^k{}_{l(i,j)}} - \frac{\partial I}{\partial \Gamma^k{}_{ij,l}} \right)_{,l} = 0 \tag{42}$$

and thus

$$(Z_k^{l(ij)} - Z_k^{(ij)l})_{,l} = 0. \tag{43}$$

Finally, using the above equation it is easy to see the antisymmetry of $U^{il}{}_k$,

$$U^{il}{}_k = U^{[il]}{}_k, \tag{44}$$

and then $U^{il}{}_k$ is a superpotential for the conserved complex $V^l{}_k$.

From the definitions of $V_k^a$, Eq. (34), and $\Theta_k^a$, Eq. (27),

$$V^i{}_k = 2\Theta^i{}_k. \tag{45}$$

Therefore, we may generalize the von Freud superpotential, i.e., the superpotential that satisfies

$$U_{(vF)\ k,j}^{[ij]} = \Theta^i{}_k, \tag{46}$$

by defining

$$U_{(vF)\ k}^{ij} \equiv \frac{1}{2} U^{ij}{}_k. \tag{47}$$

As is well known, the addition of any antisymmetric quantity $\Omega_{k,b}^{[ab]}$ to $\Theta_k^a$ yields a new differential conservation law

$$\Theta'^{a}_{k,a} = 0, \tag{48}$$

where

$$\Theta'^{a}_{k} = \Theta_k^a + \Omega_{k\ ,b}^{[ab]}. \tag{49}$$

The new pseudotensor and superpotential become, respectively,

$$t'^{a}_{k} = t_k^a + \Omega_{k\ ,b}^{[ab]} \tag{50}$$

and

$$U'^{[ab]}_{k} = U_{vF\ k}^{[ab]} + \Omega_k^{[ab]}. \tag{51}$$

Finally, Eq. (46) becomes

$$\Theta'^{a}_{k} = U'^{[ab]}_{k,b} = \sqrt{-g}(t'^{a}{}_{k} + T_k^a). \tag{52}$$

Now, $U^{[lm]}{}_k$ is equal to

$$\begin{aligned} U^{[lm]}{}_k &= W^{[lm]}{}_k - \frac{1}{3} Z^{[l|i|m]}{}_{k,i} \\ &= -\sqrt{-g} f' \Bigg[ 2\Gamma^c{}_{bk} \frac{\partial R}{\partial \Gamma^c{}_{b[l,m]}} - \Gamma^{[l}_{ab} \frac{\partial R}{\partial \Gamma^k_{|ab|,m]}} \\ &\quad + \frac{1}{3} \left( \frac{\partial R}{\partial \Gamma^k_{i[m,l]}} \right)_{,i} \Bigg] - \frac{1}{3} (\sqrt{-g} f')_{,i} \frac{\partial R}{\partial \Gamma^k_{i[m,l]}}. \end{aligned} \tag{53}$$

The scalar curvature depends on the derivative of the connection only through the term $g^{ij}\delta^u{}_v(\Gamma^v{}_{iu,j} - \Gamma^v{}_{ij,u})$. Then, Eq. (53) becomes

$$U^{[ml]}{}_k = \sqrt{-g} f'(g_{ik} g^{im,l} - g_{ik} g^{il,m} + B^l \delta^m{}_k - B^m \delta^l{}_k), \tag{54}$$

where

$$2B^l \equiv \frac{1}{2} g^{ij} C^l{}_{ij} - \frac{5}{2} (\ln f')^{,l} \tag{55}$$

and $C^l{}_{ij}$ is the metric connection.

Finally, by adding to the von Freud superpotential the skew-symmetric term

$$\Omega^{ik}{}_j \equiv U^{ik}_{(vF)\ j} - 2\sqrt{-g} f' \Lambda^{[i} \delta^{k]}{}_j, \tag{56}$$

with

$$\Lambda^i \equiv -[(\ln f')^{,i} + B^i], \tag{57}$$

we obtain the analogue of the Møller superpotential:

$$U_{(M)\ k}^{[ij]} = \sqrt{-g} g^{il} g^{jm} [(g_{km} f')_{,l} - (g_{kl} f')_{,m}]. \tag{58}$$

The last expression with $f' = 1$ is equal to the Møller superpotential of GR.

## IV. SINGLE-INDEX CONSERVATION LAWS

Single-index conservation laws can be derived by considering Eqs. (16), (18), and (30) to obtain a differential conservation law

$$\begin{aligned} &\Bigg\{ -t^l + \xi^j \Bigg[ \frac{\delta I}{\delta g_{ab}} \gamma_{abj}{}^l + \frac{\delta I}{\delta \Gamma^c_{ab}} \gamma^c_{ab\ j}{}^l + \left( \frac{\delta I}{\delta \Gamma^c_{ab}} \gamma^c_{ab\ j}{}^{kl} \right)_{,k} \Bigg] \\ &\quad + \xi^j_{,m} \gamma^c_{ab\ j}{}^{lm} \frac{\delta I}{\delta \Gamma^c_{ab}} \Bigg\}_{,l} = 0. \end{aligned} \tag{59}$$

From the above identity and requiring the gravitational field equations to hold, we can introduce a double-index superpotential

$$[-t^l + 2\sqrt{-g} T^l_j \xi^j] = \overline{U}^{[lj]}_{,j}, \tag{60}$$

where we have used Eq. (20).

From the equality (52) we can write

$$\xi^j (U'^{[ik]}_{j})_{,k} = \xi^j \Theta'^{i}_{j} = \sqrt{-g} \xi^j (T^i_j + t'^{i}_{j}). \tag{61}$$

Completing the derivative on the left hand side we obtain

$$(\xi^j U'^{[ik]}_{j})_{,k} = \sqrt{-g} \xi^j (T^i_j + \overline{t^i_j}), \tag{62}$$

where we have defined

$$\overline{t^i_j} = t'^{i}{}_j + \xi_j \xi^l{}_{,k} U'^{[ik]}_{l} (\sqrt{-g} \xi^2)^{-1}. \tag{63}$$

By comparing Eq. (60) with Eq. (62), it is clear that we can choose

$$\overline{U}^{[ik]} = 2\xi^j U'^{[ik]}_{j},$$

$$t^i = -2\sqrt{-g}\,\xi^j \overline{t^i}_j. \quad (64)$$

For example, in GR, we have to take $\overline{t^i_j}$ as the Einstein pseudotensor and $U'^{[ik]}_j$ as the von Freud superpotential [26].

The conserved vector quantity given by Eq. (62),

$$\Theta^i = (\xi^k U'^{[ij]}_k)_{,j} = \sqrt{-g}\,\xi^j (T^i_j + t'^i_j) \quad (65)$$

is clearly dependent upon the choice of superpotential. In particular, when we use Mϕller's superpotential (58), we find

$$\Theta^i_M = (\xi^k U^{[ij]}_{Mk})_{,j}. \quad (66)$$

By adding a quantity

$$\overline{W}^{[ij]}_{,j} = [\sqrt{-g}(f'\xi^j_{,m} g^{im} - f'\xi^i_{,m} g^{jm})]_{,j} \quad (67)$$

to $\Theta^i_M$ we obtain one analogue of the Komar vector $\overline{\Theta}^i_K$:

$$\overline{\Theta}^i_K = \sqrt{-g}[(f'\xi^j)^{;i} - (f'\xi^i)^{;j}]_{;j}, \quad (68)$$

where (;) is the covariant derivative defined from the metric connection. Note that $\overline{\Theta}^i_K$ is locally covariant. It is obvious that this vector has an analogue of the Komar superpotential $\overline{U}^{[ij]}_K = 2\sqrt{-g}(f'\xi^{[j})^{;i]}$, which can be written as

$$\overline{U}^{[ij]}_K = 2\sqrt{-g} f' \nabla^{[i}\xi^{j]}, \quad (69)$$

and the covariant derivative $\nabla$ is defined using the connection $\Gamma$ .

In fact, formally this superpotential differs from the Komar superpotential by the conformal factor $f'(R)$, but there is a deep difference: the Komar superpotential in general relativity depends on the metric connection and not on the other connection. Also, it is worth noting, as we have just said, that the test particles follow the geodesics of the metric connection. Therefore one must choose the last one as the physical connection.

If, instead of $\overline{W}^{[ij]}_{,j}$, we add a quantity

$$W^{[ij]}_{,j} = [\sqrt{-g}(f'\xi^j_{,m} g^{im} - f'\xi^i_{,m} g^{jm})]_{,j} + [\sqrt{-g}(f'^{,j}\xi^i - f'^{,i}\xi^j)]_{,j}, \quad (70)$$

we would obtain another analogue of the Komar vector $\Theta^i_K$,

$$\Theta^i_K = 2\sqrt{-g}(f'\xi^{[j;i]})_{;j}, \quad (71)$$

and the corresponding superpotential is, using the metric connection, formally equal to Eq. (69):

$$U^{[ij]}_K = 2\sqrt{-g} f' \xi^{[j;i]} = f' U^{[ij]}_{K\ GR}, \quad (72)$$

where $U^{[ij]}_{K\ GR}$ is the Komar superpotential in general relativity. It is important to note that the last result is not casual, the difference between two quantities of the type $\sqrt{-g}\nabla^{[i}\xi^{j]}$ with different covariant derivatives being an antisymmetric pseudotensor $\sqrt{-g}C^{[ij]}{}_a\xi^a$; then it is natural that both quantities are pseudotensor superpotentials.

It is believed that a good choice of one of these two vector densities would be one in which the equality between active and inertial mass is satisfied.

From Eqs. (68) and (71) it follows that

$$\overline{\Theta}^i_K = \Theta^i_K + 2\sqrt{-g}(\xi^{[j} f'^{;i]})_{;j}. \quad (73)$$

Finally, comparing the Komar superpotential in GR, $U^{[ij]}_{K\ GR}$, with its analogues we have

$$\overline{U}^{[ij]}_K = U^{[ij]}_K + 2\sqrt{-g}\,\xi^{[j} f'^{;i]}, \quad (74)$$

$$U^{[ij]}_K = f' U^{[ij]}_{K\ GR}. \quad (75)$$

It is easy to see that in the first-order formalism the candidates to covariant energy-density flow are all equal to the GR Komar expression, for all kind of sources, only when $f'(R)$ is constant for all $R$; then $f(R) = CR$ where $C$ is a constant. Now, in order to have the correct newtonian limit the constant $C$ must be equal to $1/8\pi G$.

## V. ACTIVE MASS

In the last section we have obtained two vector densitiess, which are candidates for covariant energy flow in the first-order formalism. In this section, we shall select one of them by asking it to satisfy the equality between active mass and inertial mass.

The weak field limit of this theory [7] is

$$g_{ab} = \eta_{ab} - (4t_a t_b + 2\eta_{ab})V_N - \eta_{ab} 8\pi f''(0)\rho, \quad (76)$$

where $t^a$ is the time direction of our global inertial coordinate system and $V_N$ is the Newtonian potential; i.e., it satisfies the Poisson equation $\nabla^2 V_N = 4\pi\rho$. Suppose we have a good theory; thus there must exist solutions of the field equations which have a region, far from the sources, where the field may be considered as weak fields and the weak field limit equations are valid. In particular, such a region may be taken as the exterior of a spherical shell $S$. To construct a solution, we impose adequate boundary conditions on $S$, to match the fields to the interior of $S$, and a general multipole expansion at infinity. From this multipole expansion we obtain that the leading order terms for $g_{00}$ are

$$g_{00} = -1 + \frac{2GM}{r}, \quad (77)$$

where $M$ is a constant which can be determined from the boundary conditions on $S$. Usually, this coefficient of the leading order term in the multipole expansion of $g_{00}$ near infinity is called the active mass or gravitational mass.

As in GR, it is easy to prove from Eq. (77) that the covariant expression of the total active mass, seen at spatial infinity for a static, asymptotically flat spacetime, is given by [3]

$$M = \frac{1}{4\pi G}\oint_S \xi^{[b;a]} dS_{ab}, \quad (78)$$

where $\xi^a$ is a timelike Killing vector field, $S(r)$ is a topological two-sphere at spatial infinity, and $dS_{ab}=2n_{[a}N_{b]}dS$. In the expression for $dS_{ab}$, $dS$ is the natural surface element on $S$ induced by the spacetime metric, $N_a$ is the unit outward pointing normal to $S$ which is orthogonal to $\xi^a$, and $n^a=\xi^a/U$ [with $U^2=(-\xi^a\xi_a)$and $U\rightarrow 1$ near infinity].

For isolated gravitating systems (i.e., asymptotically flat space-time), application of Gauss's theorem shows that the integral

$$P(\xi)=\int_\Sigma \Theta^i n_i d^3x \tag{79}$$

is independent of the open spacelike hypersurface over which it is evaluated, and represents a conserved quantity. In the above equation $\Theta^i$ is any of the two vectors $\Theta^i_K$ and $\overline{\Theta}^i_K$ and $n_i$ is the orthonormal vector to $\Sigma$.

Since the conserved vectors (68) depend on the choice of $\xi^a$, there will be an infinity of conserved quantities corresponding to the infinite group of diffeomorphisms. It is generally thought, however, that physically significant conserved quantities are generated by those infinitesimal transformations which represent intrinsic symmetry properties of the gravitational field. The conserved quantities, corresponding to the temporal invariance of the solution, obtained by the above-described method, are the total energy and inertial mass of the theory.

In particular we can assume we have a static stellar object which can have a sharp boundary ($\rho$ is discontinuous throught a two-surface $\partial V$ which is the boundary of a volume $V$) or $T_{ab}\neq 0$ near infinity but approaches zero sufficiently rapidly for the space time to be asymptotically flat. In the first case across the sharp boundary we have to satisfy, obviously, the junction conditions. Also, we assume that $f'(R)$ is an analytic function; then in the case of a sharp boundary and from Eq. (4), $f'(R)$ is discontinuous through $\partial V$, its derivative is divergent over this surface, and $R=0$ outside this surface.

The volume integral of $\overline{\Theta}^b_K$ has two parts: one is the volume integral of $\Theta^b_K$. Although the vector $\overline{\Theta}^b_K$ is always obtained from a two-index superpotential, its volume integral over $\Sigma$ cannot be straightforwardly transformed, in the case of sharp boundary, by Gauss's law into a surface integral [since the second term on the right hand side of Eq. (73) is divergent on $\partial V$]. But the volume integral of $\Theta^b_K$ could be transformed into a surface integral:

$$\begin{aligned}P(\xi)&=\int_\Sigma \overline{\Theta}^i_K n_i d^3x\\&=\oint_S 2f'(R)\xi^{[b;a]}n_{[a}N_{b]}dS+\int_\Sigma 2(f'^{[,a}\xi^{b]})_{;b}n_a d^3x,\end{aligned} \tag{80}$$

where $n_a$ is a unit timelike vector and $N_b$ is a unit normal vector to $S$ and to $n_a$. But $S$ is in spatial infinity and $R=0$ on $S$, so in order to have the appropriate Newtonian limit $f'(0)=1/8\pi G$ must hold.

Now, when $T_{ab}\neq 0$ near infinity and $T_{ab}$ is a well-behaved function in the entirety of space-time, we could apply Gauss's law to the volume integral in Eq. (80), and we obtain again a surface integral in spatial infinity, but in this surface $f'^{,a}=0$; then the second term in Eq. (80) vanishes.

When the stellar object has a sharp boundary, the volume integral of Eq. (80) could be evaluated using distribution theory [27] and in this case we obtain that this volume integral is null again. Finally, in both cases, for a stellar object with sharp boundary or not, we obtain

$$\begin{aligned}P(\xi)&=\int_\Sigma \overline{\Theta}^i_K n_i d^3x\\&=\int_\Sigma \Theta^i_K n_i d^3x\\&=\frac{1}{4\pi G}\oint_S \xi^{[b;a]}n_{[a}N_{b]}dS.\end{aligned} \tag{81}$$

But according to Eq. (78) this integral is the active mass $M$.

Hence, using $\overline{\Theta}^b_K$ or $\Theta^b_K$ as the energy density flow, the conserved quantity $E=P(\xi)$ is

$$E=P(\xi)=M. \tag{82}$$

Therefore, the weak equivalence principle (equality between inertial and active mass) is satisfied for any kind of sources.

Now, the Killing vector $\xi^a$ satisfies the equation

$$\nabla^d\nabla_d\xi^c=R^c_d\xi^d, \tag{83}$$

and using the field equations, the vector density $\overline{\Theta}^a_K$ can be written

$$\overline{\Theta}^a_K=2\sqrt{-g}\left(T^a{}_c+\frac{1}{2}f(R)g^a{}_c+\nabla^a\nabla_c f'(R)\right)\xi^c. \tag{84}$$

The conserved quantity $E(\xi)$ turns out to be

$$\begin{aligned}E(\xi)&=\int_\Sigma \Theta^a n_a dV\\&=2\int_\Sigma\left(T_{ab}+\frac{1}{2}f(R)g_{ab}+\nabla_a\nabla_b f'(R)\right)\xi^b n^a dV.\end{aligned} \tag{85}$$

By expressing $R$ as a function of $T$ we obtain a generalization of Tolman's formula for the energy, i.e., the energy as an integral of the sources.

Let us suppose we have a source of compact support $V$. Because of the analyticity of $f(R)$, outside $V$, Eq. (4) can have only a discrete set of solutions, namely, $R=r_a$, and when the solutions are asymptotically flat $r_a$ must be zero. And from Eq. (4) we can also see that for any asymptotically flat solution with $T\rightarrow 0$ at spatial infinity to exist, the theory must satisfy the necessary condition $f(0)=0$.

Since the sources are confined in a compact subregion $\Sigma'\subset\Sigma$, then the total energy can be expressed as an integral over any such subregions. Thus, under this condition, the total energy defined by Eq. (82) is independent of the two-dimensional surface $S$ which encloses $\Sigma'$.

### VI. CONCLUSIONS

The conservation laws derived above appear to be natural extensions of the work of Einstein, Møller, Bergmann, and Komar to the theories obtained from an arbitrary $f(R)$ Lagrangian and using the first-order formalism. In particular, we have derived conservation laws of two kinds: double-index conservation laws and their corresponding superpotential for the conserved complex, and single-index conservation laws. In the first case we have obtained that a possible superpotential of the theory is an analogue of the Møller superpotential of general relativity. One of the main results of this part was to establish the antisymmetry of the quantity $U_k^{ab}$ introduced in Eq. (37). In the second case, we have proved that there are two conserved vector analogues of the Komar vector of GR: one using the connection and the other the metric connection. Astonishingly both of them satisfy the physical condition that the inertial mass must be equal to the gravitational (active) mass for any class of matter, even in the case of matter with a sharp boundary. The equality between energy and active mass is not necessarily true in any other alternative theory of gravity. For example, in Brans-Dicke scalar tensor theories there is no vector density deduced from symmetry properties, which gives an inertial mass (energy) equal to the active mass.

Since the tensor $T^{ab}$ satisfies $T^{ab}{}_{;b}=0$, the test particles follow the metric geodesic. Therefore, the physical connection is the metric connection. Thus, it is natural that one of the analogues of the Komar vector (that obtained using the metric connection) is associated with the active mass of the theory. But we have obtained that both vector densities correspond to the possible energy flow, because they satisfy the equality between inertial mass and active mass. Therefore we do not have a single energy flow density, and then the localization of the energy in the interior of the system is impossible. Finally, the only theories which can have asymptotically flat solutions are theories with $f(0)=0$. On the other hand, these theories share with GR the property that, for sources with a compact support, the total energy is independent of any two-dimensional surface which encloses the support of the matter distribution.

### ACKNOWLEDGMENTS

The authors would like to thank Dr. Victor Hugo Hamity for useful discussions. The authors also are very grateful to CONICET, SeCyT, and CONICOR, Argentina, for financial support.

[1] S. Weinberg, in *General Relativity*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979), p. 407.
[2] K. S. Stelle, Gen. Relativ. Gravit. **9**, 353 (1978).
[3] R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).
[4] H. J. Schmidt, Phys. Rev. D **49**, 6354 (1994).
[5] B. Shahid-Saless, Phys. Rev. D **35**, 467 (1987).
[6] D. Barraco, Ph.D thesis, 1991.
[7] D. Barraco and V. H. Hamity, Gen. Relativ. Gravit. **25**, 461 (1993).
[8] J. P. Berthias and B. Shahid-Saless, Class. Quantum Grav. **10**, 1039 (1993).
[9] G. F. Rubilar, Class. Quantum Grav. **15**, 239 (1998).
[10] D. Barraco, R. Guibert, V. Hamity, and H. Vucetich, Gen. Relativ. Gravit. **28**, 339 (1996).
[11] B. Shahid-Saless, J. Math. Phys. **31**, 2429 (1990).
[12] B. J. Shahid-Saless, J. Math. Phys. **32**, 694 (1991).
[13] P. W. Higgs, Nuovo Cimento **11**, 816 (1959); B. Whitt, Phys. Lett. **145B**, 176 (1984); A. Jakubiec and J. Kijowski, Phys. Rev. D **37**, 1406 (1988); A. Hindawi, B. Ovrut, and D. Waldram, *ibid.* **53**, 5583 (1996).
[14] G. Magnano, M. Ferraris, and M. Francaviglia, Gen. Relativ. Gravit. **19**, 465 (1987).
[15] M. Ferraris, M. Francaviglia, and G. Magnano, Class. Quantum Grav. **5**, L95 (1988).
[16] G. Magnano and L. Sokolowski, Phys. Rev. D **50**, 5039 (1994).
[17] A. Einstein, Ann. Phys. (Leipzig) **49**, 769 (1916).
[18] P. Von Freud, Ann. Math. **40**, 417 (1939).
[19] C. Møller, Ann. Phys. (N.Y.) **4**, 347 (1958).
[20] D. Barraco, E. Dominguez, R. Guibert, and V. Hamity, Gen. Relativ. Gravit. **30**, 629 (1998).
[21] D. Barraco and V. Hamity, Class. Quantum Grav. **11**, 2113 (1994).
[22] M. Ferraris, M. Francaviglia, and I. Volovich, Class. Quantum Grav. **11**, 1505 (1994).
[23] A. Trautmann, in *Lectures on General Relativity*, edited by S. Deser and K. W. Ford (Prentice-Hall, Englewood Cliffs, NJ, 1965), Chap. 7.
[24] A. Trautmann, in *Gravitation*, edited by L. Witten (Wiley, New York, 1962).
[25] J.N. Goldberg, in *General Relativity and Gravitation: One Hundred Years after the Birth of Albert Einstein*, edited by A. Held (Plenum, New York, 1980).
[26] P. G. Bergmann, Phys. Rev. **112**, 287 (1958).
[27] L. Schwartz, *Mathematics for the Physical Sciences* (Addison-Wesley, Reading, MA, 1966).