PHYSICAL REVIEW D, VOLUME 60, 044007

Generation of gravitational waves by generic sources in de Sitter space-time

H. J. de Vega
LPTHE Université Pierre et Marie Curie (Paris VI) et Denis Diderot (Paris VII), Tour 16, 1er. étage, 4, Place Jussieu, F-75252 Paris, Cedex 05, France

J. Ramirez*
Departamento de Fisica Teorica I, Universidad Complutense de Madrid, 28040 Madrid, Spain

N. Sanchez
DEMIRM, Observatoire de Paris, 61, Avenue de l'Observatoire, F-75014 Paris, France
(Received 30 December 1998; published 9 July 1999)

We study the generation of gravitational radiation by sources moving in the de Sitter background. Exploiting the maximal symmetry and the conformal flatness of de Sitter space-time we prove that the derivation of this gravitational radiation can be done along the same lines as in Minkowski space-time. A gauge is chosen in which all the physical and unphysical modes of the graviton are those of a minimally coupled massless scalar field in de Sitter space-time and a massless field in Minkowski space-time. The graviton retarded Green's function and the Schwinger commutator function are computed in this gauge using quantum field theory techniques. We obtain closed formulas for the spectral decomposition in frequencies of the linearized gravitational field produced by the source, in terms of a suitable spectral decomposition of the source energy-momentum tensor $T_{\mu}^{(1)\nu}$. This spectral decomposition is dictated by the free (sourceless) gravitational wave modes in the de Sitter background. [S0556-2821(99)00314-8]

PACS number(s): 04.30.Db, 04.62.+v

I. INTRODUCTION

de Sitter space-time is interesting for several reasons. To begin with, de Sitter space-time is the natural framework for inflationary models of the universe, in which the universe experiences a period of exponential expansion driven by the energy density of the vacuum. In addition, it is the simplest model of space-time with a nonvanishing cosmological constant Λ; i.e. it is the simplest model of empty space-time but for the energy of the vacuum. It shares with Minkowski space-time the property of being maximally symmetric, but on the other hand it has a nonvanishing curvature. Thus, it provides a tractable example of space-time in which the effects of curvature can be explicitly computed.

Gravitational radiation is a subject of fundamental importance and current interest in both astrophysics and cosmology. Its direct detection is now a realistic challenging possibility within the current and expected level of detector's sensitivity. In addition to the weak field sources (i.e. binary star systems, fissioning stars, oscillating and rotating spheroids, . . .), neutron stars, collapsing supernovae, quasars and black hole collisions are sources of intense gravitational radiation. In cosmology, gravitational waves would arise during the several phase transitions undergone by the early universe, as well as from string sources, and at the end of inflation [1]. The linearization procedure around flat Minkowski space-time is clearly not applicable in such strong field problems. Perturbation techniques around curved backgrounds, and in the presence of sources, are needed.

In this paper we consider the generation of gravitational waves by generic sources which move in the de Sitter space-time. That is, we compute the waves produced by a source described by an energy-momentum tensor $T_{\mu}^{(1)\nu}$, which is covariantly conserved with respect to the de Sitter metric. It turns out that the maximal symmetry of the de Sitter space-time, and the use of conformal coordinates to exploit its conformal flatness, allow a discussion of gravitational radiation in this space following analogous steps as in Minkowski space-time. Of course, we consider weak enough sources for the linear approximation to hold. Apart from this condition, the sources are totally generic.

As one of the main results of our study, we obtain explicit closed formulas for the spectrum—amplitude as a function of the frequency—of gravitational waves in terms of a suitable spectral decomposition of the energy-momentum tensor which is producing these waves. These formulas should be interesting for primordial cosmology, since they relate the spectrum of primordial gravity waves produced during the inflationary period to the energy-momentum tensor of the possible sources existing during this period. This is a classical mechanism of production of primordial gravity waves, and it is different to the quantum mechanical production of gravitons due to the time variation of the de Sitter metric which has been studied before [2,3].

The paper is organized in a self-contained way as follows: In Sec. II we consider a linearized gravitational perturbation around the 4D de Sitter background. Thus, we consider the perturbative expansion of the full Einstein equations—including the source and the metric perturbation—in orders of the perturbation h_{μ}^{ν}. Because of the maximal symmetry of the de Sitter space-time, the Einstein equations up to first order, can be consistently split into two equations: a zero order equation describing the de Sitter space being produced

*Email address: mittel@fis.ucm.es

by the cosmological constant, and a first order equation which describes the production of gravity waves from a source. Thanks to the maximal symmetry, the covariant conservation of the source energy-momentum tensor $T_\mu^{(1)\nu}$ with respect to the de Sitter metric coincides with the integrability condition for the first order part of the Einstein equation.

In Sec. III we discuss the gauge invariance of the first order equation. Again the maximal symmetry of the de Sitter space-time implies the gauge invariance of the first order part of the Einstein tensor $G_\mu^{(1)\nu}$ under infinitesimal coordinate transformations. Then, in order to give the first order part of the Einstein equation a definite form, we set a gauge fixing condition of the form $D_\nu\psi_\mu^\nu=B_\mu$, where ψ_μ^ν is the trace reversed graviton field. The field B_μ is chosen as to eliminate the non-diagonal terms in first derivatives from $G_\mu^{(1)\nu}$. With this choice, we arrive to the simple decoupled system of wave equations (3.23) governing the linearized gravitational field ψ_μ^ν produced by the source $T_\mu^{(1)\nu}$.

In Sec. IV we solve the homogeneous version of Eq. (3.23) which describes free (sourceless) gravitational waves. The solutions turn out to be remarkably simple as a consequence of our gauge choice (3.21). They amount to de Sitter minimally coupled massless scalar modes and Minkowski massless modes [4]. We also discuss the residual gauge invariance allowed by our gauge choice and use it to extract the two transverse traceless physical polarizations of the graviton. We conclude this section setting the conditions of validity for the linear approximation.

Section V is devoted to the computation of the retarded Green's function solving the graviton wave equation (3.23). The graviton retarded Green's function and also the Feynman propagator in the de Sitter space-time have been studied in previous papers [5,6], following a geometrical approach developed in [7]. In this paper we compute the retarded Green's function most easily using quantum field theory (QFT) techniques. By the way we also obtain a very simple expression for the Schwinger commutator function of a minimally coupled massless scalar field.

Finally in Sec. VI we compute the linearized gravitational field produced by a generic source with energy-momentum tensor $T_\mu^{(1)\nu}$. We show that the spectral decomposition of this gravitational field in frequencies, can be easily expressed in terms of a suitable spectral decomposition of the energy momentum tensor $T_\mu^{(1)\nu}$. This spectral decomposition is dictated by the form of the free gravitational wave modes. We also show that the linearized gravitational field produced by localized sources takes the form of free gravitational waves being radiated away from the source, and whose amplitudes are easily related to $T_\mu^{(1)\nu}$.

We end up with the conclusions in Sec. VII, followed by an Appendix with the formulas for covariant derivatives, curvature tensors and d'Alembertians for the de Sitter space-time which are needed in the paper.

II. GRAVITATIONAL PERTURBATIONS IN de SITTER SPACE-TIME

In this section we study the linearized gravitational perturbations around the 4D de Sitter background. Thus, we start from the metric

$$g_{\mu\nu}\equiv\gamma_{\mu\nu}+h_{\mu\nu}=\frac{1}{H^2\eta^2}(\eta_{\mu\nu}+\phi_{\mu\nu}) \tag{2.1}$$

where $\eta_{\mu\nu}=\text{diag}(-,+,+,+)$ is the Minkowski metric, $\gamma_{\mu\nu}$ is the de Sitter metric, and $h_{\mu\nu}=(H^2\eta^2)^{-1}\phi_{\mu\nu}$ is a small perturbation, in the sense that the components of the tensor density $\phi_{\mu\nu}$ are much smaller than 1.

As it is less involved to work with tensors rather than with tensor densities, it is convenient to introduce the tensor field

$$h_\mu^\nu=\gamma^{\nu\rho}h_{\mu\rho} \tag{2.2}$$

where $\gamma^{\nu\rho}$ is the inverse de Sitter metric, and so

$$h_0^0=-\phi_{00},\quad h_i^0=-h_0^i=-\phi_{0i},\quad h_i^j=h_j^i=\phi_{ij}. \tag{2.3}$$

It is also convenient to introduce the trace reversed graviton field

$$\psi_\mu^\nu=h_\mu^\nu-\frac{1}{2}\delta_\mu^\nu h \tag{2.4}$$

where $h\equiv h_\lambda^\lambda$.

A remark about the conventions for the raising and lowering of tensor indices is here in order. The indices of h_μ^ν (or ψ_μ^ν) are raised and lowered with the de Sitter metric, $\gamma^{\mu\nu}$ denotes the inverse of $\gamma_{\mu\nu}$, D_μ denotes the covariant derivative with respect to the de Sitter metric, and $D^\nu=\gamma^{\nu\rho}D_\rho$. For the rest of the tensors, their indices are raised and lowered with the full metric and its inverse $g^{\mu\nu}$, which up to first order reads $g^{\mu\nu}=\gamma^{\mu\nu}-h^{\mu\nu}+0(h^2)$. Notice however, that when a tensor has a perturbative expansion, the former rule applies to the full tensor, and not to each of the terms in the expansion. Thus if we consider, for example, the expansions for the full Ricci tensor up to first order in h_μ^ν: $R_\mu^\nu=R_\mu^{(0)\nu}+R_\mu^{(1)\nu}+\cdots$ and $R_{\mu\kappa}=R_{\mu\kappa}^{(0)}+R_{\mu\kappa}^{(1)}+\cdots$, we have $R_\mu^\nu=g^{\nu\rho}R_{\mu\rho}$, which implies for the first order term $R_\mu^{(1)\nu}=\gamma^{\nu\rho}R_{\mu\rho}^{(1)}-h^{\nu\rho}R_{\mu\rho}^{(0)}$.

Now we consider the Einstein equations

$$G_\mu^\nu\equiv R_\mu^\nu-\frac{1}{2}\delta_\mu^\nu\ R=-8\pi\mathcal{G}\,T_\mu^\nu \tag{2.5}$$

and deal with them in the following way: we develop the Einstein tensor G_μ^ν up to first order in h_μ^ν,

$$G_\mu^\nu=G_\mu^{(0)\nu}+G_\mu^{(1)\nu}+\cdots \tag{2.6}$$

and for the second member we set

$$T_\mu^\nu=T_\mu^{(0)\nu}+T_\mu^{(1)\nu}+\cdots \tag{2.7}$$

where

$$T_\mu^{(0)\ \nu}=-\Lambda\ \delta_\mu^\nu \tag{2.8}$$

is the energy-momentum tensor producing the de Sitter background with cosmological constant $\Lambda=3H^2/8\pi\mathcal{G}$, and $T_\mu^{(1)\nu}$ is the energy-momentum tensor of a source. This source is moving in the de Sitter background. So, $T^{(1)\nu}_{\mu}$ is covariantly conserved with respect to the de Sitter metric, i.e.

$$D_\nu T^{(1)\nu}_{\mu}=0. \tag{2.9}$$

Thus, up to first order in h^ν_μ, the Einstein equations (2.5) give the two equations

$$G^{(0)\nu}_{\mu}=-8\pi\mathcal{G}\,T^{(0)\nu}_{\mu} \tag{2.10}$$

and

$$G^{(1)\nu}_{\mu}=-8\pi\mathcal{G}\,T^{(1)\nu}_{\mu}. \tag{2.11}$$

Then, Eq. (2.10) describes how the background is produced by the energy-momentum tensor $T^{(0)\nu}_{\mu}$, and Eq. (2.11) describes the production of a gravitational perturbation by the energy-momentum tensor $T^{(1)\nu}_{\mu}$, which is to be considered small, i.e. of first order. This is exactly the point of view adopted when computing gravitational radiation by astrophysical sources in Minkowski space-time [8,9], with the only difference that, for Minkowski space-time, $T^{(0)\nu}_{\mu}$ vanishes. There is however a catch: we should make sure that Eq. (2.11) is consistent, i.e. integrable. Now, the integrability condition for the full Einstein equation (2.5) takes the form

$$\begin{aligned}\mathcal{D}_\nu T^\nu_\mu &= D_\nu T^{(0)\nu}_{\mu}+D_\nu T^{(1)\nu}_{\mu}+\Gamma^{(1)\nu}_{\nu\rho}\,T^{(0)\rho}_{\mu}\\ &\quad -\Gamma^{(1)\rho}_{\nu\mu}\,T^{(0)\nu}_{\rho}+\cdots\\ &=0\end{aligned} \tag{2.12}$$

where $\mathcal{D}_\mu$ is the covariant derivative with respect to the full metric $g_{\mu\nu}$. In Eq. (2.12), we have developed the metric connection up to first order as

$$\Gamma^\nu_{\mu\lambda}=\Gamma^{(0)\nu}_{\mu\lambda}+\Gamma^{(1)\nu}_{\mu\lambda}+\cdots \tag{2.13}$$

with $\Gamma^{(0)\nu}_{\mu\lambda}$ being the metric connection for the de Sitter metric given in Eq. (A3), and $\Gamma^{(1)\nu}_{\mu\lambda}$ its first order correction

$$\Gamma^{(1)\nu}_{\mu\lambda}=\frac{1}{2}(D_\mu h^\nu_\lambda+D_\lambda h^\nu_\mu-D^\nu h_{\mu\lambda}). \tag{2.14}$$

Now the zero order part of Eq. (2.12), $D_\nu T^{(0)\nu}_{\mu}=0$, is obviously satisfied, and the first order part yields the integrability condition

$$D_\nu T^{(1)\nu}_{\mu}+\Gamma^{(1)\nu}_{\nu\rho}\,T^{(0)\rho}_{\mu}-\Gamma^{(1)\rho}_{\nu\mu}\,T^{(0)\nu}_{\rho}=0. \tag{2.15}$$

However, due to the maximally symmetric form (2.8) of $T^{(0)\nu}_{\mu}$, the terms proportional to $\Gamma^{(1)}$ identically cancel each other, and we are left with Eq. (2.9) as integrability condition for Eq. (2.11). Thus, Eq. (2.11) can be consistently solved for any de Sitter covariantly conserved source $T^{(1)\nu}_{\mu}$, and it describes, as we shall explain in this paper, the production of gravitational radiation by this source. In particular, for $T^{(1)\nu}_{\mu}=0$, we have the equation

$$G^{(1)\nu}_{\mu}=0 \tag{2.16}$$

describing the propagation of free gravitational waves in the de Sitter space-time.

III. GRAVITATIONAL WAVE EQUATION AND CHOICE OF THE GAUGE

Once we have set Eq. (2.11), our next step in order to solve it, will be to give it a definite form. This entails a combination of two things: to obtain the expression of $G^{(1)\nu}_{\mu}$ in terms of h^ν_μ and to fix the gauge. For the first, we start with the expression of the Ricci tensor in terms of the metric connection

$$R_{\mu\kappa}=\partial_\kappa\Gamma^\lambda_{\mu\lambda}-\partial_\lambda\Gamma^\lambda_{\mu\kappa}+\Gamma^\rho_{\mu\lambda}\,\Gamma^\lambda_{\kappa\rho}-\Gamma^\rho_{\mu\kappa}\,\Gamma^\lambda_{\rho\lambda} \tag{3.1}$$

and expand it up to first order

$$R_{\mu\kappa}=R^{(0)}_{\mu\kappa}+R^{(1)}_{\mu\kappa}+\cdots \tag{3.2}$$

where $R^{(0)}_{\mu\kappa}$ is given in Eq. (A9) and

$$R^{(1)}_{\mu\kappa}=\frac{1}{2}[D^\lambda D_\lambda h_{\mu\kappa}+D_\kappa D_\mu h-D_\lambda D_\mu h^\lambda_\kappa-D_\lambda D_\kappa h^\lambda_\mu]. \tag{3.3}$$

Then the left-hand side (LHS) of Eq. (2.11) is given by

$$G^{(1)\nu}_{\mu}=R^{(1)\nu}_{\mu}-\frac{1}{2}\,\delta^\nu_\mu\,R^{(1)}=\gamma^{\nu\kappa}R^{(1)}_{\mu\kappa}-h^{\nu\kappa}R^{(0)}_{\mu\kappa}-\frac{1}{2}\,\delta^\nu_\mu\,R^{(1)} \tag{3.4}$$

with $R^{(1)}=R^{(1)\lambda}_{\lambda}$.

Next, we come to the gauge fixing. As a consequence of the covariance of the Einstein equations (2.5), and the maximal symmetry of the de Sitter space-time, the equation for the gravitational perturbation (2.11) is gauge invariant under the transformations

$$h_{\mu\kappa}\rightarrow h'_{\mu\kappa}=h_{\mu\kappa}+D_\mu\xi_\kappa+D_\kappa\xi_\mu \tag{3.5}$$

where ξ^μ is an infinitesimal vector field to be considered of the same order as h^ν_μ, i.e. first order. Let us explain how this gauge invariance arises. Under an infinitesimal coordinate transformation

$$x^\mu\rightarrow x'^\mu=x^\mu-\xi^\mu(x) \tag{3.6}$$

the metric $g_{\mu\nu}$ transforms as

$$g_{\mu\nu}\rightarrow g'_{\mu\nu}=g_{\mu\nu}+\mathcal{L}_\xi\,g_{\mu\nu}=g_{\mu\nu}+D_\mu\xi_\nu+D_\nu\xi_\mu \tag{3.7}$$

where $\mathcal{L}_\xi$ is the Lie derivative for the vector field ξ^μ, and we have omitted second order terms. Thus, we can take the point of view in which the background metric $\gamma_{\mu\nu}$ remains invariant under the infinitesimal coordinate transformations (3.6), while the perturbation $h_{\mu\nu}$ transforms as stated in Eq. (3.5). Then, for the tensors G^ν_μ and T^ν_μ, the zero order parts remain invariant under Eq. (3.6), while the first order parts transform as

$$G^{(1)\nu}_{\mu}\rightarrow G'^{(1)\nu}_{\mu}=G^{(1)\nu}_{\mu}+\mathcal{L}_\xi\,G^{(0)\nu}_{\mu} \tag{3.8}$$

and a similar expression for $T_\mu^{(1)\nu}$. But now, the maximally symmetric forms $G_\mu^{(0)\nu} \propto \delta_\mu^\nu$ and $T_\mu^{(0)\nu} \propto \delta_\mu^\nu$, imply $\mathcal{L}_\xi G_\mu^{(0)\nu} = 0$ and $\mathcal{L}_\xi T_\mu^{(0)\nu} = 0$. Thus, $G_\mu^{(1)\nu}$ is invariant under the infinitesimal coordinate transformations (3.6), as can also be directly checked by substitution of the gauge transformation (3.5) in the expression (3.4). Hence, Eq. (2.11) is a second order partial differential equation for the unknown function h_μ^ν, with a gauge invariant source $T_\mu^{(1)\nu}$, which is gauge invariant under the transformation (3.5). Therefore, in order to solve it, we need to set a gauge fixing condition on h_μ^ν (or ψ_μ^ν). Of course, if we were to solve only the homogeneous equation (2.16), the simplest procedure would be to impose the gauge condition [2,3]

$$D_\nu \psi_\mu^\nu = 0, \tag{3.9}$$

and then use the residual gauge invariance allowed by Eq. (3.9), for going to the so called TT gauge [9], by means of the additional condition $\psi_\mu^0 = 0$. However, this cannot be done in the presence of a non-vanishing source term $T_\mu^{(1)\nu}$. The situation here is similar to the Maxwell equations with a source term: $\Box A_\mu - \partial_\mu(\partial_\nu A^\nu) = J_\mu$. Once we have chosen the Lorentz gauge $\partial_\nu A^\nu = 0$, we cannot impose in addition $A_0 = 0$, unless $J_0 = 0$. It is true that we could set indeed the gauge condition (3.9), and then proceed to solve the resulting gauge fixed form of Eq. (2.11). However, as we shall explain below, this introduces spurious (gauge) complications in the mode solutions to Eq. (2.11). Thus, we shall impose the more general gauge fixing condition

$$D_\nu \psi_\mu^\nu = B_\mu \tag{3.10}$$

and let the equations choose their favorite gauge fixing field B_μ instead. [Notice that Eq. (3.10) implies that the gauge fixing field B_μ is to be considered of first order in the perturbation expansion.]

Imposing the gauge condition (3.10), the first order part of the Ricci tensor (3.3) can be written as

$$R_{\mu\kappa}^{(1)} = \frac{1}{2}[\, D^\lambda D_\lambda\, h_{\mu\kappa} + 2\, h^{\lambda\sigma} R_{\sigma\mu\kappa\lambda}^{(0)} + h_\mu^\sigma R_{\sigma\kappa}^{(0)} + h_\kappa^\sigma R_{\sigma\mu}^{(0)} - D_\kappa B_\mu - D_\mu B_\kappa\,]. \tag{3.11}$$

Then from Eqs. (3.11) and (3.4), the first order part of the Einstein tensor is

$$G_\mu^{(1)\nu} = \frac{1}{2} D^\lambda D_\lambda\, \psi_\mu^\nu + R_{\lambda\sigma\mu}^{(0)\nu}\, h^{\lambda\sigma} + \frac{1}{2}\, \delta_\mu^\nu\, R_{\lambda\sigma}^{(0)}\, h^{\lambda\sigma} + \frac{1}{2}[-D^\nu B_\mu - D_\mu B^\nu + \delta_\mu^\nu\, D_\lambda B^\lambda]. \tag{3.12}$$

Now according to Eq. (A7), the gauge fixing condition (3.10) reads

$$\partial_\lambda \psi_\mu^\lambda - \frac{4}{\eta}\, \psi_\mu^0 + \frac{1}{\eta}\, \delta_\mu^0\, \psi = B_\mu . \tag{3.13}$$

This allows us to rewrite the two terms in the second line of the tensor d'Alembertian (A13) as

$$\delta_\mu^0\, \eta^{\nu\alpha}\, \partial_\beta \psi_\alpha^\beta - \delta_0^\nu\, \partial_\beta \psi_\mu^\beta = -\frac{4}{\eta}(\, \delta_\mu^0\, \psi_0^\nu + \delta_0^\nu\, \psi_\mu^0) + \frac{2}{\eta}\, \delta_\mu^0\, \delta_0^\nu\, \psi + \delta_\mu^0\, \eta^{\nu\alpha}\, B_\alpha - \delta_0^\nu\, B_\mu . \tag{3.14}$$

So, for a tensor field ψ_μ^ν that satisfies the gauge fixing condition (3.13), the tensor d'Alembertian can be recast as

$$\frac{1}{H^2\eta^2} D^\lambda D_\lambda\, \psi_\mu^\nu = \Box\, \psi_\mu^\nu + \frac{2}{\eta}[\, \partial_\eta \psi_\mu^\nu + \partial_\mu \psi_0^\nu - \eta^{\nu\kappa}\, \partial_\kappa \psi_\mu^0\,] + \frac{2}{\eta^2}[\, \psi_\mu^\nu + \delta_\mu^0\, \delta_0^\nu\, \psi - 2\, \delta_\mu^0\, \psi_0^\nu - 2\, \delta_0^\nu\, \psi_\mu^0 - \delta_\mu^\nu\, \psi_0^0\,] + \frac{2}{\eta}[\, \delta_\mu^0\, \eta^{\nu\alpha}\, B_\alpha - \delta_0^\nu\, B_\mu\,]. \tag{3.15}$$

In addition, from Eqs. (A8), (A9) and (2.4), we have

$$\frac{1}{H^2\eta^2} R_{\lambda\sigma\mu}^{(0)\nu}\, h^{\lambda\sigma} = -\frac{1}{\eta^2}\left(\psi_\mu^\nu + \frac{1}{2}\, \delta_\mu^\nu\, \psi\right) \tag{3.16}$$

and

$$\frac{1}{H^2\eta^2} R_{\lambda\sigma}^{(0)}\, h^{\lambda\sigma} = \frac{3}{\eta^2}\, \psi. \tag{3.17}$$

Also, using the expressions (A4) and (A5), the second line in Eq. (3.12) takes the form

$$\frac{1}{H^2\eta^2}[-D^\nu B_\mu - D_\mu B^\nu + \delta_\mu^\nu\, D_\lambda B^\lambda] = -\eta^{\nu\alpha}\, \partial_\mu B_\alpha - \eta^{\nu\alpha}\, \partial_\alpha B_\mu + \delta_\mu^\nu\, \eta^{\alpha\beta}\, \partial_\alpha B_\beta + \frac{2}{\eta}(\, \delta_0^\nu\, B_\mu - \delta_\mu^0\, \eta^{\nu\alpha}\, B_\alpha). \tag{3.18}$$

Finally, adding up the expressions (3.15), (3.16), (3.17) and (3.18), we obtain

$$W_\mu^\nu \equiv \frac{2}{H^2\eta^2} G_\mu^{(1)\nu} = \Box\, \psi_\mu^\nu + \frac{2}{\eta}\, \partial_\eta \psi_\mu^\nu + \frac{2}{\eta}(\, \partial_\mu \psi_0^\nu - \eta^{\nu\kappa}\, \partial_\kappa \psi_\mu^0) + \frac{2}{\eta^2}[(\, \delta_\mu^\nu + \delta_\mu^0\, \delta_0^\nu)\psi - \delta_\mu^\nu\, \psi_0^0 - 2\, \delta_\mu^0\, \psi_0^\nu - 2\, \delta_0^\nu\, \psi_\mu^0\,] - \eta^{\nu\kappa}\, \partial_\mu B_\kappa - \eta^{\nu\kappa} \partial_\kappa B_\mu + \delta_\mu^\nu\, \eta^{\lambda\kappa}\, \partial_\lambda B_\kappa . \tag{3.19}$$

This expression suggests a suitable choice for B_μ, namely

$$B_\mu = 2H\, u_\nu \psi_\mu^\nu \tag{3.20}$$

where $u^{\nu}=H\eta\,\delta_0^{\nu}$ is a unit time-like vector in the background. With this choice, the non-diagonal terms in first derivatives in the first line of Eq. (3.19), cancel against the first two terms of the third line. Moreover, with the choice (3.20), the gauge fixing condition (3.10) takes the form

$$D_\lambda \psi^\lambda_\mu - 2H\, u_\lambda \psi^\lambda_\mu \equiv \partial_\lambda \psi^\lambda_\mu - \frac{2}{\eta}\psi^0_\mu + \frac{1}{\eta}\delta^0_\mu\, \psi = 0. \tag{3.21}$$

Then, using Eqs. (3.20) and (3.21), we also have

$$\eta^{\lambda\kappa}\partial_\lambda B_\kappa = \frac{2}{\eta^2}(\psi^0_0 - \psi) \tag{3.22}$$

which produces a further cancellation of two terms in Eq. (3.19). Therefore, we arrive at the following final gauge fixed form for the gravitational wave equations with a generic source term

$$\Box\, \psi^\nu_\mu + \frac{2}{\eta}\partial_\eta \psi^\nu_\mu + \frac{2}{\eta^2}[\,\delta^0_\mu\,\delta^\nu_0\,\psi - \delta^0_\mu\,\psi^\nu_0 - \delta^\nu_0\,\psi^0_\mu\,] = -\frac{16\pi\mathcal{G}}{H^2\eta^2}T^{(1)\nu}_\mu. \tag{3.23}$$

Notice that this wave equation—which we have obtained by imposing the gauge condition (3.21)—is much simpler than the one we would have obtained imposing Eq. (3.9), i.e. setting $B_\mu = 0$ in Eq. (3.19). The reader can also directly check that the solutions to Eq. (3.23), satisfy the gauge condition (3.9) provided $T^{(1)\nu}_\mu$ is de Sitter covariantly conserved.

Once we have obtained the graviton wave equation (3.23), our next task will be to solve it for a generic $T^{(1)\nu}_\mu$. For this purpose, we need the retarded Green's function for the differential operator in the left hand side of Eq. (3.23). This Green's function will be computed in Sec. V as an appropriate superposition of solutions to the homogeneous version of Eq. (3.23). Thus, we discuss first the solutions of the homogeneous equation, which are interesting in their own right, since they represent the free gravitational waves in de Sitter space-time.

IV. FREE GRAVITATIONAL WAVES IN de SITTER SPACE-TIME

In this section we set the source $T^{(1)\nu}_\mu = 0$, and solve the homogeneous version of Eq. (3.23):

$$\Box\, \psi^\nu_\mu + \frac{2}{\eta}\partial_\eta \psi^\nu_\mu + \frac{2}{\eta^2}[\,\delta^0_\mu\,\delta^\nu_0\,\psi - \delta^0_\mu\,\psi^\nu_0 - \delta^\nu_0\,\psi^0_\mu\,] = 0. \tag{4.1}$$

We shall see that in the gauge (3.21) we have a very simple basis of mode solutions to the free gravitational wave equations (4.1). To describe the solutions of Eq. (4.1) it is convenient to introduce the tensor density

$$\chi_{\mu\nu} = H^2\eta^2\,\gamma_{\nu\rho}\,\psi^\rho_\mu = \eta_{\nu\rho}\,\psi^\rho_\mu \tag{4.2}$$

whose components are related to those of ψ^ν_μ by

$$\chi_{00} = -\psi^0_0, \quad \chi_{0i} = -\psi^0_i = \psi^i_0, \quad \chi_{ij} = \psi^j_i = \psi^i_j. \tag{4.3}$$

In terms of $\chi_{\mu\nu}$ the metric perturbation $\phi_{\mu\nu}$ is

$$\phi_{\mu\nu} = \chi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\,\eta^{\alpha\beta}\chi_{\alpha\beta}. \tag{4.4}$$

We shall also use the notation

$$\hat{\chi} = \chi_{11} + \chi_{22} + \chi_{33}, \quad \tilde{\chi} = \chi_{00} + \hat{\chi}. \tag{4.5}$$

Then, splitting Eq. (4.1) into its time-time, time-space, and space-space components, we obtain

$$\Box\,\chi_{00} + \frac{2}{\eta}\partial_\eta \chi_{00} - \frac{2}{\eta^2}\tilde{\chi} = 0 \tag{4.6}$$

$$\Box\,\chi_{0i} + \frac{2}{\eta}\partial_\eta \chi_{0i} - \frac{2}{\eta^2}\chi_{0i} = 0 \tag{4.7}$$

$$\left(\Box + \frac{2}{\eta}\partial_\eta\right)\chi_{ij} = 0. \tag{4.8}$$

From Eqs. (A10) and (4.8), we see that the space components χ_{ij} behave as minimally coupled massless scalar fields in de Sitter space-time. As it is known [3], the two physical TT components of a free gravitational wave in de Sitter space-time—which count among the χ_{ij}—behave as minimally coupled massless scalar fields. As we mentioned above, this result can be easily obtained in the absence of sources by sequentially imposing the two gauge conditions $D_\nu \psi^\nu_\mu = 0$ and $\psi^0_\mu = 0$. Then, one good property of the gauge condition (3.21)—which can be applied in the presence of sources—is that it alone is sufficient to capture the minimally coupled massless scalar field behavior (4.8). In addition, the two other equations (4.7) and (4.6) are very easy to solve. In fact, Eq. (4.7) can be rewritten as

$$\Box\left(\frac{1}{\eta}\chi_{0i}\right) = 0. \tag{4.9}$$

Moreover, adding Eq. (4.6) and the trace of eq. (4.8) we also have

$$\Box\left(\frac{1}{\eta}\tilde{\chi}\right) = 0. \tag{4.10}$$

Thus, the time-space components χ_{0i} and the combination $\tilde{\chi}$ behave as free massless Minkowski fields rescaled by a factor η.

Now it is very easy to write down the general solution for $\chi_{\mu\nu}$ as a superposition of plane waves

$$\chi_{\mu\nu}(\eta,\vec{x})=\int d^3\vec{k}\,[f_{\mu\nu}(\eta;\vec{k})\exp(i\vec{k}\vec{x})+f^*_{\mu\nu}(\eta;\vec{k})\exp(-i\vec{k}\vec{x})]. \tag{4.11}$$

For χ_{0i} and $\tilde{\chi}$ we obviously have

$$\chi_{0i}(\eta,\vec{x})=\int d^3\vec{k}[e_{0i}(\vec{k})\,\eta\exp(-i\omega\eta+i\vec{k}\vec{x})+e^*_{0i}(\vec{k})\,\eta\exp(i\omega\eta-i\vec{k}\vec{x})] \tag{4.12}$$

and

$$\tilde{\chi}(\eta,\vec{x})=\int d^3\vec{k}[\tilde{e}(\vec{k})\,\eta\exp(-i\omega\eta+i\vec{k}\vec{x})+\tilde{e}^*(\vec{k})\,\eta\exp(i\omega\eta-i\vec{k}\vec{x})] \tag{4.13}$$

where $\omega=|\vec{k}|$.

In addition, the ordinary differential equation for the modes $f_{ij}(\eta;\vec{k})$ can be solved in terms of Hankel functions of index 3/2 [10]. So, we have

$$\chi_{ij}(\eta,\vec{x})=\int d^3\vec{k}\left[e_{ij}(\vec{k})\left(\eta-\frac{i}{\omega}\right)\exp(-i\omega\eta+i\vec{k}\vec{x})+e^*_{ij}(\vec{k})\left(\eta+\frac{i}{\omega}\right)\exp(i\omega\eta-i\vec{k}\vec{x})\right]. \tag{4.14}$$

And from Eqs. (4.5) and (4.13)

$$\chi_{00}(\eta,\vec{x})=\int d^3\vec{k}\left\{\left[\tilde{e}(\vec{k})\,\eta-\hat{e}(\vec{k})\left(\eta-\frac{i}{\omega}\right)\right]\times\exp(-i\omega\eta+i\vec{k}\vec{x})+\left[\tilde{e}^*(\vec{k})\,\eta-\hat{e}^*(\vec{k})\times\left(\eta+\frac{i}{\omega}\right)\right]\exp(i\omega\eta-i\vec{k}\vec{x})\right\} \tag{4.15}$$

where we have defined $\hat{e}(\vec{k})=\delta_{ij}e_{ij}(\vec{k})$.

Equations (4.12)–(4.14) provide the general solution for a free gravity wave in de Sitter space-time in the gauge (3.21). This solution identically coincides with the one presented in [4], and we have derived it here (in a different way) to make the paper self-contained. In fact, general scalar mode solutions in de Sitter space-time have been known since the sixties [11]. As we shall see in the next section, the simplicity of solutions (4.12)–(4.14) will allow us a direct derivation of the graviton retarded Green's function. In this respect, it is enlightening to point out, that if we had imposed $D_\nu\psi^\nu_\mu=0$ as our gauge fixing condition, we would have obtained a much more difficult coupled system of partial differential equations for $\chi_{\mu\nu}$ than Eqs. (4.6)–(4.8). In particular, the solutions to those equations involve Bessel functions of index $\nu=\sqrt{33}/2$.

In the rest of this section, we show how the physical degrees of freedom for the gravitational waves are contained in expressions (4.12), (4.13), and (4.14). Due to the linearity, it is enough to consider just one mode with wave vector $\vec{k}$. To begin with, the gauge fixing condition (3.21) reduces the ten polarizations that we have in $\chi_{\mu\nu}$ down to six. More precisely: for the field $\chi_{\mu\nu}$, the gauge fixing condition (3.21) reads

$$-\partial_\eta\chi_{00}+\partial_i\chi_{0i}+\frac{1}{\eta}\tilde{\chi}=0 \tag{4.16}$$

$$-\partial_\eta\chi_{0i}+\partial_j\chi_{ji}+\frac{2}{\eta}\chi_{0i}=0 \tag{4.17}$$

which translated to the modes $f_{\mu\nu}(\eta;\vec{k})\exp(i\vec{k}\vec{x})$ gives the following constraints among polarizations:

$$n_ie_{0i}+\tilde{e}-\hat{e}=0 \tag{4.18}$$

and

$$e_{0i}+n_je_{ji}=0 \tag{4.19}$$

where $n_i=k_i/\omega$ is the unit wave vector.

Next, the residual gauge invariance in the gauge fixing (3.21) allows to reduce the six independent polarizations e_{ij} to the two physical polarizations. Under the gauge transformation (3.5), the trace reversed graviton field ψ^ν_μ transforms as

$$\psi^\nu_\mu\to\psi'^{\,\nu}_\mu=\psi^\nu_\mu-D_\mu\xi^\nu-D^\nu\xi_\mu+\delta^\nu_\mu D_\lambda\xi^\lambda. \tag{4.20}$$

Then, the invariance of the gauge condition (3.21) under the transformation (4.20) requires ξ_μ to be a solution of the equation

$$D^\lambda D_\lambda\,\xi_\mu-\xi^\lambda R^{(0)}_{\lambda\mu}+2H(u_\mu D_\lambda\xi^\lambda-u_\lambda D_\mu\xi^\lambda-u_\lambda D^\lambda\xi_\mu)=0. \tag{4.21}$$

From Eqs. (A4)–(A6), (A9), and (A12) it reduces to

$$\Box\,\xi_\mu-\frac{2}{\eta}\partial_\eta\xi_\mu+\frac{2}{\eta^2}(\xi_\mu-\delta^0_\mu\,\xi_0)=0 \tag{4.22}$$

which yields the simple decoupled equations

$$\Box\left(\frac{1}{\eta}\xi^0\right)=0 \tag{4.23}$$

and

$$\left(\Box+\frac{2}{\eta}\partial_\eta\right)\left(\frac{1}{\eta}\xi^i\right)=0. \tag{4.24}$$

The solutions to Eqs. (4.23) and (4.24) decompose in modes in the form

$$\xi^0(\eta,\vec{x})=\int d^3\vec{k}[i\,\varepsilon_0(\vec{k})\,\eta\exp(-i\omega\eta+i\vec{k}\vec{x})-i\,\varepsilon^*_0(\vec{k})\,\eta\exp(i\omega\eta-i\vec{k}\vec{x})] \tag{4.25}$$

and

$$\xi^i(\eta,\vec{x}) = \int d^3\vec{k}\left[i\,\varepsilon_i(\vec{k})\left(\eta - \frac{i}{\omega}\right)\exp(-i\omega\eta + i\vec{k}\vec{x}) - i\,\varepsilon_i^*(\vec{k})\left(\eta + \frac{i}{\omega}\right)\exp(i\omega\eta - i\vec{k}\vec{x})\right]. \tag{4.26}$$

On the other hand, from the transformation law (4.20) we find

$$\delta_\xi \chi_{0i} = \partial_i \xi^0 - \partial_\eta \xi^i \tag{4.27}$$

$$\delta_\xi \chi_{ij} = -\partial_i \xi^j - \partial_j \xi^i + \delta_{ij}\left(\partial_\eta \xi^0 + \partial_l \xi^l - \frac{2}{\eta}\xi^0\right) \tag{4.28}$$

$$\delta_\xi \tilde{\chi} = 4\,\partial_\eta \xi^0 - \frac{4}{\eta}\xi^0. \tag{4.29}$$

Then, from Eqs. (4.27)–(4.29), the residual gauge transformations given by Eqs. (4.25),(4.26), amount to the following transformation laws for polarizations

$$\begin{aligned} &\tilde{e} \to \tilde{e}' = \tilde{e} + 4\omega\varepsilon_0 \\ &e_{0i} \to e'_{0i} = e_{0i} - k_i\varepsilon_0 - \omega\varepsilon_i \\ &e_{ij} \to e'_{ij} = e_{ij} + k_i\varepsilon_j + k_j\varepsilon_i + \delta_{ij}(\omega\varepsilon_0 - \vec{k}\vec{\varepsilon}). \end{aligned} \tag{4.30}$$

Thus, it is possible to eliminate the polarizations e_{0i} and $\tilde{e}$ by suitably choosing the parameters ε_μ. Therefore, we can set the supplementary gauge conditions

$$\chi_{0i} = \tilde{\chi} = 0. \tag{4.31}$$

Of course, the gauge transformations (4.30) are compatible with the constraints (4.18) and (4.19), and by replacing the supplementary conditions $e_{0i} = \tilde{e} = 0$ in Eqs. (4.18) and (4.19) we find

$$\hat{e} = n_j e_{ji} = 0. \tag{4.32}$$

This leaves us with the traceless transverse (TT) graviton physical polarizations. Notice that since $\hat{e} = 0$, Eq. (4.15) implies $\chi_{00} = 0$. Therefore, $\phi_{\mu\nu}$ coincides with $\chi_{\mu\nu}$, and the TT conditions are satisfied in this gauge by the metric perturbation itself.

We conclude this section with a comment about the validity of the linear approximation we apply. Let us consider just one physical mode for $\phi_{\mu\nu}$, with amplitude A and wave vector $\vec{k}$,

$$A\left(\eta - \frac{i}{\omega}\right)\exp(-i\omega\eta + i\vec{k}\vec{x}) + A^*\left(\eta + \frac{i}{\omega}\right)\exp(i\omega\eta - i\vec{k}\vec{x}). \tag{4.33}$$

As stated at the beginning of Sec. II, the validity of the linear approximation requires $|\phi_{\mu\nu}| \ll 1$. Thus, according to the functional form of the mode (4.33), the amplitude A must satisfy the conditions

$$|A| \ll \frac{1}{|\eta|}, \quad |A| \ll \omega \tag{4.34}$$

that is, the linear approximation holds for high frequencies and small conformal time. Let us remind that the cosmic region for the de Sitter space-time which describes an exponentially expanding space-time, corresponds to $-1/H < \eta < 0$. Thus, the linear approximation holds throughout this region provided that

$$A \ll \min\{H, \omega\}. \tag{4.35}$$

V. RETARDED GREEN'S FUNCTION FOR THE MINIMALLY COUPLED MASSLESS SCALAR FIELD AND THE GRAVITON

We come back now to the inhomogeneous equation (3.23). Splitting Eq. (3.23) we have

$$\Box\,\chi_{00} + \frac{2}{\eta}\partial_\eta \chi_{00} - \frac{2}{\eta^2}\tilde{\chi} = -16\,\pi\,\mathcal{G}\,T_{00}^{(1)} \tag{5.1}$$

$$\Box\,\chi_{0i} + \frac{2}{\eta}\partial_\eta \chi_{0i} - \frac{2}{\eta^2}\chi_{0i} = -16\,\pi\,\mathcal{G}\,T_{0i}^{(1)} \tag{5.2}$$

$$\left(\Box + \frac{2}{\eta}\partial_\eta\right)\chi_{ij} = -16\,\pi\,\mathcal{G}\,T_{ij}^{(1)}. \tag{5.3}$$

As in the previous section, Eq. (5.2) can be rewritten as

$$\Box\left(\frac{1}{\eta}\chi_{0i}\right) = -\frac{16\,\pi\,\mathcal{G}}{\eta}T_{0i}^{(1)} \tag{5.4}$$

and adding up Eq. (5.1) with the trace of Eq. (5.3) we also have

$$\Box\left(\frac{1}{\eta}\tilde{\chi}\right) = -\frac{16\,\pi\,\mathcal{G}}{\eta}\tilde{T}^{(1)}. \tag{5.5}$$

Here

$$\hat{T}^{(1)} = T_{11}^{(1)} + T_{22}^{(1)} + T_{33}^{(1)}, \quad \tilde{T} = T_{00}^{(1)} + \hat{T}^{(1)}. \tag{5.6}$$

Thus, the retarded solution to Eqs. (5.4) and (5.5) can be obtained using the well known Minkowski massless retarded Green's function

$$G_R^{(M)}(x,x') = \frac{1}{4\,\pi\,|\vec{x} - \vec{x}'|}\,\delta(\eta - \eta' - |\vec{x} - \vec{x}'|) \tag{5.7}$$

which has support on the past light cone and satisfies

$$\Box\, G_R^{(M)}(x,x')=-\delta^{(4)}(x-x'). \tag{5.8}$$

On the other hand, in order to solve Eq. (5.3) we need the retarded Green's function for the scalar d'Alembertian in de Sitter space-time, i.e. we need the retarded solution to

$$D^{\lambda}D_{\lambda}\, G_R(x,x')=-H^4\eta^4\,\delta^{(4)}(x-x'). \tag{5.9}$$

Here $D^{\lambda}D_{\lambda}$ is the scalar d'Alembertian in de Sitter space-time given in Eq. (A10). Thus, Eq. (5.9) takes the form

$$\left(\Box+\frac{2}{\eta}\partial_{\eta}\right)G_R(x,x')=-H^2\eta^2\,\delta^{(4)}(x-x'). \tag{5.10}$$

The retarded Green's function $G_R(x,x')$ solving Eq. (5.10) is given in Eq. (5.30) as can be checked by direct substitution of Eq. (5.30) in Eq. (5.10). This retarded Green's function was obtained for the first time in [6], using the geometric techniques developed in [7]. However, although the retarded Green's function is a purely classical object, it can be most easily obtained using QFT techniques. Thus, we consider a scalar field

$$\phi(\eta,\vec{x})=\int d^3\vec{k}[u_{\vec{k}}(\eta,\vec{x})\,a(\vec{k})+u_{\vec{k}}^{*}(\eta,\vec{x})\,a^{\dagger}(\vec{k})] \tag{5.11}$$

obeying the homogeneous equation

$$\left(\Box+\frac{2}{\eta}\partial_{\eta}\right)\phi=0 \tag{5.12}$$

i.e. $\phi(\eta,\vec{x})$ is a minimally coupled scalar field in de Sitter space-time. Then, from the previous section, the modes $u_{\vec{k}}(\eta,\vec{x})$ take the form

$$u_{\vec{k}}(\eta,\vec{x})=\frac{H}{(2\pi)^{3/2}\sqrt{2\omega}}\left(\eta-\frac{i}{\omega}\right)\exp(-i\omega\eta+i\vec{k}\vec{x}). \tag{5.13}$$

These modes satisfy the normalization

$$(u_{\vec{k}'},u_{\vec{k}})=\delta(\vec{k}-\vec{k}') \tag{5.14}$$

with respect to the scalar product

$$(\phi_2,\phi_1)=\frac{i}{H^2\eta^2}\int d^3\vec{x}\,\phi_2^{*}(\eta,\vec{x})\overleftrightarrow{\partial}_{\eta}\phi_1(\eta,\vec{x}). \tag{5.15}$$

Now, if we canonically quantize the scalar field ϕ according to

$$[a(\vec{k}),a^{\dagger}(\vec{k}')]=\delta(\vec{k}-\vec{k}') \tag{5.16}$$

its retarded Green's function is given by

$$G_R(x,x')=-\,\theta(\eta-\eta')\,G(x,x'). \tag{5.17}$$

Here $\theta(\eta)$ is the Heaviside step function, and $G(x,x')$ is the Schwinger function for the field $\phi(\eta,\vec{x})$

$$G(x,x')=-i\langle 0|[\phi(x),\phi(x')]|0\rangle. \tag{5.18}$$

Then, by inserting the mode expansion (5.11) in Eq. (5.18), a straightforward computation yields

$$G(x,x')=-\frac{H^2}{8\pi^2|\vec{x}-\vec{x}'|}(\eta\eta' I_1-i(\eta-\eta')I_2+I_3) \tag{5.19}$$

where

$$I_n=\int_{-\infty}^{\infty}\frac{du}{u^{n-1}}(\exp(iuy_-)-\exp(iuy_+));\quad n=1,2,3 \tag{5.20}$$

and

$$y_{\pm}=\eta-\eta'\pm|\vec{x}-\vec{x}'|. \tag{5.21}$$

The integral I_1 gives

$$I_1=2\pi(\delta(y_-)-\delta(y_+)). \tag{5.22}$$

The integral I_2 is clearly convergent, and its evaluation using residues theorem yields

$$I_2=2\pi i(\theta(y_-)-\theta(y_+)). \tag{5.23}$$

Notice that the integral I_3 is apparently logarithmically divergent in the infrared—the integration variable being $u=|\vec{k}|$—since the integrand behaves for small u as

$$\frac{1}{u^2}(\exp(iuy_-)-\exp(iuy_+))=-\frac{i}{u}2|\vec{x}-\vec{x}'|+\cdots. \tag{5.24}$$

However, this divergence does not really exist because I_3 can be rewritten as

$$I_3=I_3'+\hat{I}_3 \tag{5.25}$$

where

$$I_3'=\int_{-\infty}^{\infty}\frac{du}{u^2}(\exp(iuy_-)-\exp(iuy_+)+2i|\vec{x}-\vec{x}'|\sin u) \tag{5.26}$$

and

$$\hat{I}_3=-2i|\vec{x}-\vec{x}'|\int_{-\infty}^{\infty}\frac{du}{u^2}\sin u. \tag{5.27}$$

Then I_3' is convergent, and $\hat{I}_3$ vanishes—in the sense of Cauchy principal value—because the integrand is an odd function. Therefore, $I_3=I_3'$, and its evaluation using residues theorem, yields

$$I_3=\pi(|y_+|-|y_-|). \tag{5.28}$$

Thus, the Schwinger function $G(x,x')$ is perfectly finite in the infrared limit, and replacing expressions (5.22), (5.23), and (5.28) in Eq. (5.19), it takes the form

$$G(x,x')=-\frac{H^2}{4\pi|\vec{x}-\vec{x}'|}+\left[\eta\eta'(\delta(y_-)-\delta(y_+))+(\eta-\eta')(\theta(y_-)-\theta(y_+))+\frac{1}{2}(|y_+|-|y_-|)\right]. \tag{5.29}$$

Finally, from Eqs. (5.17) and (5.29) we obtain the following simple expression for the de Sitter scalar retarded Green's function:

$$G_R(x,x')=\frac{H^2\,\eta\eta'}{4\pi|\vec{x}-\vec{x}'|}\delta(\eta-\eta'-|\vec{x}-\vec{x}'|)+\frac{H^2}{4\pi}\theta(\eta-\eta'-|\vec{x}-\vec{x}'|). \tag{5.30}$$

Notice that to obtain this expression, some cancellations have been produced in multiplying Eq. (5.29) by $\theta(\eta-\eta')$. This is due to the following identities among distributions:

$$\begin{aligned}
&\theta(\eta-\eta')\,\delta(y_+)=0\\
&\theta(\eta-\eta')\,\delta(y_-)=\delta(y_-)\\
&\theta(\eta-\eta')\,\theta(y_+)=\theta(\eta-\eta')\\
&\theta(\eta-\eta')\,\theta(y_-)=\theta(y_-)\\
&\theta(\eta-\eta')(|y_+|-|y_-|)=2\,(\eta-\eta')\,\theta(\eta-\eta')-2\,y_-\,\theta(y_-).
\end{aligned} \tag{5.31}$$

In particular, the advanced variable y_+ has disappeared from the arguments of the distributions entering Eq. (5.30). Thus, the retarded Green's function (5.30) has support on the past light cone and its interior, as it should be according to the causality of the classical theory. Notice also that the main difference with the Minkowski retarded Green's function is the term proportional to $\theta(y_-)$. This term tells us that—although the free gravitational waves propagate at the speed of light—in the production of gravity waves from sources, there is information about these sources, which propagates at a lower speed.

Expressions (5.7) and (5.30) completely solve the problem of the retarded graviton propagator in the de Sitter space-time. With this propagator at hand, we are going to discuss in the next section the production of gravity waves by a generic source. Notice that from now on we shall suppress the upper label (1) from the energy-momentum tensor $T_{\mu\nu}$ of the source, in order to alleviate the notation.

VI. GRAVITATIONAL WAVES PRODUCED BY SOURCES IN de SITTER SPACE-TIME

We proceed to obtain the gravitational field produced—in the linear approximation—by a generic source with energy-momentum tensor $T_{\mu\nu}$. From Eqs. (5.7), (5.8) and (5.4) the time-space components for this field are

$$\begin{aligned}
\chi_{0i}(\eta,\vec{x})&=\eta\,16\pi\,\mathcal{G}\int\frac{d^3\vec{x}'\,d\eta'}{\eta'}G_R^{(M)}(x,x')T_{0i}(x')\\
&=4\,\mathcal{G}\,\eta\int\frac{d^3\vec{x}'}{|\vec{x}-\vec{x}'|}\frac{1}{\eta-|\vec{x}-\vec{x}'|}T_{0i}(\eta-|\vec{x}-\vec{x}'|,\vec{x}').
\end{aligned} \tag{6.1}$$

Similarly, for the tilde component we have

$$\tilde{\chi}(\eta,\vec{x})=4\,\mathcal{G}\,\eta\int\frac{d^3\vec{x}'}{|\vec{x}-\vec{x}'|}\frac{1}{\eta-|\vec{x}-\vec{x}'|}\tilde{T}(\eta-|\vec{x}-\vec{x}'|,\vec{x}'). \tag{6.2}$$

On the other hand, for the space-space components, Eqs. (5.3), (5.10), and (5.30) give

$$\begin{aligned}
\chi_{ij}(\eta,\vec{x})&=16\pi\,\mathcal{G}\int\frac{d^3\vec{x}'\,d\eta'}{H^2\eta'^2}G_R(x,x')\,T_{ij}(x')\\
&=4\,\mathcal{G}\int\frac{d^3\vec{x}'}{|\vec{x}-\vec{x}'|}\frac{\eta}{\eta'}\delta(\eta-\eta'-|\vec{x}-\vec{x}'|)\\
&\quad\times T_{ij}(\eta',\vec{x}')+4\,\mathcal{G}\int d^3\vec{x}'\,d\eta'\\
&\quad\times\frac{1}{\eta'^2}\theta(\eta-\eta'-|\vec{x}-\vec{x}'|)T_{ij}(\eta',\vec{x}').
\end{aligned} \tag{6.3}$$

Then, using the identity

$$\begin{aligned}
&\frac{\eta}{\eta'}\frac{1}{|\vec{x}-\vec{x}'|}\delta(\eta-\eta'-|\vec{x}-\vec{x}'|)+\frac{1}{\eta'^2}\theta(\eta-\eta'-|\vec{x}-\vec{x}'|)\\
&=\frac{1}{|\vec{x}-\vec{x}'|}\delta(\eta-\eta'-|\vec{x}-\vec{x}'|)\\
&\quad-\partial_{\eta'}\left(\frac{1}{\eta'}\theta(\eta-\eta'-|\vec{x}-\vec{x}'|)\right)
\end{aligned} \tag{6.4}$$

and doing a partial integration, Eq. (6.3) can be recast as

$$\begin{aligned}
\chi_{ij}(\eta,\vec{x})&=4\,\mathcal{G}\int\frac{d^3\vec{x}'}{|\vec{x}-\vec{x}'|}T_{ij}(\eta-|\vec{x}-\vec{x}'|,\vec{x}')\\
&\quad+4\,\mathcal{G}\int d^3\vec{x}'\int_{-\infty}^{\eta-|\vec{x}-\vec{x}'|}\frac{d\eta'}{\eta'}\partial_{\eta'}T_{ij}(\eta'\vec{x}').
\end{aligned} \tag{6.5}$$

Equations (6.1), (6.2), and (6.5) give the linearized gravitational field $\chi_{\mu\nu}$ produced by a source with energy-momentum tensor $T_{\mu\nu}$ in de Sitter space-time. With these formulas at hand, we undertake now a double task: to do the spectral decomposition of the gravitational field $\chi_{\mu\nu}$ in frequencies, relating it to the spectral decomposition of the source $T_{\mu\nu}$, and to show that this field takes the form of radiated free gravitational waves in de Sitter space-time, when we go to the ''wave zone'' far away from the sources. For the first, we need an appropriate decomposition of the energy momentum tensor in modes of frequency ω. For the space-space components, it is suitable to decompose the energy-momentum tensor in the form

$$T_{ij}(\eta,\vec{x}) = \int_0^\infty d\omega \left[\exp(-i\omega\eta)\left(\eta - \frac{i}{\omega}\right)\mathcal{T}_{ij}(\omega,\vec{x}) + \exp(i\omega\eta)\left(\eta + \frac{i}{\omega}\right)\mathcal{T}^*_{ij}(\omega,\vec{x})\right] = \int_{-\infty}^\infty d\omega \exp(-i\omega\eta)\left(\eta - \frac{i}{\omega}\right)\mathcal{T}_{ij}(\omega,\vec{x}) \tag{6.6}$$

i.e. we decompose T_{ij} according to the modes entering the components χ_{ij} of a free gravitational wave. Notice that we have defined $\mathcal{T}_{\mu\nu}(-\omega,\vec{x}) \equiv \mathcal{T}^*_{\mu\nu}(\omega,\vec{x})$ as usual. Now, since the modes $\exp(-i\omega\eta)(\eta - i/\omega)$ satisfy

$$i\,\partial_\eta \exp(-i\omega\eta)\left(\eta - \frac{i}{\omega}\right) = \omega\,\eta\exp(-i\omega\eta) \tag{6.7}$$

the integral transform (6.6) can be easily inverted in the form

$$\mathcal{T}_{ij}(\omega,\vec{x}) = \frac{1}{2\pi\,\omega}\int_{-\infty}^\infty d\eta \exp(i\omega\eta)\frac{i}{\eta}\partial_\eta T_{ij}(\eta,\vec{x}). \tag{6.8}$$

For the other components of $T_{\mu\nu}$, since the corresponding modes of a free gravitational wave take the form $\eta\exp(i\omega\eta)$, we do the spectral decompositions

$$T_{0i}(\eta,\vec{x}) = \eta\int_{-\infty}^\infty d\omega \exp(-i\omega\eta)\mathcal{T}_{ij}(\omega,\vec{x}) \tag{6.9}$$

and

$$\tilde{T}(\eta,\vec{x}) = \eta\int_{-\infty}^\infty d\omega \exp(-i\omega\eta)\tilde{\mathcal{T}}(\omega,\vec{x}) \tag{6.10}$$

whose inverses are

$$\mathcal{T}_{0i}(\omega,\vec{x}) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{d\eta}{\eta}\exp(i\omega\eta)T_{0i}(\omega,\vec{x}) \tag{6.11}$$

and

$$\tilde{\mathcal{T}}(\omega,\vec{x}) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{d\eta}{\eta}\exp(i\omega\eta)\tilde{T}(\omega,\vec{x}). \tag{6.12}$$

Now, let us decompose the gravitational field $\chi_{\mu\nu}$ in modes of frequency ω

$$\chi_{\mu\nu}(\eta,\vec{x}) = \int_0^\infty d\omega\,\chi^{(\omega)}_{\mu\nu}(\eta,\vec{x}). \tag{6.13}$$

Then, for the space-space components of χ_{ij}, replacing the spectral decomposition (6.6) for T_{ij} in the expression (6.5), we have

$$\chi^{(\omega)}_{ij}(\eta,\vec{x}) = 4\,\mathcal{G}\int \frac{d^3\vec{x}\,'}{|\vec{x}-\vec{x}\,'|}\times\left[\exp(-i\omega\eta)\exp(i\omega|\vec{x}-\vec{x}\,'|)\times\left(\eta - \left|\vec{x}-\vec{x}\,'\right| - \frac{i}{\omega}\right)\mathcal{T}_{ij}(\omega,\vec{x}\,') + \text{c.c.}\right] + 4\,\mathcal{G}\int d^3\vec{x}\,'\left[-i\,\omega\,\mathcal{T}_{ij}(\omega,\vec{x}\,')\times\int_{-\infty}^{\eta-|\vec{x}-\vec{x}\,'|}d\eta'\exp(-i\omega\eta') + \text{c.c.}\right] \tag{6.14}$$

and doing the usual shift $\omega \to \omega + i\epsilon$, to handle the lower limit of the integral over η' in the second line of Eq. (6.14), we find

$$\chi^{(\omega)}_{ij}(\eta,\vec{x}) = 4\,\mathcal{G}\left(\eta - \frac{i}{\omega}\right)\exp(-i\omega\eta)\int \frac{d^3\vec{x}\,'}{|\vec{x}-\vec{x}\,'|}\times\exp(i\omega|\vec{x}-\vec{x}\,'|)\mathcal{T}_{ij}(\omega,\vec{x}\,') + \text{c.c.} \tag{6.15}$$

Similarly, replacing the spectral decompositions (6.9) and (6.10) for T_{0i} and $\tilde{T}$ in the expressions (6.1) and (6.2) for χ_{0i} and $\tilde{\chi}$ we have

$$\chi^{(\omega)}_{0i}(\eta,\vec{x}) = 4\,\mathcal{G}\,\eta\exp(-i\omega\eta)\int \frac{d^3\vec{x}\,'}{|\vec{x}-\vec{x}\,'|}\times\exp(i\omega|\vec{x}-\vec{x}\,'|)\mathcal{T}_{0i}(\omega,\vec{x}\,') + \text{c.c.} \tag{6.16}$$

and

$$\tilde{\chi}(\eta,\vec{x}) = 4\,\mathcal{G}\,\eta\exp(-i\omega\eta)\int \frac{d^3\vec{x}\,'}{|\vec{x}-\vec{x}\,'|}\times\exp(i\omega|\vec{x}-\vec{x}\,'|)\tilde{\mathcal{T}}(\omega,\vec{x}\,') + \text{c.c.} \tag{6.17}$$

The expressions (6.15)–(6.17) are one of the main results of this paper. Just as in Minkowski space-time, these formulas relate the ω frequency component of the gravitational field produced by the source $T_{\mu\nu}$, with the ω frequency component of $T_{\mu\nu}$ itself. On the other hand, the η-time dependent factors in front of the integrals in these formulas, exactly coincide with the η-time dependent factors for a free gravitational wave of frequency ω. This is due to the appropriate decomposition of the energy momentum tensor which has been done in Eqs. (6.6), (6.9), and (6.10). This decomposition is of course different of the plain Fourier transformation which is done in Minkowski space-time, since the form of the spectral transform is dictated by the form of the free gravitational waves themselves. Moreover, we can choose a source localized in a finite spatial region, and consider the gravitational field $\chi_{\mu\nu}$ produced by this source in the ''wave zone.'' To do it, we go far away from the source and take the region of points $\vec{x}$ that satisfy

$$|\vec{x}| \gg \max\{a,\, \omega^2 a\} \tag{6.18}$$

where a is the size of the source. In this limit we have as usual

$$\frac{1}{|\vec{x}-\vec{x}'|} \sim \frac{1}{r}, \quad \exp(i\omega|\vec{x}-\vec{x}'|) \sim \exp(i\vec{k}\vec{x}-i\vec{k}\vec{x}') \tag{6.19}$$

with $r=|\vec{x}|$, and $\vec{k}=\omega\,\vec{x}/|\vec{x}|$. Thus, the expressions (6.15), (6.16), and (6.17) give in this limit

$$\chi_{ij}^{(\omega)}(\eta,\vec{x}) \sim \frac{4\,\mathcal{G}}{r}\left(\eta-\frac{i}{\omega}\right)\exp(-i\omega\eta+i\vec{k}\vec{x})\,A_{ij}(\vec{k}) \tag{6.20}$$

$$\chi_{0i}^{(\omega)}(\eta,\vec{x}) \sim \frac{4\,\mathcal{G}}{r}\,\eta\exp(-i\omega\eta+i\vec{k}\vec{x})\,A_{0i}(\vec{k}) \tag{6.21}$$

$$\tilde{\chi}^{(\omega)}(\eta,\vec{x}) \sim \frac{4\,\mathcal{G}}{r}\,\eta\exp(-i\omega\eta+i\vec{k}\vec{x})\,\tilde{A}(\vec{k}) \tag{6.22}$$

which correspond to free gravitational spherical waves in de Sitter space-time, being radiated away from the source, and whose amplitudes are given in terms of the energy-momentum tensor of the source by

$$A_{\mu\nu}(\vec{k}) = \int d^3\vec{x}'\,\exp(-i\vec{k}\vec{x}')\,\mathcal{T}_{\mu\nu}(\omega,\vec{x}'). \tag{6.23}$$

As a final remark, it is interesting to write down the spectral decomposition in frequencies of the covariant conservation law for the source energy momentum tensor $T_{\mu\nu}$. Using the expression (A4) for the de Sitter covariant derivative, the conservation equation $D^\nu T_{\mu\nu}=0$ splits into the two equations

$$-\partial_\eta T_{00}+\partial_j T_{0j}+\frac{1}{\eta}\tilde{T}=0 \tag{6.24}$$

$$-\partial_\eta T_{0i}+\partial_j T_{ji}+\frac{2}{\eta}T_{0i}=0. \tag{6.25}$$

Notice that these equations formally coincide with the gauge fixing conditions for $\chi_{\mu\nu}$ (4.16) and (4.17). Nevertheless, the geometrical meaning is not the same for both, because $T_{\mu\nu}$ is a tensor while $\chi_{\mu\nu}$ is a tensor density. Now, using the spectral decompositions in frequencies (6.6), (6.9) and (6.10), the conservation laws for the ω frequency component of the energy-momentum tensor takes the form

$$i\,\omega(\tilde{\mathcal{T}}(\omega,\vec{x})-\mathcal{T}_{ii}(\omega,\vec{x}))+\partial_j\mathcal{T}_{0j}(\omega,\vec{x})=0 \tag{6.26}$$

$$i\,\omega\mathcal{T}_{0i}(\omega,\vec{x})+\partial_j\mathcal{T}_{ji}(\omega,\vec{x})=0. \tag{6.27}$$

Notice that Eq. (6.27) has exactly the same form as in Minkowski space-time, while Eq. (6.26) is different because $\tilde{\mathcal{T}}(\omega,\vec{x})-\mathcal{T}_{ii}(\omega,\vec{x})\neq\mathcal{T}_{00}(\omega,\vec{x})$ [in fact there is no suitable way to define $\mathcal{T}_{00}(\omega,\vec{x})$]. Finally, using the conservation laws (6.26) and (6.27), one can check that the ω frequency component $\chi_{\mu\nu}^{(\omega)}$ of the gravitational field $\chi_{\mu\nu}$, given by Eqs. (6.15), (6.16) and (6.17), satisfy the gauge fixing conditions (4.16) and (4.17).

VII. CONCLUSIONS

We have shown that the production of gravitational radiation from sources moving in the 4D de Sitter background can be studied along the same lines as for Minkowski space-time. The maximal symmetry and the conformal flatness of the de Sitter space-time are found to be two key ingredients in order to achieve this goal. In addition, we have shown that—although the general equations for linear gravitational perturbations are rather cumbersome—choosing the gauge (3.21), the equations for all (physical and unphysical) polarizations of the graviton decouple, and amount to the equations for a de Sitter minimally coupled massless scalar field and a Minkowski massless field. In this respect, it is worth remarking that the minimally coupled massless scalar field behavior can be easily obtained, in the case of the physical polarizations, by imposing the traditional synchronous transverse traceless gauge conditions $D_\nu\psi_\mu^\nu=u_\nu\psi_\mu^\nu=0$. However, these two conditions cannot be simultaneously imposed in the presence of a source, and if one imposes only the condition $D_\nu\psi_\mu^\nu=0$ instead of Eq. (3.21), one is led to a much more difficult coupled graviton wave equation than Eq. (3.23), whose solutions contain spurious complications. The same thing happens for the residual gauge invariance allowed by the gauge condition $D_\nu\psi_\mu^\nu=0$. While the infinitesimal coordinate transformations preserving the gauge condition (3.21) are given by vector fields whose equations for the time and space components decouple, and give very simple mode solutions; the vector fields corresponding to the residual gauge invariance allowed by $D_\nu\psi_\mu^\nu=0$, satisfy a much more difficult coupled system of partial differential equations.

The main new results of this paper are given in Sec. VI. In that section we have shown that decomposing the energy-momentum tensor of a given generic source $T_\mu^\nu(\eta,\vec{x})$ in frequencies, by using a spectral transform dictated by the modes of the free gravitational waves in the curved background, we have very simple closed formulas relating the ω frequency component of the linearized gravitational field produced by the source with the transform $\mathcal{T}_{\mu\nu}(\omega,\vec{x})$ of the energy-momentum tensor. We also show that for localized sources, the produced gravitational field takes the form of free gravitational waves in de Sitter space-time being radiated away from the source. Thus, the generation of gravitational radiation by sources in de Sitter space-time resembles very closely to the same process in Minkowski space-time, the main difference being in the form of the energy-momentum frequency transform, which enter the formulas for the amplitudes of the radiated waves.

As a previous step we have also shown in Sec. V, how the graviton retarded Green's function in the Sitter space-time—needed to solve the graviton wave equation—can be easily obtained using QFT techniques. The most prominent feature of this retarded Green's function is that in addition to a delta function term in the retarded time, it also contains a term proportional to the Heaviside step function of the retarded time. This second term shows that the information about the sources in de Sitter space-time propagates not only at the speed of light but also at a lower speed.

In our opinion, it would be very interesting to apply the general formulas (6.15)–(6.17) that we have derived in this paper, to sources that could exist during the inflationary period of the universe. As a first example we have work in progress [12] concerning string sources, whose equations of motion in de Sitter space-time have been solved in the case of a ring ansatz [13].

ACKNOWLEDGMENTS

One of the authors (J.R.) would like to acknowledge the warm hospitality at DEMIRM (Observatoire de Paris), where this work was carried to completion, and the financial support of DIRECCION GENERAL DE ENSEÑANZA SUPERIOR E INVESTIGACION CIENTIFICA of spanish MEC, which made this research remain possible.

APPENDIX

In this appendix we collect a number of formulas for covariant derivatives, curvature tensors and d'Alembertians for de Sitter space-time, which are used in this paper. The four dimensional de Sitter metric in conformal coordinates $x^\mu = (\eta, \vec{x})$ reads

$$ds^2 = \gamma_{\mu\nu}\, dx^\mu dx^\nu = \frac{1}{H^2\eta^2}(-d\eta^2 + d\vec{x}^2) \tag{A1}$$

i.e.

$$\gamma_{\mu\nu} = \frac{1}{H^2\eta^2}\,\eta_{\mu\nu} \tag{A2}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$ is the Minkowski metric, and H is the Hubble constant.

Then the metric connection, can be written in these coordinates as

$$\Gamma^{(0)\nu}_{\mu\lambda} = -\frac{1}{\eta}(\delta^0_\mu\, \delta^\nu_\lambda + \delta^0_\lambda\, \delta^\nu_\mu + \delta^\nu_0\, \eta_{\mu\lambda}). \tag{A3}$$

Thus, the covariant derivatives for covariant and contravariant vectors read

$$D_\mu V_\kappa = \partial_\mu V_\kappa + \frac{1}{\eta}(\delta^0_\mu\, V_\kappa + \delta^0_\kappa\, V_\mu + \eta_{\mu\kappa}\, V_0) \tag{A4}$$

$$D_\mu V^\nu = \partial_\mu V^\nu - \frac{1}{\eta}(\delta^\nu_\mu\, V^0 + \delta^0_\mu\, V^\nu + \delta^\nu_0\, \eta_{\mu\lambda}\, V^\lambda). \tag{A5}$$

In particular we have

$$D_\lambda V^\lambda = \partial_\lambda V^\lambda - \frac{4}{\eta} V^0 \tag{A6}$$

and

$$D_\lambda \psi^\lambda_\mu = \partial_\lambda \psi^\lambda_\mu - \frac{4}{\eta}\psi^0_\mu + \frac{1}{\eta}\delta^0_\mu\, \psi. \tag{A7}$$

Since the de Sitter space-time is maximally symmetric, the Riemann and Ricci tensors take the form

$$R^{(0)}_{\mu\kappa\lambda\rho} = H^2(\gamma_{\mu\rho}\, \gamma_{\kappa\lambda} - \gamma_{\mu\lambda}\, \gamma_{\kappa\rho}) \tag{A8}$$

and

$$R^{(0)}_{\mu\kappa} = -3H^2\, \gamma_{\mu\kappa}. \tag{A9}$$

In addition to these tensors, we need the scalar, vector and tensor d'Alembertians, which can be computed using the expression (A3) for the metric connection. For a scalar field ϕ, we have the scalar d'Alembertian

$$\frac{1}{H^2\eta^2} D^\lambda D_\lambda\, \phi = \left(\Box + \frac{2}{\eta}\partial_\eta\right)\phi \tag{A10}$$

where $\Box$ is the Minkowski d'Alembertian

$$\Box \equiv -\partial^2_\eta + \vec{\nabla}^2. \tag{A11}$$

For a vector field ξ_μ, the vector d'Alembertian is

$$\frac{1}{H^2\eta^2} D^\lambda D_\lambda\, \xi_\mu = \Box\, \xi_\mu + \frac{2}{\eta}[\partial_\mu \xi_0 + \delta^0_\mu\, \eta^{\alpha\beta}\, \partial_\alpha \xi_\beta] + \frac{1}{\eta^2}[3\, \xi_\mu + 2\, \delta^0_\mu\, \xi_0]. \tag{A12}$$

Finally, for a tensor field ψ^ν_μ, a long but straightforward computation yields

$$\frac{1}{H^2\eta^2} D^\lambda D_\lambda\, \psi^\nu_\mu = \Box\, \psi^\nu_\mu + \frac{2}{\eta}[\partial_\eta \psi^\nu_\mu + \partial_\mu \psi^\nu_0 - \eta^{\nu\kappa}\, \partial_\kappa \psi^0_\mu] + \frac{2}{\eta}[\delta^0_\mu\, \eta^{\nu\alpha}\, \partial_\beta \psi^\beta_\alpha - \delta^\nu_0\, \partial_\beta \psi^\beta_\mu] + \frac{2}{\eta^2}[\psi^\nu_\mu + 2\, \delta^0_\mu\, \psi^\nu_0 + 2\, \delta^\nu_0\, \psi^0_\mu - \delta^0_\mu\, \delta^\nu_0\, \psi - \delta^\nu_\mu\, \psi^0_0]. \tag{A13}$$

[1] Ya. B. Zeldovich, *My Universe* (Harwood Academic, Chur, Switzerland, 1992), Chap. 6, and references therein.

[2] L. P. Grishchuk, Zh. Éksp. Teor. Fiz. **67**, 825 (1974) [Sov. Phys. JETP **40**, 409 (1975)]; L. P. Grishchuk and Y. V. Sidorov, Phys. Rev. D **42**, 3413 (1990); L. P. Grishchuk and M. Solokhin, *ibid.* **43**, 2566 (1991).

[3] L. H. Ford and L. Parker, Phys. Rev. D **16**, 1601 (1977).

[4] N. C. Tsamis and R. P. Woodard, Phys. Lett. B **292**, 269 (1992).

[5] B. Allen, in *Field Theory, Quantum Gravity and Strings II*, edited by H. J. de Vega and N. Sanchez, Lecture Notes in Physics Vol. 280 (Springer-Verlag, Berlin, 1986); B. Allen and M. Turyn, Nucl. Phys. **B292**, 813 (1987).

[6] N. C. Tsamis and R. P. Woodard, Commun. Math. Phys. **162**, 162 (1994).

[7] B. Allen and T. Jacobson, Commun. Math. Phys. **103**, 669 (1986).

[8] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[9] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1972).

[10] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, London, 1980).

[11] G. Börner and H. P. Dürr, Nuovo Cimento A **64**, 664 (1969).

[12] H. J. de Vega, J. Ramirez, and N. Sanchez, ‘‘Gravitational radiation from strings evolving in de Sitter space-time’’ (in preparation).

[13] H. J. de Vega, A. V. Mikhailov, and N. Sanchez, Teor. Mat. Fiz. **94**, 232 (1993); H. J. de Vega, A. L. Larsen, and N. Sanchez, Nucl. Phys. **B427**, 643 (1994).