Neutrino-nucleon interactions in magnetized neutron-star matter: The effects of parity violation

Phil Arras* and Dong Lai†
Center for Radiophysics and Space Research, Space Sciences Building, Cornell University, Ithaca, New York 14853
(Received 24 November 1998; published 22 July 1999)

We study neutrino-nucleon scattering and absorption in a dense, magnetized nuclear medium. These are the most important sources of neutrino opacity governing the cooling of a proto-neutron star in the first tens of seconds after its formation. Because the weak interaction is parity violating, the absorption and scattering cross sections depend asymmetrically on the directions of the neutrino momenta with respect to the magnetic field. We develop the moment formalism of neutrino transport in the presence of such asymmetric opacities and derive explicit expressions for the neutrino flux and other angular moments of the Boltzmann transport equation. For a given neutrino species, there is a drift flux of neutrinos along the magnetic field in addition to the usual diffusive flux. This drift flux depends on the deviation of the neutrino distribution function from thermal equilibrium. Hence, despite the fact that the neutrino cross sections are asymmetric throughout the star, the asymmetric neutrino flux can be generated only in the outer region of the proto-neutron star where the neutrino distribution deviates significantly from thermal equilibrium. The deviation from equilibrium is similarly altered by the asymmetric scattering and absorption, although its magnitude will still be quite small in the interior of the star. We clarify two reasons why previous studies have led to misleading results. First, inelasticity must be included in the phase space integrals in order to satisfy detail balance. Second, nucleon recoil must be included in order to find the leading order asymmetric cross sections correctly, even though it can be ignored to leading order to get the zero field opacities. In addition to the asymmetric absorption opacity arising from nucleon polarization, we also derive the contribution of the electron (or positron) ground state Landau level. For neutrinos of energy less than a few times the temperature, this is the dominant source of asymmetric opacity. Last, we discuss the implication of our result to the origin of pulsar kicks: in order to generate kick velocity of a few hundred $\mathrm{km\,s^{-1}}$ from asymmetric neutrino emission using the parity violation effect, the proto-neutron star must have a dipole magnetic field of at least $10^{15}-10^{16}$ G. [S0556-2821(99)02616-8]

PACS number(s): 97.80.Fk, 04.25.Dm, 04.40.Dg, 97.60.Jd

I. INTRODUCTION

A. Astrophysical motivation

Neutrinos play an essential role in core-collapse supernovas and the formation of neutron stars [1,2]. It is through neutrino emission that a hot proto-neutron star releases its gravitational binding energy and cools. The explosion itself also relies on the neutrinos for its success. The most important ingredient of neutrino transport in proto-neutron stars is the neutrino opacity in a dense nuclear medium. Much effort has been devoted to understanding various effects of neutrino-matter interactions at supra-nuclear density (e.g., [3–11] and references therein). Neutron stars, however, possess strong magnetic fields. While the present-day, dipolar magnetic fields of most radio pulsars lie in the range of $10^{12}-10^{13}$ G, it has been suggested that fields of 10^{15} G or larger can be generated by a dynamo process in proto-neutron stars [12]. Several recent observations [13–15] have lent support to the idea that soft gamma-ray repeaters and slowly spinning x-ray pulsars (''anomalous x-ray pulsars'') in supernova remnants are neutron stars endowed with superstrong magnetic fields $B \gtrsim 10^{14}$ G [16,17]. It is therefore necessary to understand how neutrino opacities are modified by the presence of a strong magnetic field. This is the subject of our paper.

A direct motivation of our study is to explore whether strong magnetic fields can induce asymmetric neutrino emission from proto-neutron stars to explain pulsar ''kicks.'' It has long been recognized that neutron stars have space velocities that are about an order of magnitude greater than their progenitors' (e.g., [18,19]). Recent studies of pulsar proper motion give $200-500\ \mathrm{km\,s^{-1}}$ as the mean 3D velocity of neutron stars at birth [20–23], with possibly a significant population having velocity of order or greater than $700\ \mathrm{km\,s^{-1}}$. Direct evidence for pulsars with velocities $\gtrsim 1000\ \mathrm{km\,s^{-1}}$ comes from observations of the bow shock produced by PSR B2224+65 in the interstellar medium [24] and studies of pulsar-supernova remnant associations [25]. A natural explanation for such high velocities is that supernova explosions are asymmetric, and provide kicks to nascent neutron stars. Support for supernova kicks has come from the detections of geodetic precession in the binary pulsar PSR 1913+16 [26,27] and the orbital plane precession in the PSR J0045-7319/B star binary and its fast orbital decay [28,29]. In addition, evolutionary studies of the neutron star binary population imply the existence of pulsar kicks [30–33].

Two classes of mechanisms for the *natal kicks* have been suggested. The first class relies on hydrodynamical instabilities in the collapsed stellar core [34–38] that lead to asymmetric matter ejection and/or asymmetric neutrino emission, but numerical simulations indicate that these instabilities are

*Also at Department of Physics, Cornell University. Email address: arras@spacenet.tn.cornell.edu

†Also at Department of Astronomy, Cornell University. Email address: dong@spacenet.tn.cornell.edu

not adequate to account for kick velocities $\gtrsim 100\ \mathrm{km\,s^{-1}}$ [36,39,40]. Global asymmetric perturbations in the presupernova cores are required in order to produce the observed kicks [39,41]. In this paper we are concerned with the second class of models in which the pulsar kicks arise from magnetic field induced asymmetry in neutrino emissions from proto-neutron stars. The fractional asymmetry α in the radiated neutrino energy required to generate a kick velocity $V_{\rm kick}$ is

$$\alpha = 0.028\left(\frac{M}{1.4M_\odot}\right)\left(\frac{3\times 10^{53}\ \mathrm{erg}}{E_{\rm tot}}\right)\left(\frac{V_{\rm kick}}{1000\ \ \mathrm{km\,s^{-1}}}\right), \tag{1.1}$$

where M is the mass of the neutron star and $E_{\rm tot}$ is the total neutrino energy radiated from the neutron star. Since 99% of the neutron star binding energy (a few times 10^{53} erg) is released in neutrinos, tapping the neutrino energy would appear to be an efficient means to kick the newly formed neutron star. Magnetic fields are naturally invoked to break the spherical symmetry in neutrino emission, but the actual mechanism is unclear. We first review previous work related to this subject.

B. Review of previous work

Beta decay in a strong magnetic field was first investigated in Refs. [42,43] (see also Refs. [44,45]). A number of authors have noted that parity violation in weak interactions may lead to asymmetric neutrino emission from proto-neutron stars [46–49]. Chugai [46] and Vilenkin [48] (see also Ref. [50]) considered neutrino-electron scattering and concluded that the effect is extremely small[1] (e.g., to obtain $V_{\rm kick}=300\ \mathrm{km\,s^{-1}}$ would require a magnetic field of at least 10^{16} G). However, neutrino-electron scattering is less important than neutrino-nucleon scattering in determining the characteristics of neutrino transport in proto-neutron stars (e.g., [51,52]). Similarly, Dorofeev et al. [47] considered neutrino emission by Urca processes in strong magnetic fields. However, as we shall see below (see Ref. [53]), the asymmetry in neutrino emission is cancelled by the asymmetry associated with neutrino absorption for young proto-neutron stars where the neutrinos are nearly in thermal equilibrium. The size of the asymmetric flux due to absorption and emission processes is then dependent on the *deviations* from thermal equilibrium at the neutrino photosphere.

Horowitz and Li [54] suggested that large asymmetries in the neutrino flux could result from the *cumulative* effect of multiple scatterings of neutrinos by nucleons which are slightly polarized by the magnetic field. In particular, they found that the size of the asymmetry was proportional to the optical depth of the star to neutrinos ($\tau\sim 10^4$). The result was that kick velocities of a few hundred $\mathrm{km\,s^{-1}}$ could be generated by field strengths of only a few times 10^{12} G. Initial neutrino cooling calculation [53] of a proto-neutron star in magnetic fields appeared to indicate that a dipole field of order 10^{14} G is needed to produce a kick velocity of $200\ \mathrm{km\,s^{-1}}$. The larger magnetic field required results from cancellations of the asymmetries associated with ν_μ, ν_τ and their antiparticles as well as the opposite signs of polarizations of neutrons and protons. A preliminary numerical study reported in Ref. [40] drew a similar conclusion although it was claimed that only 10^{13} G is needed to produce $200\ \mathrm{km\,s^{-1}}$.

As appealing as the cumulative effect may be, we now believe that it does not work in the bulk interior of the neutron star [55,56], and the conclusions reached in Refs. [40,53,54] are incorrect [57]. In spite of the fact that the scattering cross section is asymmetric with respect to the magnetic field for individual neutrinos, detailed balance requires that no asymmetric neutrino flux can arise in the stellar interior where neutrinos are in thermal equilibrium to a good approximation. Since this issue is somewhat subtle and counter-intuitive, we discuss it in detail in Sec. II where we also point out where previous studies went wrong.

A related, but different kick mechanism relies on the asymmetric magnetic field distribution in proto-neutron stars [58–60]. Since the cross section for ν_e ($\bar{\nu}_e$) absorption on neutrons (protons) depends on the local magnetic field strength due to the quantization of energy levels for the e^- (e^+) produced in the final state, the local neutrino fluxes emerged from different regions of the stellar surface are different. It was found [60] that to generate a kick velocity of $\sim 300\ \mathrm{km\,s^{-1}}$ using this mechanism alone would require that the difference in the field strengths at the two opposite poles of the star be at least 10^{16} G.

There have also been several interesting ideas on pulsar kicks which rely on nonstandard neutrino physics. It was suggested [61] that asymmetric ν_τ emission could result from the Mikheyev-Smirnov-Wolfenstein flavor transformation between ν_τ and ν_e inside a magnetized proto-neutron star because a magnetic field changes the resonance condition for the flavor transformation. Another similar idea [62] relies on both the neutrino mass and the neutrino magnetic moment to facilitate the flavor transformation. A more detailed analysis [63,64], however, indicates that even with favorable neutrino parameters (such as mass and magnetic moment) for neutrino oscillation, the induced pulsar kick is much smaller than previously estimated. We will not consider the issues related to nonstandard physics in this paper.

Finally, we mention that previous calculations of neutrino processes in magnetic fields have generally neglected nucleon recoils (e.g., [40,42,43,47,53,54]). Although this simplification is justified in most cases without a magnetic field, it is invalid in a magnetic field because the *asymmetric part* of the opacity depends sensitively on the phase space available in the scattering and absorption. This and other technical issues (such as Landau levels) will be addressed in our paper.

C. Plan of this paper

In this paper, we carry out a systematic study of neutrino-nucleon (ν,N) scattering and electron neutrino absorption

[1]Note that Chugai's estimate for the electron polarization in the relativistic and degenerate regime (the relevant physical regime) is incorrect. This error leads to an overestimate of the effect as compared to Vilenkin's result.

and emission ($\nu_e+n\rightleftharpoons p+e^-$ and $\overline{\nu}_e+p\rightleftharpoons n+e^+$) in a dense, magnetized nuclear medium. These are the most important sources of opacity for neutrino cooling of the proto-neutron star in the first tens of seconds after its formation, when most of the binding energy of the neutron star is radiated as neutrinos. Our study goes beyond merely calculating differential cross sections of the neutrinos in a magnetized medium in that we derive the expression for the neutrino flux and other angular moments from the Boltzmann equation. This is necessary in order to determine whether the effect of parity violation introduces any asymmetric ''drift flux'' in addition to the usual ''diffusive flux.'' Indeed, there are a number of subtleties in these derivations such that several previous studies have arrived at incorrect results (see Secs. I B and II). We show that, despite the fact that the scattering and absorption cross sections are asymmetric with respect to the magnetic field for individual neutrinos, there is no ''drift flux'' when the neutrinos are in thermal equilibrium; the drift flux is proportional to the *deviations* from equilibrium, which are small below the neutrino-matter decoupling layer. Hence, asymmetric neutrino emission can be generated only near the surface layer of the star.

In Sec. II we discuss a simplified calculation and point out its problems. This serves as an illustration of various issues that one must pay attention to in order to obtain the correct answers. We begin our formal theoretical development in Sec. III, where the relevant cross sections are defined starting from the Boltzmann transport equation. It is important to distinguish different cross sections (e.g., those related to scattering into the beam and scattering out of the beam in the Boltzmann equation) in order to satisfy the principle of detailed balance, which states that in complete thermal equilibrium the collisional term in the Boltzmann equation vanishes.

Section IV contains a detailed calculation of ν-N scattering in magnetic fields. Starting from the weak interaction Hamiltonian, we compute the scattering opacity, carefully including the effect of nuclear motion to lowest nonvanishing order. This opacity is then used to find the contribution to the angular moments of the transport equation. Explicit expressions are obtained for the outer layer of the neutron star where nucleons are nondegenerate and where parity-violating asymmetric flux can be generated. A technical complication arises from the effect of small inelasticity: Even in the regime where the nucleon recoil energy is much smaller than the neutrino energy and temperature, phase space considerations require that the inelasticity effect be included when deriving the asymmetric flux and other moments (the situation is similar to that found in the derivation of the Kompaneets equation in electron-photon scattering; see Ref. [65]).

In Sec. V we calculate the cross sections for the absorption of ν_e and $\overline{\nu}_e$ by nucleons. As in the scattering case, it is necessary to include nucleon recoil in the absorption calculation. Additional complications arise from the quantized Landau levels of the final state electrons (or positrons). In particular, the ground state Landau level of the electron introduces an effective electron ''polarization'' term in the asymmetric part of the opacity. For certain parameter regimes, this electron ''polarization'' term dominates the asymmetry in the absorption opacity. We also demonstrate explicitly that the Landau levels of protons have no effect on the absorption opacity since many levels are summed over for the situation of interest.

In Sec. VI we combine the results of Secs. IV and V to derive the angular moments of the Boltzmann transport equation. These moment equations are truncated at quadrupole order, since we expect that the contributions of the higher order terms to the asymmetric flux are smaller. As expected, our explicit expression for the neutrino flux contains the usual diffusive flux plus a drift flux along the magnetic field. This drift flux, however, depends on the deviation for the neutrino distribution from thermal equilibrium. Although strictly speaking our truncated moment equations break down near the neutron star surface, these equations are accurate below the neutrinosphere, and provide a reasonable physical description of the neutrino radiation field throughout the star. Finally in Sec. VII we use the moment equations to obtain an order-of-magnitude estimate of the asymmetric neutrino emission from the proto-neutron star due to the parity violating processes considered in this paper.

Throughout this paper, we treat nucleons as noninteracting particles. This is clearly a simplifying assumption. In reality, strong interaction correlations may significantly change the neutrino opacities (e.g., [10,11] and references therein). However, the goal of this paper is to consider whether there is any new effect associated with strong magnetic fields. For this purpose, it is certainly appropriate to focus on noninteracting nucleons, particularly since there are still large uncertainties in our understanding of matter at super-nuclear densities. Moreover, for certain nuclear potentials, the nuclear medium effects merely amount to giving the nucleon an effective mass, and therefore the result of this paper can be easily extended. For general nuclear interactions, it is likely that the qualitative conclusion reached in this paper will remain valid, although this issue is beyond the scope of this paper.

Unless noted otherwise, we shall use units in which $\hbar$, c and the Boltzmann constant k_B are unity.

II. SIMPLIFIED CALCULATION OF THE SCATTERING EFFECT AND ITS PROBLEMS

As indicated above, the effect of asymmetric neutrino-nucleon scattering is sufficiently subtle that several previous studies have led to a wrong conclusion. It is therefore instructive to consider a simplified treatment of the problem to understand how previous work went wrong. This section also serves as an illustration of the various issues that one must pay attention to in doing such a calculation (our systematic calculation is presented in Secs. III–VI).

A. Calculation

We shall follow the treatment as given in Ref. [53] (Refs. [40,54] used a Monte Carlo method for the neutrino transport, which is less transparent for our analysis). For the purpose of clarity, we consider the scattering of neutrinos by nondegenerate neutrons. Assuming the scattering is elastic (which is a good approximation since the neutrino energy k is much less than the neutron mass), one can easily obtain the matrix elements when the initial neutron has spin along the magnetic axis (the z axis):

$$|\mathcal{H}_{\mathbf{k}\uparrow\to\mathbf{k}'\uparrow}|^2 = 2\,G_F^2 c_V^2\left[\cos\frac{1}{2}(\theta'-\theta)+\lambda\cos\frac{1}{2}(\theta+\theta')\right]^2, \tag{2.1}$$

$$|\mathcal{H}_{\mathbf{k}\uparrow\to\mathbf{k}'\downarrow}|^2 = 2\,G_F^2 c_A^2\left(2\sin\frac{1}{2}\theta'\cos\frac{1}{2}\theta\right)^2, \tag{2.2}$$

where $\mathbf{k}$ ($\mathbf{k}'$) is the initial (final) neutrino momentum, θ (θ') is the angle between $\mathbf{k}$ ($\mathbf{k}'$) and the z axis (assuming azimuthal angle $\phi=0$), and G_F, c_V, c_A, λ are weak interaction constants as defined in Appendix A. The differential cross section, for the initial nucleon with spin along the z axis, is given by

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega'}\right)_{\mathbf{k}\uparrow\to\mathbf{k}'} &= \frac{k^2}{(2\pi)^2}\left[|\mathcal{H}_{\mathbf{k}\uparrow\to\mathbf{k}'\uparrow}|^2+|\mathcal{H}_{\mathbf{k}\uparrow\to\mathbf{k}'\downarrow}|^2\right]\\ &= \frac{k^2}{(2\pi)^2}G_F^2c_V^2[(1+3\lambda^2)+2\lambda(\lambda+1)\cos\theta\\ &\quad -2\lambda(\lambda-1)\cos\theta'+(1-\lambda^2)\cos(\theta-\theta')].\end{aligned} \tag{2.3}$$

One can similarly obtain $(d\sigma/d\Omega')_{\mathbf{k}\downarrow\to\mathbf{k}'}$ when the initial nucleon spin is anti-parallel to the z axis. For general nucleon spin polarization $P=\langle\sigma_z\rangle$, the differential cross section is given by

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega'}\right)_{\mathbf{k}\to\mathbf{k}'} &= \left(\frac{1+P}{2}\right)\left(\frac{d\sigma}{d\Omega'}\right)_{\mathbf{k}\uparrow\to\mathbf{k}'}+\left(\frac{1-P}{2}\right)\left(\frac{d\sigma}{d\Omega'}\right)_{\mathbf{k}\downarrow\to\mathbf{k}'}\\ &= \left(\frac{G_Fc_Vk}{2\pi}\right)^2(1+3\lambda^2)\left[1+\epsilon_{\rm in}\cos\theta+\epsilon_{\rm out}\cos\theta'\right.\\ &\quad\left.+\frac{1-\lambda^2}{1+3\lambda^2}\cos(\theta-\theta')\right],\end{aligned} \tag{2.4}$$

where

$$\epsilon_{\rm in}=2P\frac{\lambda(\lambda+1)}{(1+3\lambda^2)},\quad \epsilon_{\rm out}=-2P\frac{\lambda(\lambda-1)}{(1+3\lambda^2)}. \tag{2.5}$$

This clearly indicates that the scattering is asymmetric with respect to the magnetic field, a direct consequence of parity violation in weak interaction. A similar expression was derived in Ref. [54], although a different sign in $\epsilon_{\rm in}$ and $\epsilon_{\rm out}$ was given.

Next we study the consequence of the asymmetric cross section on the neutrino flux. The Boltzmann transport equation for the neutrino distribution function $f_\nu(\mathbf{k})$ can be written in the form

$$\begin{aligned}\frac{\partial f_\nu(\mathbf{k})}{\partial t}+\mathbf{\Omega}\cdot\nabla f_\nu(\mathbf{k}) &= \int d\Omega'\left(\frac{d\kappa}{d\Omega}\right)_{\mathbf{\Omega}'\to\mathbf{\Omega}} f_\nu(\mathbf{k}')[1-f_\nu(\mathbf{k})]\\ &\quad -\int d\Omega'\left(\frac{d\kappa}{d\Omega'}\right)_{\mathbf{\Omega}\to\mathbf{\Omega}'}\\ &\quad \times f_\nu(\mathbf{k})[1-f_\nu(\mathbf{k}')],\end{aligned} \tag{2.6}$$

where $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ are unit vectors along $\mathbf{k}$ and $\mathbf{k}'$, respectively, and the (elastic) differential cross section per unit volume can be written in the form

$$\begin{aligned}\left(\frac{d\kappa}{d\Omega'}\right)_{\mathbf{\Omega}\to\mathbf{\Omega}'} &= \frac{\kappa}{4\pi}[1+\epsilon_{\rm in}\mathbf{\Omega}\cdot\hat{\mathbf{B}}+\epsilon_{\rm out}\mathbf{\Omega}'\cdot\hat{\mathbf{B}}\\ &\quad +{\rm const}\times(\mathbf{\Omega}\cdot\mathbf{\Omega}')],\end{aligned} \tag{2.7}$$

with $\hat{\mathbf{B}}$ the unit vector along the magnetic field. Note that in Eq. (2.6) we have neglected neutrino absorption for simplicity. The first order moment of the transport equation is obtained by multiplying Eq. (2.6) by $\mathbf{\Omega}$ and then integrating over $d\Omega$. The specific neutrino flux is then given by[2]

$$\mathbf{F}_\nu = -\frac{c}{3\kappa}\nabla U_\nu+\frac{1}{3}(\epsilon_{\rm out}-\epsilon_{\rm in})cU_\nu(1-f_\nu)\hat{\mathbf{B}}, \tag{2.8}$$

where U_ν is the specific neutrino energy density. According to Eq. (2.8), the neutrino flux consists of the usual diffusive flux, $\mathbf{F}_{\rm diff}\propto\nabla U_\nu$, and a ''drift'' flux $\mathbf{F}_{\rm drift}$ along the magnetic field. The drift flux induces asymmetric neutrino transport. One can easily see that the ratio $F_{\rm diff}/F_{\rm drift}$ is of order $(\epsilon_{\rm out}-\epsilon_{\rm in})\tau$, where $\tau\sim\kappa R$ is the optical depth of the star (R is the stellar radius); i.e., the asymmetry increases with τ. This is the origin of the cumulative effect discussed in Refs. [40,53,54].

B. Problems

The calculation presented above, while physically motivated, is actually incorrect. There are two problems: first, the asymmetry terms (those proportional to $\epsilon_{\rm in}$ and $\epsilon_{\rm out}$) in Eq. (2.4) are incomplete. Even in the regime where the elastic approximation is highly accurate from the energy point of view, small inelasticity will affect the asymmetric part of the cross section. This comes about because the asymmetric terms depend in a subtle way on the available phase space of the scattering. Indeed, our full calculation presented in Sec. IV reveals additional terms in the expressions of $\epsilon_{\rm in}$ and $\epsilon_{\rm out}$. Moreover, to obtain the correct expression for the asymmetric neutrino flux, the elastic cross section is inadequate; one must incorporate the full inelastic effect in the Boltzmann equation (see Sec. IV). A similar comment can be made for neutrino-nucleon absorption, where one must incorporate the

[2] An overall factor of $(1-{\rm const}/3)^{-1}$ has been dropped in this equation.

recoil motion of the nucleon as well as the Landau levels for the electron in order to obtain the correct cross section (see Sec. V).

Second, and more importantly, Eq. (2.6) is incorrect; therefore Eq. (2.8) is also incorrect and there is no drift flux proportional to U_ν. The problem with Eq. (2.6) can easily be seen by considering detailed balance (e.g., Ref. [66]): The right-hand side (RHS) of Eq. (2.6) must vanish when neutrinos are in thermal equilibrium with the matter. Substituting $f_\nu(\mathbf{k})$ and $f_\nu(\mathbf{k}')$ by the equilibrium distribution $f_\nu^{(0)}(k)$ and using Eq. (2.7), we find the RHS of Eq. (2.6) to be $\kappa f_\nu^{(0)}(1-f_\nu^{(0)})(\epsilon_{\rm out}-\epsilon_{\rm in})\mathbf{\Omega}\cdot\hat{\mathbf{B}}$. It is exactly this violation of detailed balance that gives rise to the drift flux term in Eq. (2.8). It is also clear that any physical drift flux must depend on the *deviation* from the equilibrium distribution.

Equation (2.6) is the starting point of almost all astrophysical radiative transport studies (e.g., Ref. [67]). However, it is invalid in the presence of asymmetric scattering. A crucial (but incorrect) assumption implicit in Eq. (2.6) is that the cross section for scattering into the beam (propagating along $\mathbf{\Omega}$), $(d\kappa/d\Omega)_{\mathbf{\Omega}'\to\mathbf{\Omega}}$, is related to that for scattering into the beam, $(d\kappa/d\Omega')_{\mathbf{\Omega}\to\mathbf{\Omega}'}$, by merely switching $\mathbf{\Omega}$ and $\mathbf{\Omega}'$. In reality, however, the two cross sections have slightly different forms such that detailed balance is satisfied in equilibrium [see Eq. (3.9)]. In other words, although the first (second) term in Eq. (2.6) represents a good approximation to the actual probability of scattering into (out of) the beam, the error in their difference is significant. It will become clear in our study presented the following sections that to properly take into account the detailed balance constraint, one must incorporate inelasticity—no matter how small—into the Boltzmann equation when deriving the asymmetric neutrino flux.

III. GENERAL FORMALISM

In this section, we set up the general framework to study neutrino transport in magnetic fields. As Sec. II shows, it is important to properly define the relevant cross sections which enter the transport equation. The actual calculations of the cross sections are given in Secs. IV and V.

The Boltzmann equation for neutrino transport is written in the form

$$\frac{\partial f_\nu(\mathbf{k})}{\partial t}+\mathbf{\Omega}\cdot\nabla f_\nu(\mathbf{k})=\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm sc}+\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm abs} \tag{3.1}$$

where $\mathbf{k}=k\mathbf{\Omega}$ is the neutrino momentum, both scattering and absorption collisions terms are included on the right-hand side of the equation and we have suppressed the position and time dependence in the neutrino distribution function f_ν.

A. Scattering term

The collision term due to neutrino-nucleon scattering can be written as (e.g., Ref. [68])

$$\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm sc}=\sum_{ss'}\int\frac{d^3k'}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}(2\pi)^4\times\delta^4(P+K-P'-K')|M_{ss'}(\mathbf{\Omega},\mathbf{\Omega}')|^2\times[(1-f_\nu)(1-f_N)f_N'f_\nu'-f_\nu f_N(1-f_N')(1-f_\nu')], \tag{3.2}$$

where $f_\nu=f_\nu(\mathbf{k})$ and $f_\nu'=f_\nu(\mathbf{k}')$ are the initial and final state neutrino distribution functions, $f_N=f_N(E)$ and $f_N'=f_N(E')$ are the initial and final state nucleon distribution functions, P (P') and K (K') are the initial (final) state nucleon and neutrino four vectors, respectively, and $s,s'=\pm1$ specify the initial and final nucleon spins. Time-reversal symmetry has been used to relate the matrix element for scattering in and out of the beam. Note that in the case of neutrino-proton scattering, we neglect the Landau levels of proton, and therefore the proton momentum is a well-defined quantity. This is justified because many Landau levels are occupied for the conditions in a proto-neutron star, and the change in the available phase space due to the Landau levels is negligible. Nucleons are always in thermal equilibrium, and the nucleon distribution function is given by

$$f_N(E)=\frac{1}{\exp[(E-\mu_N)/T]+1}, \tag{3.3}$$

where μ_N is the nucleon chemical potential (excluding rest mass). As the neutrinos exchange energy with matter only through the weak interactions, their distribution can deviate from equilibrium, especially in the outer layer of the star.

The scattering rate can be rewritten in a more conventional form as follows. Define the differential cross section (per unit volume) to be

$$\frac{d\Gamma}{dk'd\Omega'}=\frac{k'^2}{(2\pi)^3}\sum_{ss'}|M_{ss'}(\mathbf{\Omega},\mathbf{\Omega}')|^2S_{ss'}(q_0,q), \tag{3.4}$$

where the ''nucleon response function,'' $S_{ss'}(q_0,q)$, is

$$S_{ss'}(q_0,q)=\int\frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}(2\pi)^4\times\delta^4(P+K-P'-K')f_N(1-f_N'). \tag{3.5}$$

Here we have defined the energy transfer q_0 and the momentum transfer q via

$$q_0=k-k',\quad q=|\mathbf{k}-\mathbf{k}'|=(k^2+k'^2-2kk'\mathbf{\Omega}\cdot\mathbf{\Omega}')^{1/2}. \tag{3.6}$$

For nucleons in thermal equilibrium, energy conservation gives

$$\frac{(1-f_N)f_N'}{f_N(1-f_N')}=\exp[(E-E')/T]=\exp[-(k-k')/T]=\exp(-q_0/T). \tag{3.7}$$

Using this expression to relate the scattering into and out of the beam and plugging in the differential cross section, the scattering rate becomes

$$\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm sc}=\int_0^\infty dk'\int d\Omega'\frac{d\Gamma}{dk'd\Omega'}\times[e^{-q_0/T}(1-f_\nu)f'_\nu-f_\nu(1-f'_\nu)]. \quad (3.8)$$

Note that one can also explicitly define a differential cross section, $[d\Gamma/(dk'd\Omega')]_+$, for scattering into the beam by writing

$$\left[\frac{\partial f(\mathbf{k})}{\partial t}\right]_{\rm sc}=\int_0^\infty dk'\int d\Omega'\left[\left(\frac{d\Gamma}{dk'd\Omega}\right)_+\times(1-f_\nu)f'_\nu-\left(\frac{d\Gamma}{dk'd\Omega'}\right)f_\nu(1-f'_\nu)\right]. \quad (3.9)$$

Clearly we have

$$\left(\frac{d\Gamma}{dk'd\Omega}\right)_+=e^{-q_0/T}\left(\frac{d\Gamma}{dk'd\Omega'}\right). \quad (3.10)$$

All the microphysics is now contained in $d\Gamma/(dk'd\Omega')$. It depends only on k, k', and $\mathbf{\Omega}\cdot\mathbf{\Omega}'$, or equivalently, k, q_0 and q, as can be seen from Eq. (3.5). Note that if one sets $q_0/T=0$, then the neutrino Fermi blocking terms proportional to $f_\nu(\mathbf{k})f_\nu(\mathbf{k}')$ cancel (e.g., Ref. [69]). Hence, the neutrino degeneracy does not enter the scattering rate if the elastic limit is taken in the phase space integrals.

It is instructive to compare Eq. (3.9) with the (wrong) equation (2.6). In general, the cross section for scattering into the beam has a different form as that for scattering out of the beam. This difference, even though numerically small, is essential for maintaining detailed balance in thermal equilibrium (see Sec. III C below). Also, as we show in the next few sections, one cannot trivially take the elastic limit in Eq. (3.10), because this will lead to zero drift flux even when the neutrino distribution deviates from thermal equilibrium.

B. Absorption term

The collision term in Eq. (3.1) for absorption and emission is

$$\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm abs}=\int d\Pi_e d\Pi_n d\Pi_p\, W_{if}^{(\rm abs)}[f_pf_e(1-f_n)(1-f_\nu)-(1-f_p)(1-f_e)f_\nu f_n] \quad (3.11)$$

where $W_{if}^{(\rm abs)}$ is the transition rate (S matrix squared divided by time) for absorption, and we have used time-reversal invariance of the weak Hamiltonian to relate the S matrix for each direction. The notation $d\Pi$ denotes sum of states (including spins). Note that since we will include Landau levels for electrons and protons, $d\Pi_{e,p}$ is not equal to $d^3p_{e,p}/(2\pi)^3$ (see Appendix C). The components of the transverse momentum (perpendicular to the magnetic field) are not conserved, although we still have conservation of the z momentum and energy conservation,

$$k+E_n+Q=E_e+E_p, \quad (3.12)$$

where Q is the mass difference between neutron and proton (recall that E_n and E_p do not include rest mass). Since the electron, proton, and neutron are in thermal equilibrium with Fermi-Dirac distributions, we have the equality

$$\frac{f_pf_e(1-f_n)}{(1-f_p)(1-f_e)f_n}=\exp\left[-\left(\frac{k-\mu_\nu}{T}\right)\right], \quad (3.13)$$

where we have defined the neutrino chemical potential

$$\mu_\nu\equiv\mu_e+\mu_p-\mu_n-Q. \quad (3.14)$$

Equation (3.11) then takes on the standard form

$$\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm abs}=-\kappa^{*(\rm abs)}\delta f_\nu, \quad (3.15)$$

where $\delta f_\nu=f_\nu(\mathbf{k})-f_\nu^{(0)}(k)$ measures the deviation of neutrino distribution from thermal equilibrium (see below). Here

$$\kappa^{*(\rm abs)}=\kappa^{(\rm abs)}\left[1+\exp\left(\frac{\mu_\nu-k}{T}\right)\right], \quad (3.16)$$

and $\kappa^{(\rm abs)}$ is the absorption opacity:

$$\kappa^{(\rm abs)}=\int d\Pi_e d\Pi_n d\Pi_p\, W_{if}^{(\rm abs)}(1-f_p)(1-f_e)f_n. \quad (3.17)$$

The factor $[1+e^{(\mu_\nu-k)/T}]$ in $\kappa^{*(\rm abs)}$ takes into account the effect of stimulated absorption (e.g., Ref. [70]).

C. Detailed balance

In thermal equilibrium, the neutrino has the Fermi-Dirac distribution function

$$f_\nu=f_\nu^{(0)}(k)=\frac{1}{\exp[(k-\mu_\nu)/T]+1}. \quad (3.18)$$

We then find

$$\frac{f_\nu^{(0)}(1-f_\nu'^{(0)})}{(1-f_\nu^{(0)})f_\nu'^{(0)}}=\exp[(k'-k)/T]=\exp(-q_0/T), \quad (3.19)$$

so that the scattering rate in Eq. (3.8) is zero, as required by detailed balance. Similarly, for $f_\nu=f_\nu^{(0)}$, the absorption rate in Eq. (3.15) vanishes. Therefore the only nonzero contribution to $(\partial f_\nu/\partial t)_{\rm sc}$ and $(\partial f_\nu/\partial t)_{\rm abs}$ must be proportional to the deviation of neutrino distribution from thermal equilibrium. This implies that there is no drift flux along the magnetic field proportional to $f_\nu^{(0)}$ (see Sec. II A).

As noted before (Sec. III A), when one writes the scattering rate in the form of Eq. (3.9), it is essential to distinguish $(d\Gamma/dk'd\Omega)_+$ from $(d\Gamma/dk'd\Omega')$ in order to satisfy detailed balance.

D. Deviation from thermal equilibrium

In order to calculate the size of the asymmetric flux, we must consider the deviation of neutrino distribution from thermal equilibrium:

$$\delta f_\nu(\mathbf{k}) = f_\nu(\mathbf{k}) - f_\nu^{(0)}(k). \tag{3.20}$$

For ν_e and $\overline{\nu_e}$, the neutrino-matter energy exchange is mediated primarily by absorption and emission via the URCA processes (and to a lesser extent by electron-neutrino scattering), while neutrino transport is affected by both absorption and ν-N scattering. The result is that the decoupling sphere of electron type neutrinos lies only slightly deeper than the neutrinosphere and there is only a small region over which the cumulative asymmetry can develop. For μ and τ neutrinos, the transport opacity is primarily from ν-N scattering, while energy exchange is due to inelastic ν-e^- scattering. As a consequence, the decoupling layer is much deeper than the neutrinosphere and the asymmetric flux has a large optical depth over which to develop. Unfortunately, as we will show, the flux asymmetry due to the μ and τ neutrinos is cancelled by that from the corresponding antineutrinos.

For the purpose of deriving the moment equations of neutrino transport, we shall expand $\delta f_\nu(\mathbf{k})$ in spherical harmonics up to quadrupole order as

$$\delta f_\nu(\mathbf{k}) \equiv g(k) + 3\mathbf{\Omega}\cdot\mathbf{h}(k) + \frac{10}{3} I_{ij}(k)\mathcal{P}_{ij}(\mathbf{\Omega}) + \cdots \tag{3.21}$$

where

$$\mathcal{P}_{ij} = \frac{1}{2}(3\Omega_i\Omega_j - \delta_{ij}) \tag{3.22}$$

(the components of $\mathcal{P}_{ij}$ can be explicitly expressed solely in terms of the quadrupole spherical harmonics Y_{2m}). In Eq. (3.21), $g(k)$ is the spherically symmetric deviation from equilibrium, $\mathbf{h}(k)$ represents the dipole deviation which leads to the flux, and $I_{ij}(k)$ is the tensor describing the pressure asymmetry. Since $\mathcal{P}_{ij}$ is symmetric, we can choose I_{ij} to be symmetric, leaving six independent elements. Moreover, we shall choose I_{ij} to be traceless (the nonzero trace can always be incorporated into g_ν).

To be explicit about the physical meaning of each component of δf_ν, one can relate g, $\mathbf{h}$, and I_{ij} to more commonly used quantities. The energy density per unit energy interval is

$$U_k = \int \frac{k^2 d\Omega}{(2\pi)^3} k f_\nu = \frac{4\pi k^3}{(2\pi)^3}[f_\nu^{(0)}(k) + g(k)]. \tag{3.23}$$

The energy flux per unit energy interval is

$$\mathbf{F}_k = \int \frac{k^2 d\Omega}{(2\pi)^3} k\mathbf{\Omega} f_\nu = \frac{4\pi k^3}{(2\pi)^3}\mathbf{h}(k). \tag{3.24}$$

Using the identity

$$\int \frac{d\Omega}{4\pi}\Omega_i\Omega_j\mathcal{P}_{kl} = -\frac{1}{15}\delta_{ij}\delta_{kl} + \frac{1}{10}(\delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}) \tag{3.25}$$

and the fact that I_{ij} is traceless, the pressure tensor per unit energy interval is

$$\begin{aligned}[\mathbf{P}_k]_{ij} &= \int \frac{k^2 d\Omega}{(2\pi)^3} k\Omega_i\Omega_j f_\nu \\ &= \frac{4\pi k^3}{(2\pi)^3}\left[\frac{1}{3}\delta_{ij}(f_\nu^{(0)}(k) + g(k)) + \frac{2}{3}I_{ij}(k)\right] \\ &= \frac{1}{3}\delta_{ij}U_{\mathbf{k}} + \frac{2}{3}\frac{4\pi k^3}{(2\pi)^3}I_{ij}(k).\end{aligned} \tag{3.26}$$

Thus I_{ij} is the anisotropic portion of the pressure tensor.

Note that we have truncated our expansion at the quadrupole order angular dependence, since each successive term will be smaller than the previous by a factor of [71] $\sim\tau^{-1}$ (where τ is the optical depth). It will be shown that $h \propto f_\nu^{(0)}\tau^{-1}$, and g and I_{ij} both scale as $f_\nu^{(0)}\tau^{-2}$ (for $B=0$). The lth spherical harmonic would have coefficients which scale as $f_\nu^{(0)}\tau^{-l}$.

IV. NEUTRINO-NUCLEON SCATTERING

In this section, we calculate the differential cross section for ν-N scattering in magnetic fields for general conditions of nucleons (non-relativistic but arbitrary degeneracy). We also obtain explicit expressions for the angular moments of the scattering term of the Boltzmann equation. These moments are then used in (Sec. VI) to obtain the neutrino flux, as well as the spherical and quadrupole deviations from thermodynamic equilibrium, on which the asymmetric flux depends.

We approximate the nucleons as nonrelativistic particles with energy (excluding rest mass)

$$E(\mathbf{p},s) = \frac{p^2}{2m} - s\mu_B B, \tag{4.1}$$

where B is the magnetic field strength, $s=\pm1$ is the nucleon spin projection along the z axis (in the direction $\hat{\mathbf{B}}$), and $\mu_B = ge\hbar/(2mc)$ is the nucleon magnetic moment ($g_n = -1.913$ for the neutron and $g_p = 2.793$ for the proton). In this section, we shall omit the label n or p whenever possible, denoting final-state quantities by a prime. Also, the quantization of the proton energy levels and nucleon-nucleon interactions are neglected (see Sec. II A), although we shall include proton Landau levels in our calculation of the absorption opacity (Sec. V).

A. Differential cross section

The differential cross section, defined in Eq. (3.4), can be evaluated analytically to linear order in B for general conditions of nucleons. Since $\mu_B B = 3.15\times 10^{-4} g B_{14}$ MeV (where B_{14} is the field strength in units of 10^{14} G) is much smaller than the temperature or nucleon Fermi energy in the proto-neutron star, an expansion in the lowest nonvanishing power of B is an excellent approximation.

The matrix element, $|M_{ss'}(\mathbf{\Omega},\mathbf{\Omega}')|^2$, for the case in which both the initial and final nucleon states are polarized has been derived in Appendix A with the result [Eq. (A11)]

$$|M_{ss'}(\mathbf{\Omega},\mathbf{\Omega}')|^2 = \frac{1}{2} G_F^2 c_V^2 \{(1+3\lambda^2) + (1-\lambda^2)\mathbf{\Omega}\cdot\mathbf{\Omega}' + 2\lambda(\lambda+1)(s\mathbf{\Omega}+s'\mathbf{\Omega}')\cdot\hat{\mathbf{B}} - 2\lambda(\lambda-1) \times (s\mathbf{\Omega}'+s'\mathbf{\Omega})\cdot\hat{\mathbf{B}} + ss'[(1-\lambda^2) \times (1+\mathbf{\Omega}\cdot\mathbf{\Omega}') + 4\lambda^2 \mathbf{\Omega}\cdot\hat{\mathbf{B}}\mathbf{\Omega}'\cdot\hat{\mathbf{B}}]\} \tag{4.2}$$

where G_F, c_V, and $\lambda = c_A/c_V$ are the weak interaction constants defined in Appendix A. The nucleon response function, defined in Eq. (3.5), has been calculated in Appendix B. It can be written as $S_{ss'} = S_0 + \delta S_{ss'}$, where S_0 is the spin-independent $B=0$ result and $\delta S_{ss'}$ is the correction linear in B. Combining the expressions for $|M_{ss}(\mathbf{\Omega},\mathbf{\Omega}')|^2$ and $S_{ss'}$ into Eq. (3.4), we find

$$\frac{d\Gamma}{dk'\,d\Omega'} = A_0(k,k',\mu') + \delta A_+(k,k',\mu')\mathbf{\Omega}\cdot\hat{\mathbf{B}} + \delta A_-(k,k',\mu')\mathbf{\Omega}'\cdot\hat{\mathbf{B}}, \tag{4.3}$$

where $\mu' = \mathbf{\Omega}\cdot\mathbf{\Omega}'$ (not to be confused with the nucleon magnetic moment, μ_B, or the nucleon chemical potential, μ_N). The first term in Eq. (4.3) is the $B=0$ result:

$$A_0(k,k',\mu') = \frac{k'^2}{(2\pi)^3} \sum_{s,s'} |M_{ss'}(\mathbf{\Omega},\mathbf{\Omega}')|^2 S_0(q_0,q) = \frac{k'^2}{(2\pi)^3} 2 G_F^2 c_V^2 [(1+3\lambda^2) + (1-\lambda^2)\mu'] S_0(q_0,q), \tag{4.4}$$

with

$$S_0(q_0,q) = \frac{m^2 T}{2\pi q} \frac{1}{1-e^{-z}} \ln\left[\frac{1+\exp(-x_0)}{1+\exp(-x_0-z)}\right], \tag{4.5}$$

and we have defined

$$x_0 = \frac{(q_0 - q^2/2m)^2}{4T(q^2/2m)} - \frac{\mu_N}{T} \quad \text{and} \quad z = \frac{q_0}{T}. \tag{4.6}$$

The second and third terms in Eq. (4.3) correspond to the corrections arising from nonzero B:

$$\delta A_+ \mathbf{\Omega}\cdot\hat{\mathbf{B}} + \delta A_- \mathbf{\Omega}'\cdot\hat{\mathbf{B}} = \frac{k'^2}{(2\pi)^3} \sum_{s,s'} |M_{ss'}(\mathbf{\Omega},\mathbf{\Omega}')|^2 \delta S_{ss'}(q_0,q), \tag{4.7}$$

with the coefficients

$$\delta A_\pm(k,k',\mu') = \frac{k'^2}{(2\pi)^3} \frac{2 G_F^2 c_V^2 m^2 \lambda \mu_B B}{\pi q} \times \frac{1}{[\exp(x_0)+1][1+\exp(-x_0-z)]} \times \left(1 \pm \lambda \frac{2 m q_0}{q^2}\right). \tag{4.8}$$

The reason for writing the cross section in the form of Eq. (4.3) is that the angular dependence needed to find the moment equations (see Sec. IV C) is now manifest. Note that the cross section in Eq. (4.3) exhibits parity violation. If the parity operation is taken, the vectors $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ reverse sign and the pseudovector $\hat{\mathbf{B}}$ keeps the same sign so that the cross section does not retain the same form. Also note that the cross section for scattering from the state $\mathbf{\Omega}$ to the state $\mathbf{\Omega}'$ does not have the same numerical value as the reverse process. However, this does not mean that time reversal invariance is violated. The inequality arises from averaging the matrix element over the nucleon distribution functions. Indeed, the matrix element in Eq. (4.2) can be explicitly shown to satisfy time reversal invariance by simultaneously interchanging all initial and final state labels.

B. Differential cross section: Nondegenerate nucleon limit

Even after expanding the cross section in Eqs. (4.4) and (4.8) for small magnetic fields, the expressions are still quite difficult to evaluate in general. However, as discussed in Sec. III, asymmetric drift flux can develop only when the neutrino distribution deviates from thermal equilibrium (i.e., above the decoupling sphere). This occurs in the regime where nucleons are nondegenerate (at density $\rho \sim 10^{12} - 10^{13}\ \mathrm{g\,cm^{-3}}$). In this subsection we derive simplified expressions of A_0 and $\delta A_\pm$ which will be useful for obtaining the angular moments of the scattering term (Secs. IV C and IV D) and neutrino flux.

For nondegenerate nucleons, the characteristic neutrino energy transfer in each scattering is of order $q_0 \sim k(T/m)^{1/2} \ll k$. The cross section peaks sharply around $k'=k$, and we can evaluate A_0, $\delta A_\pm$ in a series in the small parameter $(T/m)^{1/2}$. Define the dimensionless quantities

$$\epsilon = [4(1-\mu')T/m]^{1/2}, \quad u = \frac{k'-k}{\epsilon k}, \tag{4.9}$$

so that the range of u, the dimensionless neutrino energy, is from $-1/\epsilon$ to ∞. Using the expansion of the nucleon response function derived in Appendix B, we have, to linear order in ϵ,

$$A_0 \simeq \left(\frac{G_F c_V k}{2\pi}\right)^2 [1+3\lambda^2+(1-\lambda^2)\mu'] \frac{n}{k\epsilon\pi^{1/2}} \times e^{-u^2}\left[1+\frac{3}{2}\epsilon u+\epsilon u^3-2\frac{k(1-\mu')}{\epsilon m}u\right] \quad (4.10)$$

and

$$\delta A_\pm \simeq \left(\frac{G_F c_V k}{2\pi}\right)^2 \frac{2\mu_B B}{T}\lambda \frac{n}{k\epsilon\pi^{1/2}} \times e^{-u^2}\left[1+\frac{3}{2}\epsilon u+\epsilon u^3-2\frac{k(1-\mu')}{\epsilon m}u \mp \frac{\lambda\epsilon m}{k(1-\mu')}\left(u+\frac{1}{2}\epsilon u^2+\epsilon u^4-2\frac{k(1-\mu')}{\epsilon m}u^2\right)\right]. \quad (4.11)$$

In deriving these expressions we have used the $B=0$ equation

$$\exp\left(\frac{\mu_N}{T}\right) = n\left(\frac{2\pi^3}{m^3T^3}\right)^{1/2} \quad (4.12)$$

to relate the nucleon chemical potential μ_N to its number density n (the corrections due to finite B are of order B^2). These expansions of A_0, $\delta A_\pm$ are valid under the conditions (see Appendix B for details) $T \ll m$, $k \ll (mT)^{1/2}$, $\mu_B B \ll T$ and $k \gtrsim k_{min} = \mu_B B(m/T)^{1/2} \simeq 10^{-2}|g|B_{14}T^{-1/2}$ MeV. These conditions are satisfied for the conditions of interest in our study.[3]

Note that, dimensionally, $\delta A_\pm$ is smaller than A_0 by a factor of order $\mu_B B/T$, but also note that the quantity $\epsilon m/k$ [which appears on the second line of Eq. (4.11)] is of order $\sqrt{mT}/k \sim \sqrt{m/T}$, which can be quite large. This point will be important when we consider the size of the neutrino drift flux.

C. Moments of the scattering rate

To derive the expression for the neutrino flux, one needs to take the angular moments of the Boltzmann transport equation (i.e., multiply the equation by some power of $\mathbf{\Omega}$ and then integrate over $d\Omega$). In this subsection we derive the general expressions for the moments of the scattering rate in the Boltzmann equation. In the next subsection we shall evaluate these expressions explicitly for the regime when the nucleons are nondegenerate and the scattering is approximately elastic.

[3]For $k \lesssim k_{min}$, the expansions leading to Eqs. (4.10) and (4.11) are no longer valid, so that different approximations must be made (the $k \to 0$ limit). Since this is a relatively small range of neutrino energy, we ignore this complication here.

We first write the scattering rate [Eq. (3.8)] in terms of $\delta f_\nu \equiv f_\nu - f_\nu^{(0)}$:

$$\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm sc} = \int_0^\infty dk' \int d\Omega' \frac{d\Gamma}{dk'd\Omega'}[C(k,k')\delta f'_\nu + D(k,k')\delta f_\nu + E(k,k')\delta f'_\nu \delta f_\nu], \quad (4.13)$$

where the C, D, E coefficients, and their $q_0/T \ll 1$ expansions, are

$$C(k,k') = e^{-q_0/T}(1-f_\nu^{(0)}) + f_\nu^{(0)} \simeq 1 + \left(-\frac{q_0}{T}+\frac{q_0^2}{2T^2}\right)(1-f_\nu^{(0)}), \quad (4.14)$$

$$D(k,k') = -[e^{-q_0/T}f_\nu^{(0)\prime} + 1 - f_\nu^{(0)\prime}] \simeq -1 + \frac{q_0}{T}f_\nu^{(0)} - \frac{q_0^2}{2T^2}\left(f_\nu^{(0)} + 2T\frac{\partial f_\nu^{(0)}}{\partial k}\right), \quad (4.15)$$

$$E(k,k') = 1 - e^{-q_0/T} \simeq \frac{q_0}{T} - \frac{q_0^2}{2T^2}. \quad (4.16)$$

The nonlinear terms $E(k,k')\delta f_\nu \delta f'_\nu$ in Eq. (4.13) will be dropped since we consider the regime where the deviation of f_ν from thermal equilibrium is relatively small (the regime where the moment formalism is valid); for the electron neutrinos this requires $\rho \gtrsim 10^{11-12}$ g cm^{-3} and $T \gtrsim 3$ MeV, and for the mu and tau neutrinos the required temperature and density are slightly larger (e.g., [72]).

Plugging Eqs. (3.21) and (4.3) into Eq. (4.13) and then performing the azimuthal integrals using the indentities

$$\int_0^{2\pi}\frac{d\phi'}{2\pi}\Omega'_i = \mu'\Omega_i$$

$$\int_0^{2\pi}\frac{d\phi'}{2\pi}\Omega'_i\Omega'_j = P_2(\mu')\Omega_i\Omega_j + \frac{1}{3}[1-P_2(\mu')]\delta_{ij},$$

$$\int_0^{2\pi}\frac{d\phi'}{2\pi}\mathcal{P}_{ij}(\mathbf{\Omega}') = P_2(\mu')\mathcal{P}_{ij}(\mathbf{\Omega})$$

$$\int_0^{2\pi}\frac{d\phi'}{2\pi}\Omega'_i\mathcal{P}_{jk}(\mathbf{\Omega}') = \frac{3}{5}\mathcal{P}_{ijk}(\mathbf{\Omega}) - \frac{1}{5}\mu'\Omega_i\delta_{jk} + \frac{3}{10}\mu'[\Omega_j\delta_{ki}+\Omega_k\delta_{ij}], \quad (4.17)$$

where the three-index tensor equivalent to the Y_{3m} is defined by $\mathcal{P}_{ijk}(\mathbf{\Omega}) = (5\Omega_i\Omega_j\Omega_k - \Omega_i\delta_{jk} - \Omega_j\delta_{ki} - \Omega_k\delta_{ij})/2$, we find

$$\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm sc}=\int_0^\infty dk'\int_{-1}^1 d\mu'\int_0^{2\pi}d\phi'[A_0+\delta A_+\mathbf{\Omega}\cdot\hat{\mathbf{B}}+\delta A_-\mathbf{\Omega}'\cdot\hat{\mathbf{B}}]$$
$$\times\left[C\left(g'+3\mathbf{\Omega}'\cdot\mathbf{h}'+\frac{10}{3}I'_{ij}\mathcal{P}'_{ij}\right)+D\left(g+3\mathbf{\Omega}\cdot\mathbf{h}+\frac{10}{3}I_{ij}\mathcal{P}_{ij}\right)\right]$$
$$=2\pi\int_0^\infty dk'\int_{-1}^1 d\mu'[A_0+\delta A_+\mathbf{\Omega}\cdot\hat{\mathbf{B}}]\left\{(Cg'+Dg)+3(C\mu'\mathbf{h}'+D\mathbf{h})\cdot\mathbf{\Omega}+\frac{10}{3}[CP_2(\mu')I'_{ij}+DI_{ij}]\mathcal{P}_{ij}\right\}$$
$$+2\pi\int_0^\infty dk'\int_{-1}^1 d\mu'\,\delta A_-\Big\{\mu'(Cg'+Dg)\mathbf{\Omega}\cdot\hat{\mathbf{B}}+3CP_2(\mu')(\mathbf{\Omega}\cdot\hat{\mathbf{B}})(\mathbf{\Omega}\cdot\mathbf{h}')$$
$$+C[1-P_2(\mu')]\hat{\mathbf{B}}\cdot\mathbf{h}'+3D\mu'(\mathbf{\Omega}\cdot\hat{\mathbf{B}})(\mathbf{\Omega}\cdot\mathbf{h})$$
$$+\frac{10}{3}C\hat{B}_iI'_{jk}\left[\frac{3}{5}\mathcal{P}_{ijk}-\frac{1}{5}\mu'\Omega_i\delta_{jk}+\frac{3}{10}\mu'(\Omega_j\delta_{ki}+\Omega_k\delta_{ij})\right]+\frac{10}{3}D\mu'I_{ij}\mathcal{P}_{ij}\mathbf{\Omega}\cdot\hat{\mathbf{B}}\Big\}. \tag{4.18}$$

To calculate the moments of the scattering rate, the following identities are needed:

$$\int\frac{d\Omega}{4\pi}\Omega_i\Omega_j=\frac{1}{3}\delta_{ij}$$
$$\int\frac{d\Omega}{4\pi}\mathcal{P}_{ij}=0$$
$$\int\frac{d\Omega}{4\pi}\mathcal{P}_{ij}\mathcal{P}_{kl}=-\frac{1}{10}\delta_{ij}\delta_{kl}+\frac{3}{20}(\delta_{ik}\delta_{lj}+\delta_{il}\delta_{jk})$$
$$\int\frac{d\Omega}{4\pi}\Omega_i\Omega_j\mathcal{P}_{kl}=-\frac{1}{15}\delta_{ij}\delta_{kl}+\frac{1}{10}(\delta_{ik}\delta_{lj}+\delta_{il}\delta_{jk})$$
$$\int\frac{d\Omega}{4\pi}\mathcal{P}_{ijk}=0$$
$$\int\frac{d\Omega}{4\pi}\mathcal{P}_{ijk}\Omega_l=0. \tag{4.19}$$

Also note that any integral of an odd number of Ω_i's over the solid angle gives zero. The zeroth moment of Eq. (4.18) is

$$\int\frac{d\Omega}{4\pi}\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm sc}=2\pi\int_0^\infty dk'\int_{-1}^1 d\mu'[A_0(Cg'+Dg)$$
$$+\delta A_+(C\mu'\mathbf{h}'+D\mathbf{h})\cdot\hat{\mathbf{B}}+\delta A_-$$
$$\times(D\mu'\mathbf{h}+C\mathbf{h}')\cdot\hat{\mathbf{B}}], \tag{4.20}$$

the first moment is

$$\int\frac{d\Omega}{4\pi}\Omega_i\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm sc}$$
$$=2\pi\int_0^\infty dk'\int_{-1}^1 d\mu' A_0[C\mu'h'_i+Dh_i]$$
$$+\frac{2\pi}{3}\int_0^\infty dk'\int_{-1}^1 d\mu'(Cg'+Dg)(\delta A_++\mu'\delta A_-)\hat{B}_i$$
$$+\frac{4\pi}{3}\int_0^\infty dk'\int_{-1}^1 d\mu'\{\delta A_+[CP_2(\mu')I'_{ij}+DI_{ij}]$$
$$+\mu'\delta A_-(CI'_{ij}+DI_{ij})\}\hat{B}_j, \tag{4.21}$$

and the second moment is

$$\int\frac{d\Omega}{4\pi}\mathcal{P}_{ij}(\mathbf{\Omega})\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm sc}$$
$$=2\pi\int_0^\infty dk'\int_{-1}^1 d\mu' A_0[CP_2(\mu')I'_{ij}+DI_{ij}]$$
$$+2\pi\frac{3}{10}\int_0^\infty dk'\int_{-1}^1 d\mu'\Big[C[\mu'\delta A_++P_2(\mu')\delta A_-]$$
$$\times\left(h'_i\hat{B}_j+h'_j\hat{B}_i-\frac{2}{3}\delta_{ij}\mathbf{h}'\cdot\hat{\mathbf{B}}\right)+D(\delta A_++\mu'\delta A_-)$$
$$\times\left(h_i\hat{B}_j+h_j\hat{B}_i-\frac{2}{3}\delta_{ij}\mathbf{h}\cdot\hat{\mathbf{B}}\right)\Big]. \tag{4.22}$$

D. Moments of the scattering rate: Nondegenerate nucleon limit

We now evaluate the moments in Eqs. (4.20)–(4.22) explicitly for nondegenerate nucleons (near the stellar surface) using the expressions of A_0 and $\delta A_\pm$ [Eqs. (4.10) and (4.11)] as derived in Sec. IV B for small inelasticity. After substituting Eqs. (4.14) and (4.15), it will be necessary to evaluate moments of q_0/T against A_0 and $\delta A_\pm$. These moments are defined as

$$M_0^n \equiv \int_0^\infty dk' A_0(k,k',\mu')\left(\frac{q_0}{T}\right)^n = \epsilon k\left(\frac{-\epsilon k}{T}\right)^n \int_{-\infty}^\infty du A_0 u^n \tag{4.23}$$

and

$$\delta M_\pm^n \equiv \int_0^\infty dk'\, \delta A_\pm \left(\frac{q_0}{T}\right)^n = \epsilon k\left(\frac{-\epsilon k}{T}\right)^n \int_{-\infty}^\infty du\, \delta A_\pm u^n, \tag{4.24}$$

where we have used Eq. (4.9) and in the du integrals we have extended the lower limit $(-\epsilon^{-1})$ to $-\infty$ (since $\epsilon \ll 1$). Only the following values will be needed:[4]

$$M_0^0 = \left(\frac{G_F c_V k}{2\pi}\right)^2 n[1+3\lambda^2+(1-\lambda^2)\mu']$$

$$M_0^1 = \left(\frac{G_F c_V k}{2\pi}\right)^2 n[1+3\lambda^2+(1-\lambda^2)\mu'] \times\left[\frac{k}{m}(1-\mu')\left(\frac{k}{T}-6\right)\right]$$

$$M_0^2 = \left(\frac{G_F c_V k}{2\pi}\right)^2 n[1+3\lambda^2+(1-\lambda^2)\mu'] \times\left[2\frac{k^2}{mT}(1-\mu')\right]$$

$$\delta M_\pm^0 = \left(\frac{G_F c_V k}{2\pi}\right)^2 n\frac{2\mu_B B}{T}\left(\lambda \pm \lambda^2 \mp 4\lambda^2\frac{T}{k}\right)$$

$$\delta M_\pm^1 = \pm\left(\frac{G_F c_V k}{2\pi}\right)^2 n\frac{2\mu_B B}{T}\; 2\lambda^2. \tag{4.25}$$

Let the notation $\mathcal{O}(n)$ mean a term which contains a factor of $(q_0/T)^n$ in the integrands of Eqs. (4.20)–(4.22). In the zeroth moment, the $\mathcal{O}(0)$ term is zero and M_0^1 is of the same size as M_0^2 so we need to expand to $\mathcal{O}(2)$ for the $B=0$ piece. For the $B\neq 0$ piece, the $\mathcal{O}(0)$ term is not zero, but $\delta M_\pm^0$ is of the same size as $\delta M_\pm^1$ so we need to expand to $\mathcal{O}(1)$. The zeroth moment is then given by

$$\int \frac{d\Omega}{4\pi}\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm sc} = \kappa_0^{(\rm sc)}\frac{k}{m}\left\{6\left[T\frac{\partial g}{\partial k}+(1-2f_\nu^{(0)})g\right] + \frac{k}{T}\left[T^2\frac{\partial^2 g}{\partial k^2}+T\frac{\partial g}{\partial k}(1-2f_\nu^{(0)})-2gT\frac{\partial f_\nu^{(0)}}{\partial k}\right]\right\} + \epsilon_{\rm sc}\kappa_0^{(\rm sc)}\left(T\frac{\partial \mathbf{h}}{\partial k}+4\frac{T}{k}\mathbf{h}\right)\cdot\hat{\mathbf{B}}, \tag{4.26}$$

where we have defined the zero-field scattering opacity (per unit volume)

$$\kappa_0^{(\rm sc)} = \frac{8\pi}{3}\left(\frac{G_F c_V k}{2\pi}\right)^2(1+5\lambda^2)n \tag{4.27}$$

and the dimensionless asymmetry parameter

$$\epsilon_{\rm sc} = \frac{6\lambda^2}{(1+5\lambda^2)}\frac{\mu_B B}{T}. \tag{4.28}$$

For the first moment, the $B=0$ terms are nonvanishing at $\mathcal{O}(0)$ so that only the $q_0/T=0$ terms are needed. The terms involving g cancel at $\mathcal{O}(0)$ and so only the $\mathcal{O}(1)$ terms are needed (note that $\delta M_\pm^1 \gg \delta M_\pm^2$). Last, both the $\mathcal{O}(0)$ and the $\mathcal{O}(1)$ terms are needed for the I_{ij} terms since $\delta M_\pm^0$ is the same size as $\delta M_\pm^1$. The first moment is then given by

$$\int \frac{d\Omega}{4\pi}\Omega_i\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm sc} = -\kappa_0^{(\rm sc)}h_i - \frac{1}{3}\epsilon_{\rm sc}\kappa_0^{(\rm sc)}\left\{\left[T\frac{\partial g}{\partial k}+(1-2f_\nu^{(0)})g\right]\hat{B}_i + \left(1-2f_\nu^{(0)}+\frac{1}{\lambda}-4\frac{T}{k}\right)I_{ij}\hat{B}_j\right\}. \tag{4.29}$$

For the second moment, the $B=0$ piece is nonzero at $\mathcal{O}(0)$ so that only lowest order is needed. The $B\neq 0$ terms must be kept at both $\mathcal{O}(0)$ and $\mathcal{O}(1)$. The result is

$$\int \frac{d\Omega}{4\pi}\mathcal{P}_{ij}(\mathbf{\Omega})\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm sc} = -\frac{3}{2}\left(\frac{1+3\lambda^2}{1+5\lambda^2}\right)\kappa_0^{(\rm sc)}I_{ij} - \frac{3}{20}\epsilon_{\rm sc}\kappa_0^{(\rm sc)}\left(1-2f_\nu^{(0)}+\frac{1}{\lambda}-4\frac{T}{k}\right) \times\left(h_i\hat{B}_j+h_j\hat{B}_i-\frac{2}{3}\delta_{ij}\mathbf{h}\cdot\hat{\mathbf{B}}\right). \tag{4.30}$$

E. Elastic cross section

As discussed before (see Sec. III A), it is essential to retain the inelasticity in the differential cross section in order

[4]In A_0 [Eq. (4.10)], the leading order term in ϵ is an even function of u, and the higher order term is an odd function of u. Hence, when $q_0^n \propto u^n$ is integrated against A_0 to find the moments M_0^n, the $n=0$ moment is larger than the $n=1,2$ moments by a factor of $\sim k/m$, which are larger than the $n=3,4$ moments by a factor of $\sim k/m$, etc. On the other hand, the leading order term in $\delta A_\pm$ is an odd function of u. The moments of q_0^n against $\delta A_\pm$, called $\delta M_\pm^n$, will then have the $n=0,1$ terms larger than $n=2,3$ by a factor of k/m and so on.

to derive the correct neutrino flux (as we have done in the previous subsections). Nevertheless, from the energetics point of view, ν-N scattering is highly elastic near the surface of the proto-neutron star,[5] and it is instructive to consider the ''elastic'' scattering rate, $d\Gamma/d\Omega'$, obtained by integrating $(d\Gamma/dk'd\Omega')$ over all final neutrino energies k'. Using Eqs. (4.10) and (4.11), together with Eq. (4.25), we find

$$\begin{aligned}\frac{d\Gamma}{d\Omega'} &= \int_0^\infty dk' \frac{d\Gamma}{dk'd\Omega'} \\ &= M_0^0 + \delta M_+^0 \boldsymbol{\Omega}\cdot\hat{\mathbf{B}} + \delta M_-^0 \boldsymbol{\Omega}'\cdot\hat{\mathbf{B}} \\ &= \left(\frac{G_F c_V k}{2\pi}\right)^2 n\left\{1+3\lambda^2+(1-\lambda^2)\mu' \right. \\ &\quad + \frac{2\mu_B B}{T}\left[\lambda\left(1+\lambda-4\lambda\frac{T}{k}\right)\boldsymbol{\Omega}\cdot\hat{\mathbf{B}} \right. \\ &\quad \left.\left. +\lambda\left(1-\lambda+4\lambda\frac{T}{k}\right)\boldsymbol{\Omega}'\cdot\hat{\mathbf{B}}\right]\right\}. \end{aligned} \tag{4.31}$$

The resulting cross section per particle, $(d\Gamma/d\Omega)/n$, is similar to Eq. (2.4) (recall that polarization $P=\mu_B B/T$ for nondegenerate nucleons), obtained by assuming complete elasticity of the scattering process. The difference involves the terms with $4\lambda T/k$ in Eq. (4.31). These terms appear in the phase space integral of the asymmetric part of the cross section, which can be affected by even a small inelasticity. In fact, for low energy neutrinos (with $k\lesssim 4T$), the T/k terms dominates the asymmetry in the cross section.[6]

F. Antineutrino cross section

The expressions derived in previous sections apply only for neutrinos. For $\bar{\nu}+N\rightarrow\bar{\nu}+N$, the differential cross section of the form Eq. (4.3) still applies, except that one needs to switch the coefficients in front of $\boldsymbol{\Omega}\cdot\hat{\mathbf{B}}$ and $\boldsymbol{\Omega}'\cdot\hat{\mathbf{B}}$. This is due to the crossing symmetry of the tree-level Feynman diagram.

V. NEUTRINO ABSORPTION BY NUCLEONS

In this section we derive an explicit expression the cross section for neutrino absorption $(\nu_e+n\rightarrow p+e^-)$ in magnetic fields as formulated in Sec. III B.

In the regime where the neutrino energy is much smaller than the nucleon rest mass, one might be tempted to consider an ''elastic'' cross section, obtained by treating the nucleons infinitely massive (i.e., neglecting nucleon recoil).[7] Further neglecting the effects of electron Landau levels and Fermi blocking, we obtain (e.g., Ref. [53])

$$\sigma = \frac{1}{\pi}(G_F c_V k)^2[(1+3\lambda^2)+2P_n\lambda(\lambda+1)\boldsymbol{\Omega}\cdot\hat{\mathbf{B}}], \tag{5.1}$$

where P_n is the neutron polarization, $\mathbf{k}=k\boldsymbol{\Omega}$ is the incident neutrino momentum, and G_F, c_V, λ are (charge-current) weak interaction constants (see Appendix D). This simplified treatment, however, leads to an incomplete expression for the asymmetric part (proportional to $\boldsymbol{\Omega}\cdot\hat{\mathbf{B}}$) of the cross section.[8] As we show in this section, by incorporating the effects of electron Landau levels and small inelasticity, additional asymmetric terms which relate to electron and proton polarizations are revealed. These additional terms are important in determining asymmetric flux from the proto-neutron star.

Note that the quantization of electron energy levels can induce oscillatory features in the total absorption cross section as a function of the neutrino energy [60]. This effect results purely from the modification of the electron phase space due to the Landau levels (similar to the magnetization of an electron gas at low temperatures). This oscillatory feature is particularly prominent in the low density regime where only a few electron Landau levels are filled. Our focus in this paper is the asymmetric part of the cross section, which arises from parity violation. We shall therefore restrict to the regime where more than a few electron Landau levels are filled. In this regime, we can replace the sum over Landau levels by an integral, and obtain an explicit expression for the asymmetric parameter in the cross section.

In the calculation presented below, we also include the effect of proton Landau levels. As expected, this introduces no new term in the cross-section since many proton Landau levels are filled for the typical conditions of proto-neutron stars. Our result therefore also serves as an explicit demonstration on the validity of using proton plane waves in calculating neutrino-nucleon opacities (absorption and scattering; see Sec. VI). It turns out that, after using some standard identities involving Landau wave functions, the calculation with electron and proton Landau levels is not more difficult than the calculation with only electron Landau levels.

[5]In fact, the scattering is elastic (i.e., $|q_0|\ll k$) to a good approximation in most regions of the proto-neutron star. The only exception is during the first second or so after core collapse, when ν_e's are highly degenerate in the stellar core.

[6]Note, however, that Eq. (4.31) is valid only for $k\gtrsim k_{min} = \mu_B B(m/T)^{1/2}$ (see the end of Sec. IV B). This means that the maximum asymmetry is $\sim\mu_B B/k_{min}=(T/m)^{1/2}$.

[7]Previous authors have all neglected the recoil effect. References [42,43] focus on the rate of neutron decay (including the effect of electron Landau levels) and do not address the angular distribution. Ref. [47] discusses the electron contribution to the asymmetric emission in $e^-+p\rightarrow n+\nu_e$, but the authors did not give an explicit expression. Moreover, the cancellation of asymmetric emission and absorption was not considered (see Sec. III B).

[8]In traditional laboratory experiments of parity violation (e.g., Ref. [73]), the approximation (of neglecting nucleon recoil and electron Landau levels) is valid because the temperature T is much smaller than the neutrino energy k. In the supernova context, k is comparable to T and this simple approximation leads to significant error.

A. Expression for absorption opacity

The absorption opacity is defined in Eq. (3.17). The energy of a relativistic electron in a magnetic field is given by

$$E_e=(m_e^2+2eBN_e+p_{e,z}^2)^{1/2}, \tag{5.2}$$

where $N_e=0,1,2,\ldots$ is the Landau level index and $p_{e,z}$ is the electron z momentum. The other quantum numbers specifying the electron states are $\sigma_e=\pm 1$, the spin projection along $\mathbf{\Pi}=\mathbf{p}+e\mathbf{A}$, and $R_e=0,1,2,\ldots$, which determines the radius of the electron guiding center (see Appendix C). Note that for the ground Landau level, the electron spin is opposite to the magnetic field; thus only the spin projection $\sigma_{e0}=-\mathrm{sgn}(p_{e,z})$ is allowed. The sum over electron states is then

$$\int d\Pi_e=\sum_{N_e=0}^{\infty}\sum_{\sigma_e=\pm 1}c(N_e,\sigma_e)\sum_{R_e=0}^{R_{max}}\int_{-\infty}^{\infty}\frac{L\,dp_{e,z}}{2\pi} \tag{5.3}$$

where $c(N_e,\sigma_e)=1-\delta_{N_e,0}\delta_{\sigma_e,-\sigma_{e0}}$ is 0 if both $N_e=0$ and $\sigma_e=-\sigma_{e,0}$ and 1 otherwise. The cutoff $R_{max}\simeq eBA/2\pi$ (the degeneracy of the Landau level) limits the guiding center to lie within the normalization volume $V=AL$ (where A is the area). The proton energy is given by

$$E_p=\frac{eB}{m}\left(N_p+\frac{1}{2}\right)-s_p\mu_{Bp}B+\frac{p_{p,z}^2}{2m}, \tag{5.4}$$

where N_p specifies the Landau level, $s_p=\pm 1$ is the spin projection along the z axis, μ_{Bp} is the proton magnetic moment, and $p_{p,z}$ is the proton z momentum. The summation over states for the proton takes the form

$$\int d\Pi_p=\sum_{N_p=0}^{\infty}\sum_{R_p=0}^{R_{max}}\sum_{s_p=\pm 1}\int_{-\infty}^{\infty}\frac{L\,dp_{p,z}}{2\pi} \tag{5.5}$$

where R_p is the quantum number for the proton guiding center, and R_{max} is the same as for the electron. Finally, the neutron phase space is simply that of a free particle, with energy $E_n=\mathbf{p}_n^2/(2m)-s_n\mu_{Bn}B$.

In Appendix D, we find that the transition rate for absorption (S matrix squared divided by time) takes on the form

$$W_{if}^{(\mathrm{abs})}=L^{-1}V^{-2}(2\pi)^2\delta(E_e+E_p-k-E_n-Q)\times\delta(p_{e,z}+p_{p,z}-k_z-p_{n,z})|M|^2, \tag{5.6}$$

where the matrix element, summed over the guiding center quantum numbers R_e and R_p for the electron and proton as well as electron spin σ_e, can be written as

$$\sum_{R_e=0}^{R_{max}}\sum_{R_p=0}^{R_{max}}\sum_{\sigma_e=\pm 1}c(N_e,\sigma_e)|M|^2=\frac{G_F^2}{2}A\frac{eB}{2\pi}L_{\mu\nu}N^{\mu\nu}. \tag{5.7}$$

Here $L_{\mu\nu}$ is the lepton tensor and $N^{\mu\nu}$ is the nucleon tensor, which takes on precisely the same form as in the zero field case. Plugging this back into Eq. (3.17) gives

$$\kappa^{(\mathrm{abs})}=\frac{G_F^2}{2}\frac{eB}{2\pi}\sum_{N_e=0}^{\infty}\int_{-\infty}^{\infty}\frac{dp_{e,z}}{2\pi}(1-f_e)\times\sum_{N_p=0}^{\infty}\int\frac{d^2p_{n,\perp}}{(2\pi)^2}\sum_{s_n,s_p=\pm 1}S_{s_ns_p}L_{\mu\nu}N^{\mu\nu}, \tag{5.8}$$

where we have defined a "response function" for absorption:

$$S_{s_ns_p}=\int_{-\infty}^{\infty}\frac{dp_{n,z}}{2\pi}\int_{-\infty}^{\infty}\frac{dp_{p,z}}{2\pi}(2\pi)^2\delta(E_e+E_p-k-E_n-Q)\times\delta(p_{e,z}+p_{p,z}-k_z-p_{n,z})f_n(1-f_p). \tag{5.9}$$

By integrating over the delta functions (see Appendix E), we derive a general expression for the nucleon response function:

$$S_{s_ns_p}=\frac{m}{|q_z|}\left(\frac{1}{e^y+1}\right)\left(\frac{1}{1+e^{-y-z}}\right), \tag{5.10}$$

where $q_z=k_z-p_{e,z}$, $q_\perp^2=eB(2N_p+1)-p_{n,\perp}^2$, $q^2=q_z^2+q_\perp^2$, $q_0=k-E_e$,

$$y=\frac{E_n-\mu_n}{T}=-\frac{\mu_n}{T}+\frac{p_{n,\perp}^2}{2mT}+\frac{[q_0+Q-q^2/2m+(\mu_{Bp}s_p-\mu_{Bn}s_n)B]^2}{4T(q_z^2/2m)}-\frac{\mu_{Bn}s_nB}{T} \tag{5.11}$$

and

$$z=\frac{\mu_n-\mu_p+q_0+Q}{T}. \tag{5.12}$$

The above expressions apply for arbitrary values of nucleon degeneracy, recoil energy, and magnetic field. In this general case, the nucleon tensor $N^{\mu\nu}$ depends on s_n and s_p, the lepton tensor $L_{\mu\nu}$ depends on N_e, $p_{e,z}$, and $w_\perp\equiv|\mathbf{p}_{n,\perp}+\mathbf{k}_\perp|$, and the response function $S_{s_ns_p}$ depends on s_n, s_p, N_p, N_e, $p_{e,z}$, and $w_\perp$. To evaluate Eq. (5.8), we are left with two infinite sums and three integrals left to perform.

To make progress, we shall proceed in the next subsection with an approximate method appropriate to the outer layers of the proto-neutron star in which the nucleons are nondegenerate and the recoil energies and nucleon spin energies are small in comparison to other energy scales. As discussed before (see Sec. III), only in the outer layers (where the neutrino distribution deviates from thermal equilibrium) can asymmetric neutrino flux develops.

B. Evaluation of the absorption opacity: Nondegenerate nucleon regime

For small nucleon spin energies and nucleon recoil, all dependence on s_n, s_p, and N_p can be taken out of the exponential in the nucleon response function so that these quantities can easily be summed over. Since $S_{s_n s_p}$ is expanded to linear order in $\mu_B B$, it contains only terms linear in s_n or s_p, but not both $s_n s_p$. As a consequence, any terms in $L_{\mu\nu}$ containing $s_n s_p$ can immediately be dropped, considerably simplifying this expression. We will also drop all terms in $L_{\mu\nu}$ which will give small corrections to the angle-independent, $B=0$ opacity. Last, we drop terms in $L_{\mu\nu}$ which will give zero in the sums over N_p (see Appendix D for a discussion).

1. Contribution from the $N_e=0$ state

Since the electron in the the ground Landau level can only have a spin opposite the magnetic field, the $N_e=0$ term in the opacity expression [Eq. (5.8)] requires special treatment. As Eq. (5.8) already contains a prefactor B, we can drop all nucleon polarization terms when evaluating the $N_e=0$ contribution to the asymmetric opacity (this cannot be done when summing over all the $N_e \geqslant 1$ states since $N_e = p_{e,\perp}^2/2eB$ is summed over a large number of states so that the prefactor of B effectively cancels). Since only the nucleon polarization terms contain pieces with large coefficients, the $N_e=0$ state can be evaluated to lowest order in inelasticity.

In Appendix E, it was shown that to lowest order in the inelasticity, the nucleon response function for $N_e=0$ can be written

$$S_{s_n s_p} = \frac{m}{|q_{z,0}|} \exp\left(\frac{\mu_n}{T} - \frac{p_{n,\perp}^2}{2mT} - u^2 \right) \quad (N_e=0 \text{ state}), \tag{5.13}$$

where u is a dimensionless electron z momentum defined by

$$p_{e,z} = \pm (k+Q)(1+\epsilon u), \tag{5.14}$$

and

$$\epsilon = \left(\frac{2T}{m} \right)^{1/2} \frac{|q_{z,0}|}{k+Q} \tag{5.15}$$

is a small parameter $[q_{z,0}=k_z \mp (k+Q)]$. As this expression for $S_{s_n s_p}$ is independent of N_p, we may sum over N_p in $L_{\mu\nu}$, with the result (Appendix D)

$$\sum_{N_p=0}^{\infty} N^{\mu\nu} L_{\mu\nu}(N_e=0) = \Theta(p_{e,z})(c_V^2 - c_A^2) \mathbf{\Omega} \cdot \hat{\mathbf{B}}. \tag{5.16}$$

Since this expression is independent of s_n and s_p, their sums give a factor of 4. The asymmetric opacity from the electron ground state is then given by

$$\begin{aligned}
\kappa^{(\mathrm{abs})}(N_e=0) &= \frac{G_F^2}{2} \frac{eB}{2\pi} \int_{-\infty}^{\infty} \frac{dp_{e,z}}{2\pi} (1-f_e) \\
&\quad \times \int \frac{d^2 p_{n,\perp}}{(2\pi)^2} 4 \frac{m}{|q_{z,0}|} \\
&\quad \times \exp\left(\frac{\mu_n}{T} - \frac{p_{n,\perp}^2}{2mT} - u^2 \right) \\
&\quad \times \Theta(p_{e,z})(c_V^2 - c_A^2) \mathbf{\Omega} \cdot \hat{\mathbf{B}} \\
&= \frac{G_F^2}{2} \frac{eB}{2\pi} 4 \frac{mT}{2\pi} \frac{m}{|q_{z,0}|} \\
&\quad \times \exp\left(\frac{\mu_n}{T} \right) (c_V^2 - c_A^2) \mathbf{\Omega} \cdot \hat{\mathbf{B}} \\
&\quad \times \int_0^{\infty} \frac{dp_{e,z}}{2\pi} (1-f_e) e^{-u^2} \\
&= \frac{G_F^2 eB n_n}{2\pi} [1 - f_e(k+Q)](c_V^2 - c_A^2) \mathbf{\Omega} \cdot \hat{\mathbf{B}},
\end{aligned} \tag{5.17}$$

where we have used Eq. (4.12) to relate the neutron chemical potential μ_n to its number density n_n. In evaluating the $p_{e,z}$ integral in Eq. (5.17), we have approximated $f_e(E_e)$ by $f_e(k+Q)$ since the first term in an expansion of $f_e(E_e)$ about this value is odd in u and hence gives vanishing contribution and the second order term in the expansion is down by a factor T/m.

2. Contribution from the $N_e \geqslant 1$ states

For electrons in the excited Landau levels ($N_e \geqslant 1$), the relevant matrix element can be written as (see Appendix D)

$$N^{\mu\nu} L_{\mu\nu}(N_e \geqslant 1) = \mathcal{T}_0 + \mathcal{T}_e + \mathcal{T}_n, \tag{5.18}$$

with

$$\begin{aligned}
\mathcal{T}_0 = \frac{1}{2}(c_V^2 + 3c_A^2) \Bigg[& I_{N_e-1,N_p}^2(\omega) \left(1 - \frac{p_{e,z}}{|\Lambda|} \right) \\
& + I_{N_e,N_p}^2(\omega) \left(1 + \frac{p_{e,z}}{|\Lambda|} \right) \Bigg],
\end{aligned} \tag{5.19}$$

$$\begin{aligned}
\mathcal{T}_e = \frac{1}{2} \Bigg[& -I_{N_e-1,N_p}^2(\omega) \left(1 - \frac{p_{e,z}}{|\Lambda|} \right) + I_{N_e,N_p}^2(\omega) \\
& \times \left(1 + \frac{p_{e,z}}{|\Lambda|} \right) \Bigg] (c_V^2 - c_A^2) \mathbf{\Omega} \cdot \hat{\mathbf{B}},
\end{aligned} \tag{5.20}$$

$$\begin{aligned}
\mathcal{T}_n = [& c_A(c_A + c_V) s_n \mathbf{\Omega} \cdot \hat{\mathbf{B}} - c_A(c_A - c_V) s_p \mathbf{\Omega} \cdot \hat{\mathbf{B}}] \\
& \times \left[I_{N_e-1,N_p}^2(\omega) \left(1 - \frac{p_{e,z}}{|\Lambda|} \right) + I_{N_e,N_p}^2(\omega) \left(1 + \frac{p_{e,z}}{|\Lambda|} \right) \right],
\end{aligned} \tag{5.21}$$

where $|\Lambda|=(p_{e,z}^2+2eBN_e)^{1/2}$, $\omega=w_\perp^2/(2eB)=(\mathbf{p}_{n,\perp}+\mathbf{k}_\perp)^2/(2eB)$, and the function I_{N_e,N_p} (the shape of the Landau wave function) is defined by Eq. (C4). Note that in these expressions, we have dropped all terms that will give zero contribution to the opacity (such as those terms involving $s_n s_p$).

The $N_e \geqslant 1$ contribution to the absorption opacity [Eq. (5.8)] can be written as

$$\kappa^{(\mathrm{abs})}(N_e\geqslant 1)=\kappa_0^{(\mathrm{abs})}+\kappa^{(\mathrm{abs})}(e,N_e\geqslant 1)+\kappa^{(\mathrm{abs})}(np), \tag{5.22}$$

where each piece corresponds to a different part of $N^{\mu\nu}L_{\mu\nu}(N_e\geqslant 1)$:

$$\kappa_0^{(\mathrm{abs})}=\frac{G_F^2}{2}\frac{eB}{2\pi}\sum_{N_e=1}^{\infty}\int_{-\infty}^{\infty}\frac{dp_{e,z}}{2\pi}(1-f_e)\times\sum_{N_p=0}^{\infty}\int\frac{d^2p_{n,\perp}}{(2\pi)^2}\sum_{s_p,s_n}S_{s_n s_p}\mathcal{T}_0, \tag{5.23}$$

$$\kappa^{(\mathrm{abs})}(e,N_e\geqslant 1)=\frac{G_F^2}{2}\frac{eB}{2\pi}\sum_{N_e=1}^{\infty}\int_{-\infty}^{\infty}\frac{dp_{e,z}}{2\pi}(1-f_e)\times\sum_{N_p=0}^{\infty}\int\frac{d^2p_{n,\perp}}{(2\pi)^2}\sum_{s_p,s_n}S_{s_n s_p}\mathcal{T}_e, \tag{5.24}$$

$$\kappa^{(\mathrm{abs})}(np)=\frac{G_F^2}{2}\frac{eB}{2\pi}\sum_{N_e=1}^{\infty}\int_{-\infty}^{\infty}\frac{dp_{e,z}}{2\pi}(1-f_e)\times\sum_{N_p=0}^{\infty}\int\frac{d^2p_{n,\perp}}{(2\pi)^2}\sum_{s_p,s_n}S_{s_n s_p}\mathcal{T}_n. \tag{5.25}$$

We are going to evaluate the sum Σ_{N_e} by replacing it with an integral. Such a procedure effectively eliminates any possible oscillatory behavior of the opacity as a function of energy (see the beginning of Sec. V), but is valid when more than a few electron Landau levels are filled. For infinite nucleon mass, energy conservation requires $k+Q=(p_{e,z}^2+2eBN_e)^{1/2}$ (neglecting m_e). For given N_e and $p_{e,z}$, the nucleon recoil energy is of order $|q_z|\sqrt{T/m}$. Thus it is natural to define a dimensionless recoil energy u via

$$2eBN_e=E_\perp^2(1+\epsilon u), \tag{5.26}$$

where

$$E_\perp^2=(k+Q)^2-p_{e,z}^2,\quad \epsilon=\left(\frac{8T}{m}\right)^{1/2}\frac{|q_z|(k+Q)}{E_\perp^2}. \tag{5.27}$$

The nucleon response function [Eq. (5.10)] can be expanded for small ϵu, with the result (Appendix E)

$$S_{s_n s_p}=\frac{m}{|q_z|}\exp(-y_0)(1-\delta y), \tag{5.28}$$

where

$$\exp(-y_0)\simeq\exp\left(\frac{\mu_n}{T}-\frac{p_{n,\perp}^2}{2mT}-u^2\right)\times\left[1+\frac{\epsilon u^3E_\perp^2}{2(k+Q)^2}-\frac{2u(k+Q)q^2}{\epsilon mE_\perp^2}\right] \tag{5.29}$$

and

$$\delta y=-\frac{\mu_{Bn}s_nB}{2T}\left[1-\frac{q_\perp^2}{q_z^2}-\frac{\epsilon umE_\perp^2}{(k+Q)q_z^2}\left(1-\frac{\epsilon uE_\perp^2}{4(k+Q)^2}\right)\right]-\frac{\mu_{Bp}s_pB}{2T}\left[1+\frac{q_\perp^2}{q_z^2}+\frac{\epsilon umE_\perp^2}{(k+Q)q_z^2}\left(1-\frac{\epsilon uE_\perp^2}{4(k+Q)^2}\right)\right]. \tag{5.30}$$

Similarly, for small ϵu, the electron Fermi blocking factor can be expanded to first order in ϵ as

$$1-f_e(E_e)\simeq 1-f_e(k+Q)-\frac{\partial f_e(k+Q)}{\partial E_e}\frac{\epsilon uE_\perp^2}{2(k+Q)}=[1-f_e(k+Q)]\left[1+\frac{\epsilon uE_\perp^2}{2T(k+Q)}f_e(k+Q)\right]. \tag{5.31}$$

Now consider $\kappa_0^{(\mathrm{abs})}$ in Eq. (5.23). Since only $S_{s_n s_p}$ depends on spin, the spin sums Σ_{s_n,s_p} effectively set $\delta y=0$. The factor $\exp(-y_0)$ can be evaluated at lowest order. The sum Σ_{N_p} can be calculated using the summation rule for I_{NS}^2 (Appendix D). Replacing Σ_{N_e} by $\int dN_e=\int\epsilon E_\perp^2du/(2eB)$ and integrating over $p_{n,\perp}$ (see Appendix F), we arrive at

$$\kappa_0^{(\mathrm{abs})}=\frac{G_F^2}{2}\frac{eB}{2\pi}(c_V^2+3c_A^2)\frac{mT}{2\pi}4e^{\mu_n/T}\int_{-(k+Q)}^{(k+Q)}\frac{dp_{e,z}}{2\pi}\frac{m}{|q_z|}\int_{-\infty}^{\infty}\frac{\epsilon E_\perp^2 du}{2eB}\exp(-u^2)(1-f_e)$$
$$=\frac{G_F^2}{2}\frac{eB}{2\pi}(c_V^2+3c_A^2)\frac{mT}{2\pi}4e^{\mu_n/T}m\left(\frac{8T}{m}\right)^{1/2}\frac{(k+Q)}{2eB}\pi^{1/2}\int_{-(k+Q)}^{(k+Q)}\frac{dp_{e,z}}{2\pi}(1-f_e)$$
$$=\frac{G_F^2}{\pi}(k+Q)^2 n_n(c_V^2+3c_A^2)[1-f_e(k+Q)], \qquad (5.32)$$

which is exactly the usual $B=0$ opacity.

Next consider $\kappa^{(\mathrm{abs})}(e,N_e\geqslant 1)$, the ''electron contribution'' from the $N_e\geqslant 1$ states to the opacity [Eq. (5.24)]. We may evaluate all quantities to lowest order in the inelasticity. The spin sums and nucleon response function are evaluated as before. Performing all integrals but $p_{e,z}$ gives

$$\kappa^{(\mathrm{abs})}(e,N_e\geqslant 1)\propto\int_{-(k+Q)}^{(k+Q)}dp_{e,z}p_{e,z}=0. \qquad (5.33)$$

So the electron contribution from the higher Landau levels is zero to lowest order in the inelasticity. The next order correction scales as T/m, and can be neglected.

Finally, $\kappa^{(\mathrm{abs})}(np)$ [Eq. (5.25)] gives the contribution of nucleon polarizations to the opacity. Performing the spin sums, the integral over $p_{n,\perp}$, and using the results of Appendix D for the sum over N_p yields

$$\kappa^{(\mathrm{abs})}(np)=\frac{G_F^2}{2}\frac{eB}{2\pi}4\frac{mT}{2\pi}e^{\mu_n/T}\boldsymbol{\Omega}\cdot\hat{\mathbf{B}}\int_{-\infty}^{\infty}\frac{dp_{e,z}}{2\pi}\frac{m}{|q_z|}\sum_{N_e=1}^{\infty}(1-f_e)\exp(-u^2)\Bigg\{\frac{\mu_{Bn}B}{T}c_A(c_A+c_V)-\frac{\mu_{Bp}B}{T}c_A(c_A-c_V)$$
$$+\left[\frac{\mu_{Bn}B}{T}c_A(c_A+c_V)+\frac{\mu_{Bp}B}{T}c_A(c_A-c_V)\right]\Bigg[\frac{\epsilon^2u^2mE_\perp^4}{4(k+Q)^3q_z^2}-\frac{2eBN_e+k_\perp^2+eBp_{e,z}/|\Lambda|}{q_z^2}-\frac{\epsilon umE_\perp^2}{(k+Q)q_z^2}$$
$$-\frac{\epsilon^2u^4mE_\perp^4}{2(k+Q)^3q_z^2}+2u^2\frac{q_z^2+2eBN_e+k_\perp^2+eBp_{e,z}/|\Lambda|}{q_z^2}\Bigg]\Bigg\}. \qquad (5.34)$$

Again changing the sum over N_e into an integral over u, expanding f_e, and then performing the trivial $p_{e,z}$ integral gives the final result

$$\kappa^{(\mathrm{abs})}(np)=\frac{G_F^2}{\pi}(k+Q)^2n_n\boldsymbol{\Omega}\cdot\hat{\mathbf{B}}[1-f_e(k+Q)]$$
$$\times\Bigg\{2\frac{\mu_{Bn}B}{T}c_A(c_A+c_V)-\frac{T}{(k+Q)}$$
$$\times\left[1+\frac{(k+Q)}{T}f_e(k+Q)\right]\Bigg[2\frac{\mu_{Bn}B}{T}c_A(c_A+c_V)$$
$$+2\frac{\mu_{Bp}B}{T}c_A(c_A-c_V)\Bigg]\Bigg\}. \qquad (5.35)$$

3. Result

We can summarize our result for the absorption opacity in the following transparent formula:

$$\kappa^{(\mathrm{abs})}=\kappa_0^{(\mathrm{abs})}(1+\epsilon_{\mathrm{abs}}\boldsymbol{\Omega}\cdot\hat{\mathbf{B}}), \qquad (5.36)$$

where $\kappa_0^{(\mathrm{abs})}$ is the $B=0$ opacity as given by Eq. (5.32), and the asymmetry parameter ϵ_{abs} is given by

$$\epsilon_{\mathrm{abs}}=\epsilon_{\mathrm{abs}}(e)+\epsilon_{\mathrm{abs}}(np), \qquad (5.37)$$

with

$$\epsilon_{\mathrm{abs}}(e)=\frac{1}{2}\frac{eB}{(k+Q)^2}\frac{c_V^2-c_A^2}{c_V^2+3c_A^2} \qquad (5.38)$$

$$\epsilon_{\mathrm{abs}}(np)=2\frac{c_A(c_A+c_V)}{c_V^2+3c_A^2}\frac{\mu_{Bn}B}{T}-\frac{T}{(k+Q)}$$
$$\times\left[1+\frac{(k+Q)}{T}f_e(k+Q)\right]$$
$$\times\left[2\frac{c_A(c_A+c_V)}{c_V^2+3c_A^2}\frac{\mu_{Bn}B}{T}+2\frac{c_A(c_A-c_V)}{c_V^2+3c_A^2}\frac{\mu_{Bp}B}{T}\right]. \qquad (5.39)$$

Comparing this result with Eq. (5.1), which was obtained from a simplified calculation assuming infinite nucleon mass and neglecting Landau levels for electrons, we see that the simplified calculation gave an incorrect result for the asymmetry parameter. Only a neutron polarization term was included in Eq. (5.1). The correct expression for $\epsilon_{\rm abs}$ [Eqs. (5.37)–(5.39)] contains an electron contribution (arising from from the ground-state Landau level) and an additional contribution from both neutron and proton polarizations (which arises from our more careful treatment of inelasticity). These news terms dominate the asymmetric parameter for neutrino energy $k/T \lesssim$ a few. It is interesting to note that the particles in the final state of this reaction (p and e^-) contribute to the asymmetry parameter. Previous investigators found a contribution only from the initial state particles.

C. Moments of the absorption or emission rate

With the absorption opacity given in previous subsection, it is straightforward to calculate the moments of the absorption or emission rate in the Boltzmann equation. The absorption rate is given by Eq. (3.15) with δf_ν given by Eq. (3.21). Using the opacity (5.36) and the integrals in Eq. (4.19), we obtain the zeroth, first and second moments

$$\int \frac{d\Omega}{4\pi}\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm abs} = -\kappa_0^{*({\rm abs})}(g+\epsilon_{\rm abs}\mathbf{h}\cdot\hat{\mathbf{B}}), \quad (5.40)$$

$$\int \frac{d\Omega}{4\pi}\Omega_i\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm abs} = -\kappa_0^{*({\rm abs})}\left(h_i+\frac{1}{3}\epsilon_{\rm abs}g\hat{B}_i + \frac{2}{3}\epsilon_{\rm abs}I_{ij}\hat{B}_j\right), \quad (5.41)$$

$$\int \frac{d\Omega}{4\pi}\mathcal{P}_{ij}\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm abs} = -\kappa_0^{*({\rm abs})}\left[I_{ij}+\frac{3}{10}\epsilon_{\rm abs}\left(\hat{B}_ih_j + \hat{B}_jh_i - \frac{2}{3}\delta_{ij}\mathbf{h}\cdot\hat{\mathbf{B}}\right)\right] \quad (5.42)$$

where

$$\kappa_0^{*({\rm abs})} = \kappa_0^{({\rm abs})}\left[1+\exp\left(\frac{\mu_\nu-k}{T}\right)\right] \quad (5.43)$$

[see Eq. (3.16)].

D. Absorption opacity for $\bar{\nu}_e$

So far we have been concerned with the opacity for $\nu_e + n \to p + e$. The opacity for $\bar{\nu}_e + p \to n + e^+$ can be obtained by a similar calculation. The result can be found from the equations in Sec. V B 3 by replacing Q by $-Q$, n_n by n_p, the electron distribution function by the positron distribution function (which is rather small), and μ_{Bn} by $-\mu_{Bp}$. Thus, the absorption opacity for $\bar{\nu}_e$ is given by

$$\overline{\kappa}^{({\rm abs})} = \overline{\kappa}_0^{({\rm abs})}(1+\overline{\epsilon}_{\rm abs}\mathbf{\Omega}\cdot\hat{\mathbf{B}}), \quad (5.44)$$

where $\overline{\kappa^{({\rm abs})}}_0$ is the $B=0$ opacity as given by

$$\overline{\kappa}_0^{({\rm abs})} = \frac{G_F^2}{\pi}(k-Q)^2 n_p(c_V^2+3c_A^2) \times[1-f_{e^+}(k-Q)]\Theta(k-Q), \quad (5.45)$$

and the asymmetry parameter $\overline{\epsilon}_{\rm abs}$ is given by

$$\overline{\epsilon}_{\rm abs} = \overline{\epsilon}_{\rm abs}(e^+) + \overline{\epsilon}_{\rm abs}(np), \quad (5.46)$$

with

$$\overline{\epsilon}_{\rm abs}(e^+) = \frac{1}{2}\frac{eB}{(k-Q)^2}\frac{c_V^2-c_A^2}{c_V^2+3c_A^2} \quad (5.47)$$

$$\overline{\epsilon}_{\rm abs}(np) = -2\frac{c_A(c_A-c_V)}{c_V^2+3c_A^2}\frac{\mu_{Bp}B}{T} + \frac{T}{(k-Q)} \times\left[1+\frac{(k-Q)}{T}f_{e^+}(k-Q)\right]\left[2\frac{c_A(c_A-c_V)}{c_V^2+3c_A^2}\frac{\mu_{Bp}B}{T} + 2\frac{c_A(c_A+c_V)}{c_V^2+3c_A^2}\frac{\mu_{Bn}B}{T}\right]. \quad (5.48)$$

The theta function in Eq. (5.45) comes from the fact that the reaction is not energetically allowed unless $k \geqslant Q$ since the proton is lighter than the neutron.

VI. MOMENT EQUATIONS OF NEUTRINO TRANSPORT

In the last two sections (Secs. IV and V) we have carried out detailed calculations of neutrino scattering and absorption in magnetic fields. Explicit expressions have been obtained in the nondegenerate nucleon regime, which is appropriate for the outer layer the of proto-neutron star where asymmetric neutrino flux (drift flux) is expected. We now use these results to derive the moments of the Boltzmann transport equation (3.1). We focus on ν_e below, although similar results can also be obtained for other neutrino species.

The zeroth moment is obtained by integrating Eq. (3.1) over $d\Omega$. Combining Eqs. (4.26) and (5.40) we find

$$\frac{\partial(f_\nu^{(0)}+g)}{\partial t}+\nabla\cdot\mathbf{h}$$
$$=\int\frac{d\Omega}{4\pi}\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm sc}+\int\frac{d\Omega}{4\pi}\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm abs}$$
$$=\kappa_0^{(\rm sc)}\frac{k}{m}\left\{6\left[T\frac{\partial g}{\partial k}+(1-2f_\nu^{(0)})g\right]\right.$$
$$\left.+\frac{k}{T}\left[T^2\frac{\partial^2 g}{\partial k^2}+T\frac{\partial g}{\partial k}(1-2f_\nu^{(0)})-2gT\frac{\partial f_\nu^{(0)}}{\partial k}\right]\right\}$$
$$+\epsilon_{\rm sc}\kappa_0^{(\rm sc)}\left(T\frac{\partial\mathbf{h}}{\partial k}+4\frac{T}{k}\mathbf{h}\right)\cdot\hat{\mathbf{B}}-\kappa_0^{*(\rm abs)}(g+\epsilon_{\rm abs}\mathbf{h}\cdot\hat{\mathbf{B}}). \tag{6.1}$$

The zeroth moment equation governs the energy exchange between matter and neutrinos. The $B=0$ part of the scattering opacity can be ignored in the zeroth moment equation since it is suppressed by a factor of k/m (i.e., only the inelastic part of the scattering contributes to matter-neutrino energy exchange; recall that we have not included inelastic electron-neutrino scattering which can be a much larger effect). The term $-\kappa_0^{*(\rm abs)}g$ represents the usual neutrino emission and absorption. It is of interest to note that the asymmetric parts of the scattering and absorption introduce new terms to the zeroth moment equation. The importance of these terms will depend strongly on the field strength and optical depth. In a steady state (neglecting the time derivative term) we have

$$\nabla\cdot\mathbf{h}=-\kappa_0^{*(\rm abs)}g-\epsilon_{\rm abs}\kappa_0^{*(\rm abs)}\mathbf{h}\cdot\hat{\mathbf{B}}$$
$$+\epsilon_{\rm sc}\kappa_0^{(\rm sc)}\left[T\frac{\partial\mathbf{h}}{\partial k}+4\frac{T}{k}\mathbf{h}\right]\cdot\hat{\mathbf{B}}. \tag{6.2}$$

The first moment equation is obtained by multiplying Eq. (3.1) by Ω_i and then integrating over $d\Omega$. Combining Eqs. (4.29) and (5.41) we find

$$\frac{\partial h_i}{\partial t}+\frac{1}{3}\frac{\partial(f_\nu^{(0)}+g)}{\partial x_i}+\frac{2}{3}\frac{\partial I_{ij}}{\partial x_j}$$
$$=\int\frac{d\Omega}{4\pi}\Omega_i\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm sc}+\int\frac{d\Omega}{4\pi}\Omega_i\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm abs}$$
$$=-\kappa_0^{(\rm sc)}h_i-\frac{1}{3}\epsilon_{\rm sc}\kappa_0^{(\rm sc)}\left\{\left[T\frac{\partial g}{\partial k}+(1-2f_\nu^{(0)})g\right]\hat{B}_i\right.$$
$$\left.+\left(1-2f_\nu^{(0)}+\frac{1}{\lambda}-4\frac{T}{k}\right)I_{ij}\hat{B}_j\right\}$$
$$-\kappa_0^{*(\rm abs)}\left(h_i+\frac{1}{3}\epsilon_{\rm abs}g\hat{B}_i+\frac{2}{3}\epsilon_{\rm abs}I_{ij}\hat{B}_j\right). \tag{6.3}$$

The time derivative term can almost always be dropped (this corresponds to a rapid redistribution of matter temperature, the time scale of which is of order the mean free path divided by c, much smaller than neutrino diffusion time of the star; see Ref. [74]). Defining the total $B=0$ opacity, $\kappa_0^{(\rm tot)}=\kappa^{*(\rm abs)}+\kappa_0^{(\rm sc)}$, we have

$$h_i=-\frac{1}{3\kappa_0^{(\rm tot)}}\frac{\partial(f_\nu^{(0)}+g)}{\partial x_i}-\frac{2}{3\kappa_0^{(\rm tot)}}\frac{\partial I_{ij}}{\partial x_j}$$
$$-\frac{1}{3\kappa_0^{(\rm tot)}}\left\{\epsilon_{\rm sc}\kappa_0^{(\rm sc)}\left[T\frac{\partial g}{\partial k}+(1-2f_\nu^{(0)})g\right]\right.$$
$$\left.+\kappa_0^{*(\rm abs)}\epsilon_{\rm abs}g\right\}\hat{B}_i-\frac{1}{3\kappa_0^{(\rm tot)}}\left[2\kappa_0^{*(\rm abs)}\epsilon_{\rm abs}\right.$$
$$\left.+\epsilon_{\rm sc}\kappa_0^{(\rm sc)}\left(1-2f_\nu^{(0)}+\frac{1}{\lambda}-4\frac{T}{k}\right)\right]I_{ij}\hat{B}_j. \tag{6.4}$$

Clearly, in addition to the usual diffusive flux [the first line of Eq. (6.4)], there is also a drift flux [the second and third lines of Eq. (6.4)] which depends on the direction of the magnetic field. This asymmetric drift flux is a unique feature of parity violation in weak interactions. Equation (6.4) explicitly shows that the drift flux is nonzero only when the neutrino distribution deviates from thermal equilibrium, as expected from general consideration of detailed balance (Sec. III C; see also Sec. II for a discussion). The spherical deviation, g, always gives rise to a drift flux along $\hat{B}$. However, the drift flux from I_{ij} is along the direction of the vector $I_{ij}\hat{B}_j$, which does not have to be directed along the magnetic field at all points in the star. For cylindrical symmetry, however, one would expect that the net flux produced by the $I_{ij}\hat{B}_j$ term would average to the $\hat{B}$ direction.

Finally, the second moment equation can be obtained by multiplying Eq. (3.1) by $\mathcal{P}_{ij}=(3\Omega_i\Omega_j-\delta_{ij})/2$ and then integrating over $d\Omega$. Combining Eqs. (4.30) and (5.42) and ignoring the time derivative, we find

$$\frac{3}{10}\left(\frac{\partial h_i}{\partial x_j}+\frac{\partial h_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{h}\right)$$
$$=\int\frac{d\Omega}{4\pi}\mathcal{P}_{ij}\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm sc}+\int\frac{d\Omega}{4\pi}\mathcal{P}_{ij}\left[\frac{\partial f_\nu(\mathbf{k})}{\partial t}\right]_{\rm abs}$$
$$=-\frac{3}{2}\left(\frac{1+3\lambda^2}{1+5\lambda^2}\right)\kappa_0^{(\rm sc)}I_{ij}-\frac{3}{20}\epsilon_{\rm sc}\kappa_0^{(\rm sc)}$$
$$\times\left(1-2f_\nu^{(0)}+\frac{1}{\lambda}-4\frac{T}{k}\right)\left(h_i\hat{B}_j+h_j\hat{B}_i-\frac{2}{3}\delta_{ij}\mathbf{h}\cdot\hat{\mathbf{B}}\right)$$
$$-\kappa_0^{*(\rm abs)}\left[I_{ij}+\frac{3}{10}\epsilon_{\rm abs}\left(\hat{B}_ih_j+\hat{B}_jh_i-\frac{2}{3}\delta_{ij}\mathbf{h}\cdot\hat{\mathbf{B}}\right)\right]. \tag{6.5}$$

The above equations apply to ν_e. Similar equations can be derived for other species of neutrinos. Note that since

$\nu_{\mu(\tau)}$ and $\bar{\nu}_{\mu(\tau)}$ are always created in pairs inside the proto-neutron star, they have the same energy density distribution. Because of the crossing symmetry (see Sec. IV F), the drift flux of $\nu_{\mu(\tau)}$ exactly cancels the drift flux of $\bar{\nu}_{\mu(\tau)}$.

VII. DISCUSSION

In this paper we have presented a detailed study of neutrino-nucleon scattering and absorption in strong magnetic fields. Specifically, we focused on the effect of parity violation in weak interactions which can induce asymmetric neutrino transport in the proto-neutron star. Starting from the weak interaction Hamiltonian, we found the macroscopic moment equations of neutrino transport. Explicit results applicable to the outer region of a proto-neutron star are given in Eqs. (6.2), (6.4) and (6.5). Despite the fact that the neutrino cross sections are asymmetric with respect to the magnetic field throughout the star, asymmetric neutrino flux can be generated only in the outer region of the proto-neutron star where the neutrino distribution deviates from thermal equilibrium.

Previous studies based on simplified treatments (see Sec. II) have led to misleading results. We have tried to clarify many of the subtleties in deriving the correct expressions. The main technical complication lies in the proper treatment of the inelasticity of neutrino-nucleon scattering and absorption: although these processes are highly elastic from the energetics point of view, it is essential to include the small inelastic effect in order to obtain the correct expression for the asymmetric neutrino flux. In addition, it is necessary to use Landau wave functions for the electron since the quantum mechanical ground state of the electron gives the dominant contribution to the asymmetry for low energy electron neutrinos. To obtain simple formulas for the respective opacities, we developed a method to expand phase space integrals for both small magnetic field strengths (and correspondingly small spin energies) and also for small inelasticity. This method has general applicability for computing the effects of nucleon recoil in phase space integrals in powers of T/m.

To quantitatively determine the asymmetry in neutrino emission from a magnetized proto-neutron star one has to solve the moment equations (6.2), (6.4) and (6.5) in the outer layer of the star. This is beyond the scope of this paper. Here we shall be content with an order-of-magnitude estimate. First note that the net drift flux associated with $\nu_{\mu(\tau)}$ and $\bar{\nu}_{\mu(\tau)}$ is zero, and we only need to consider ν_e and $\bar{\nu}_e$. The key quantities that determine the asymmetric flux are the dimensionless asymmetry parameters $\epsilon_{\rm sc}$ and $\epsilon_{\rm abs}$. For neutrino-nucleon scattering, we have [Eq. (4.28)]

$$\epsilon_{\rm sc} \simeq 0.006\, B_{15} T^{-1}, \tag{7.1}$$

where B_{15} is the field strength in units of 10^{15} G, and T is the temperature in MeV. The asymmetry parameter for neutrino absorption has contributions from electron and nucleons:

$$\epsilon_{\rm abs}(e) \simeq 0.6\, B_{15} E_\nu^{-2},$$

$$\epsilon_{\rm abs}(np) \simeq 0.006\, B_{15} T^{-1}[1+\mathcal{O}(T/E_\nu)], \tag{7.2}$$

where E_ν is the neutrino energy in MeV. Thus for high energy neutrinos the asymmetry is dominated by $\epsilon_{\rm sc}$ and $\epsilon_{\rm abs}(np)$ (arising from nucleon polarization, $\sim \mu_B B/T$), while for lower energy neutrinos it is dominated by $\epsilon_{\rm abs}(e)$ (arising from electrons in the ground Landau level). The electron neutrinos decouple from matter near the neutrinosphere, where the typical density and temperature are $\rho \sim 10^{12}$ g cm^{-3}, and $T \sim 3$ MeV. For a mean ν_e energy of 10 MeV, $\epsilon_{\rm abs}$ is greater than $\epsilon_{\rm sc}$. The asymmetry in the ν_e, $\bar{\nu}_e$ flux is approximately given by the ratio of the drift flux and the diffusive flux, of order $\epsilon_{\rm abs}[\kappa_0^{(\rm abs)}/\kappa_0^{(\rm tot)}]$. Averaging over all neutrino species, we find the total asymmetry in the neutrino flux, $\alpha \sim 0.2\epsilon_{\rm abs}$. To generate a kick of a few hundreds per second would require a dipole field of order $10^{15}-10^{16}$ G.

Since the asymmetric neutrino flux depends crucially on the deviation of the neutrino distribution function from thermal equilibrium, it is of interest to consider how the function $g = f_\nu - f_\nu^{(0)}$ (or I_{ij}) scales with the depth of the star measured from the surface. Without a magnetic field, we expect g to decrease exponentially toward zero below the decoupling layer (which is close to the neutrinosphere for ν_e and $\bar{\nu}_e$). In the presence of asymmetric absorption and scattering opacities, this scaling may be modified. Inspecting the zeroth moment equation (6.2), we may conclude $g \sim \epsilon \mathbf{h} \cdot \hat{\mathbf{B}}$ in the deep interior of the star (recall that in radiative equilibrium one always have $\nabla \cdot \int \mathbf{h}\, dk = 0$). This effect may increase our estimate for the asymmetric flux. We hope to address some of these issues in a future paper.

ACKNOWLEDGMENTS

We thank O. Grimsrud, Y.-Z. Qian and I. Wasserman for useful discussion. This work is partly supported by a NASA ATP grant. D.L. acknowledges support from the Alfred P. Sloan Foundation.

APPENDIX A: MATRIX ELEMENT FOR ν-N SCATTERING

In the usual case in which the spin projection of the particles is not ''measured,'' one can sum the matrix element over the final spins and average over the initial spins. However, for spin 1/2 particles in a magnetic field, the initial and final state spin dependence in the matrix element *cannot* be immediately summed over since the nucleon distribution function and the energy conservation delta function both have spin dependence of the form $-s\mu_B B/T$. To obtain the differential cross section one needs to calculate the matrix element $|M_{ss'}(\mathbf{\Omega},\mathbf{\Omega}')|^2$ for initial (final) nucleon spin s (s') and initial (final) neutrino direction $\mathbf{\Omega}$ ($\mathbf{\Omega}'$). We neglect the effect of Landau levels of proton in the ν-N scattering cross section (but see Appendixes C and D).

The low energy effective Hamiltonian density for neutral current scattering of a spin-1/2 fermion with a neutrino is given by (see, e.g., Refs. [10,75])

$$\mathcal{H}_{\rm int}=\frac{G_F}{\sqrt{2}}\overline{\Psi}'_N\gamma_\mu(c_V-c_A\gamma_5)\Psi_N\overline{\Psi}'_\nu\gamma^\mu(1-\gamma_5)\Psi_\nu+\text{H.c.}, \tag{A1}$$

where neutral current vector and axial coupling constants are [10] given by[9] $c_V=-1/2$ and $c_A=-1.23/2$ for $\nu+n\rightarrow\nu+n$ and $c_V=1/2-2\sin^2\theta_W=0.035$ and $c_A=1.23/2$ for $\nu+p\rightarrow\nu+p$. Here $G_F=1.166\times10^{-5}$ GeV^{-2} is the universal Fermi constant and $\sin^2\theta_W=0.2325$ (θ_W is the Weinberg angle).

The (nonrelativistic) nucleon wave function with four-momentum $P=(m+E,\mathbf{p})\simeq(m,\mathbf{0})$ and spin four-vector $S\simeq s\hat{B}$ is given by

$$\Psi_N=V^{-1/2}U_N e^{i\mathbf{p}\cdot\mathbf{x}-iEt}, \tag{A2}$$

(where V is the normalization volume and U_N is the 4-spinor), while the neutrino wave function with four momentum $K=(k,k\mathbf{\Omega})$ is

$$\Psi_\nu=V^{-1/2}U_\nu e^{i\mathbf{k}\cdot\mathbf{x}-ikt}. \tag{A3}$$

For the antineutrino, replace $U_\nu\exp(i\mathbf{k}\cdot\mathbf{x}-ikt)$ with $V_\nu\exp(-i\mathbf{k}\cdot\mathbf{x}+ikt)$.

Plugging the wave functions into Eq. (A1), the transition rate W (S matrix squared divided by time) can be written

$$W(\{\mathbf{p},s,\mathbf{\Omega}\},\{\mathbf{p}',s',\mathbf{\Omega}'\})=\frac{1}{V^3}(2\pi)^4\delta^4(P+K-P'-K')|M_{ss'}(\mathbf{\Omega},\mathbf{\Omega}')|^2 \tag{A4}$$

where

$$|M_{ss'}(\mathbf{\Omega},\mathbf{\Omega}')|^2=\frac{1}{2}G_F^2L_{\mu\nu}N^{\mu\nu}(s,s'), \tag{A5}$$

$$\begin{aligned}L_{\mu\nu}&=\overline{U'}_\nu\gamma_\mu(1-\gamma_5)U_\nu\overline{U}_\nu\gamma_\nu(1-\gamma_5)U'_\nu\\&=\frac{1}{4kk'}\text{Tr}[\not{K}'\gamma_\mu(1-\gamma_5)\not{K}\gamma_\nu(1-\gamma_5)]\\&=\frac{2}{kk'}K'^\alpha K^\beta X_{\alpha\mu\beta\nu},\end{aligned} \tag{A6}$$

$$\begin{aligned}X_{\alpha\mu\beta\nu}&=\frac{1}{4}\text{Tr}[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu(1-\gamma_5)]\\&=g_{\alpha\mu}g_{\beta\nu}-g_{\alpha\beta}g_{\mu\nu}+g_{\alpha\nu}g_{\beta\mu}-i\epsilon_{\alpha\mu\beta\nu},\end{aligned} \tag{A7}$$

$$N^{\mu\nu}(s,s')=\overline{U'}_N\gamma^\mu(c_V-c_A\gamma_5)U_N\overline{U}_N\gamma^\nu(c_V-c_A\gamma_5)U'_N, \tag{A8}$$

[9]Raffelt and Seckel [75,76] considered the isoscalar contributions to the scattering amplitude as well as the usual isospin pieces, and suggested $c_V=-1/2$ and $c_A=-1.15/2$ for $\nu+n\rightarrow\nu+n$ and $c_V=1/2-2\sin^2\theta_W$ and $c_A=1.37/2$ for $\nu+p\rightarrow\nu+p$.

and we use the sign conventions $g_{00}=+1$, $g_{ij}=-\delta_{ij}$, and $\epsilon_{0123}=+1$. The nucleon piece can be evaluated using the spin projection operator [77] $(1+\gamma_5\gamma_3 s)/2$ and the energy projection operator $(1+\gamma_0)/2$ so that

$$\begin{aligned}N^{\mu\nu}(s,s')=&\frac{1}{4}\text{Tr}[(1+\gamma_5\gamma_3 s')(1+\gamma_0)\gamma^\mu(c_V-c_A\gamma_5)\\&\times(1+\gamma_5\gamma_3 s)(1+\gamma_0)\gamma^\nu(c_V-c_A\gamma_5)].\end{aligned} \tag{A9}$$

Explicit computation of each component gives

$$\begin{aligned}N^{00}&=\frac{1}{2}c_V^2(1+ss')\\N^{0i}&=N^{i0}=-\frac{1}{2}c_Vc_A(s+s')\delta_{i3}\\N^{ij}&=\frac{1}{2}c_A^2[\delta_{ij}(1-ss')+2ss'\delta_{i3}\delta_{j3}\\&\quad+i\epsilon_{0ij3}(s'-s)]\end{aligned} \tag{A10}$$

where $\epsilon_{\mu\nu\lambda\sigma}$ is the completely antisymmetric tensor with $\epsilon_{0123}=+1$. The remaining traces can be evaluated by standard methods [77,78] with the result

$$\begin{aligned}|M_{ss'}(\mathbf{\Omega},\mathbf{\Omega}')|^2=&\frac{1}{2}G_F^2c_V^2\{(1+3\lambda^2)+(1-\lambda^2)\mathbf{\Omega}\cdot\mathbf{\Omega}'\\&+2\lambda(\lambda+1)(s\mathbf{\Omega}+s'\mathbf{\Omega}')\cdot\hat{\mathbf{B}}-2\lambda(\lambda-1)\\&\times(s\mathbf{\Omega}'+s'\mathbf{\Omega})\cdot\hat{\mathbf{B}}+ss'[(1-\lambda^2)\\&\times(1+\mathbf{\Omega}\cdot\mathbf{\Omega}')+4\lambda^2\mathbf{\Omega}\cdot\hat{\mathbf{B}}\mathbf{\Omega}'\cdot\hat{\mathbf{B}}]\}\end{aligned} \tag{A11}$$

where we have defined $\lambda=c_A/c_V$.

Time-reversal invariance can be explicitly checked for the matrix element in Eq. (A11), or equivalently the S matrix in Eq. (A4), by simultaneously exchanging all initial and final state labels.

For antineutrinos, one would just switch $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ in Eq. (A11).

APPENDIX B: NUCLEON RESPONSE FUNCTION FOR SCATTERING

Following the procedure outlined in [10], we first use d^3p' to integrate over $\delta^3(\mathbf{p}+\mathbf{q}-\mathbf{p}')$ and then integrate over the azimuthal angle for $\mathbf{p}$, with the result

$$\begin{aligned}S_{ss'}(q_0,q)=&\frac{1}{2\pi}\int_0^\infty dp\,p^2\int_{-1}^1 d\mu\,\delta(q_0+E-E')f_N(E)\\&\times[1-f_N(E')],\end{aligned} \tag{B1}$$

where $\mu=\mathbf{p}\cdot\mathbf{q}/pq$ and $E'=-\mu_B Bs'+(\mathbf{p}+\mathbf{q})^2/2m=-\mu_B Bs'+(p^2+q^2+2pq\mu)/2m$. Care must now be taken to correctly integrate over the energy-conservation delta function with the coordinate μ; the integral is only nonzero if the argument of the delta function is zero for $\mu\equiv\mu_0\in(-1,1)$. We find

$$\mu_0=\frac{q_0-q^2/2m-\mu_B B(s-s')}{pq/m} \tag{B2}$$

so that $\mu_0^2\leqslant 1$ for

$$p^2\geqslant p_{min}^2=\left[\frac{q_0-q^2/2m-\mu_B B(s-s')}{q/m}\right]^2. \tag{B3}$$

Changing variables from p to $E=-s\mu_B B+p^2/2m$ in the remaining integral gives

$$S_{ss'}(q_0,q)=\frac{m^2}{2\pi q}\int_{E_{\min}}^{\infty}dEf_N(E)[1-f_N(E+q_0)], \tag{B4}$$

where

$$\begin{aligned}E_{\min}&=-\mu_B Bs+\frac{p_{min}^2}{2m}\\&=-\mu_B Bs+\frac{[q_0-q^2/(2m)-\mu_B B(s-s')]^2}{4\,(q^2/2m)}\end{aligned} \tag{B5}$$

is the minimum energy allowed for the initial state nucleon in order for energy and momentum conservation to be satisfied given q_0, q, s, and s'. The last integral can be done by first defining the dimensionless variables $x=(E-\mu_N)/T$, $x_{\min}=(E_{\min}-\mu_N)/T$, and $z=q_0/T$, yielding

$$\begin{aligned}S_{ss'}(q_0,q)&=\frac{m^2T}{2\pi q}\int_{x_{\min}}^{\infty}dx\left(\frac{1}{e^x+1}\right)\left(\frac{1}{1+e^{-x-z}}\right)\\&=\frac{m^2T}{2\pi q}\frac{1}{1-e^{-z}}\ln\left[\frac{1+\exp(-x_{\min})}{1+\exp(-x_{\min}-z)}\right].\end{aligned} \tag{B6}$$

This expression agrees with [10] for $B=0$ keeping in mind that our definition of S is a factor of two smaller than theirs.

Expanding $x_{\min}$ to linear order in B we find

$$\begin{aligned}x_{\min}&\simeq x_0+\delta x\\x_0&=\frac{(q_0-q^2/2m)^2}{4T(q^2/2m)}-\frac{\mu_N}{T}\\\delta x&=\frac{-\mu_B B}{2T}\left[\left(1+\frac{2mq_0}{q^2}\right)s+\left(1-\frac{2mq_0}{q^2}\right)s'\right].\end{aligned} \tag{B7}$$

For $\delta x\ll 1$, $S_{ss'}$ can be written as a sum of S_0, the zero field value, and $\delta S_{ss'}$, the correction due to the magnetic field, i.e.,

$$\begin{aligned}S_{ss'}(q_0,q)&=S_0(q_0,q)+\delta S_{ss'}(q_0,q)\\S_0(q_0,q)&=\frac{m^2T}{2\pi q}\frac{1}{1-e^{-z}}\ln\left[\frac{1+\exp(-x_0)}{1+\exp(-x_0-z)}\right]\\\delta S_{ss'}&=-\frac{m^2T}{2\pi q}\frac{\delta x}{[\exp(x_0)+1][1+\exp(-x_0-z)]}.\end{aligned} \tag{B8}$$

Note that the asymmetry in the the coefficients in Eq. (4.8) is entirely due to the $2mq_0/q^2$ terms, which first appear in Eq. (B7) as a consequence of the energy and momentum conservation delta function. Had one initially set $m\rightarrow\infty$ in $\delta(E+q_0-E')$, these terms would not have appeared.

In the limit of nondegenerate nucleons with $\mu_N/T\ll -1$ and $\exp(\mu_N/T)=(2^{1/2}\pi^{3/2}n)/(m^{3/2}T^{3/2})$, we find

$$\begin{aligned}&\frac{1}{1-e^{-z}}\ln\left[\frac{1+\exp(-x_0)}{1+\exp(-x_0-z)}\right]\\&\simeq\frac{1}{[\exp(x_0)+1][1+\exp(-x_0-z)]}\\&\simeq\exp(-x_0)=\exp\left[\frac{\mu_N}{T}-\frac{(q_0-q^2/2m)^2}{4T(q^2/2m)}\right],\end{aligned} \tag{B9}$$

which takes the form of a Gaussian in k'. The center of the Gaussian is located at $k'\simeq k$ and has a width of order $(T/m)^{1/2}k$ due to the recoil motion of the nucleons. We can further simplify this expression by defining the small quantity ϵ and the dimensionless variable u by

$$\epsilon=\left[\frac{4(1-\mu')T}{m}\right]^{1/2},\quad u=\frac{k'-k}{\epsilon k} \tag{B10}$$

so that $k'=k(1+\epsilon u)$. Then the recoil momentum is

$$q^2=2k^2(1-\mu')(1+\epsilon u)+\mathcal{O}(\epsilon k)^2 \tag{B11}$$

and

$$\begin{aligned}\frac{(q_0-q^2/2m)^2}{4T(q^2/2m)}&\simeq\frac{\epsilon^2k^2u^2\left[1+\dfrac{2k(1-\mu')}{\epsilon mu}\right]}{4T[2k^2(1-\mu')/2m](1+\epsilon u)}\\&\simeq u^2\left[1-\epsilon u+2\frac{k(1-\mu')}{\epsilon mu}\right].\end{aligned} \tag{B12}$$

The criteria for the expansion of the $B=0$ part are $T\ll m$ (so that $\epsilon\ll 1$) and $k\ll(mT)^{1/2}$. The $B\neq 0$ terms require the additional assumptions that both $\mu_B B/T\ll 1$ and $(\mu_B B/T)(\sqrt{Tm}/k)\ll 1$. To first order in ϵ we then find

$$\begin{aligned}S_0(q_0,q)=\frac{\pi^{1/2}n}{\epsilon k}\exp(-u^2)\Bigg[&1-\frac{1}{2}\epsilon u+\epsilon u^3\\&-2\frac{k(1-\mu')}{\epsilon m}u\Bigg],\end{aligned} \tag{B13}$$

and $\delta S_{ss'}=-S_0\delta x$ involves the quantity

$$\frac{2mq_0}{q^2} \simeq -\frac{\epsilon m}{k(1-\mu')}(u-\epsilon u^2). \tag{B14}$$

The range of the variable u is from $u_{min} = -1/\epsilon \ll -1$ to ∞. Since $\epsilon \ll 1$, we may extend the lower limit to $-\infty$ with only exponentially small error.

APPENDIX C: WAVE FUNCTIONS FOR THE ABSORPTION OPACITY

The electron wave functions in cylindrical coordinates (ρ,ϕ,z) are given in [79]. In the standard representation for the Dirac matrices [77], these wave functions are written as

$$\Psi_e = L^{-1/2} e^{ip_{e,z}z - iE_e t} U_e(\rho,\phi),$$

$$E_e = (m_e^2 + 2eBN_e + p_{e,z}^2)^{1/2} \tag{C1}$$

where $p_{e,z}$ is the z momentum, and $N_e = 0,1,2,\ldots$ is the Landau level index. The 4-spinor U_e is given by

$$U_e(\rho,\phi) = \left(\frac{1}{2\pi\lambda^2}\right)^{1/2} e^{i(N_e-R_e)\phi} \begin{pmatrix} C_1 I_{N_e-1,R_e}(t) e^{-i\phi} \\ iC_2 I_{N_e,R_e}(t) \\ C_3 I_{N_e-1,R_e}(t) e^{-i\phi} \\ iC_4 I_{N_e,R_e}(t) \end{pmatrix} \tag{C2}$$

with

$$C_1 = \alpha_+ A_+$$

$$C_2 = \sigma_e \alpha_- A_+ \qquad \alpha_\pm = \sqrt{\frac{1}{2}\left(1 \pm \sigma_e \frac{p_{e,z}}{|\Lambda|}\right)}$$

$$C_3 = \sigma_e \alpha_+ A_- \qquad A_\pm = \sqrt{\frac{1}{2}\left(1 \pm \frac{m_e}{E_e}\right)}$$

$$C_4 = \alpha_- A_- \tag{C3}$$

$$I_{nr}(t) = \left(\frac{r!}{n!}\right)^{1/2} e^{-t/2} t^{(n-r)/2} L_r^{(n-r)}(t), \qquad \int_0^\infty dt I_{nr}^2(t) = 1. \tag{C4}$$

Here $\lambda = (eB)^{-1/2}$ is the cyclotron radius, $t \equiv \rho^2/2\lambda^2$ (not to be confused with the symbol for time), and $V = LA$ is the normalization volume (with length L along the z axis and area A in the x-y plane, and one particle in volume V). The operators which have been simultaneously diagonalized, and their corresponding eigenvalues, are perpendicular energy $E_\perp = (eB/m_e)(N_e + 1/2)$, z momentum $p_{e,z}$, the radius of the guiding center $R_{gc} = \lambda(2R_e+1)^{1/2}$ ($R_e = 0,1,2,\ldots$) and the ''longitudinal'' spin polarization operator $\Lambda = \mathbf{\Sigma}\cdot\mathbf{\Pi} = \sigma_e(p_{e,z}^2 + 2eBN_e)^{1/2}$, where $\mathbf{\Pi} = \mathbf{p} + e\mathbf{A}$ and $\sigma_e = \pm 1$. Note that Λ reduces to helicity when $B = 0$. The coefficients C_i are found by requiring the spinors to satisfy the the Dirac equation, the normalization condition, and by diagonalizing Λ. The functions $I_{nr}(t)$ are the same as in [79], except that we have written them in terms of the Laguerre polynomials as defined in [80].

As the properties of the ground state are crucial to our results, we now describe them in some detail. The lowest energy state has $N_e = 0$. As the functions I_{-1,R_e} are not well defined, only the spin projection $\sigma_{e,0} = -\mathrm{sgn}(p_{e,z})$ is allowed for $N_e = 0$; hence the coefficient of I_{-1,R_e} will be zero. To interpret this restriction, remember that σ_e is the projection of the spin on the vector π, *not* the magnetic field. However, by expressing π in terms of raising and lowering operators one may easily show that $(\Sigma_x\pi_x + \Sigma_y\pi_y)\Psi(N_e = 0) = 0$, and hence $\Lambda\Psi(N_e = 0) = \Sigma_z\pi_z\Psi(N_e = 0) = \Sigma_z p_z\Psi(N_e = 0) = -p_{e,z}\Psi(N_e = 0)$. Hence the $N_e = 0$ state with $\sigma_e = \sigma_{e,0}$ has spin opposite to the magnetic field, as expected of the electron ground state. Since only one value of σ_e is allowed for $N_e = 0$, this state is expected to have a different form for the matrix element than the $N_e \geq 1$ states.

We assume that the protons are non-relativistic in which case the wave functions can be written as [77]

$$\Psi_p = L^{-1/2} e^{ip_{p,z}z - iE_p t} U_p(\rho,\phi), \tag{C5}$$

where

$$U_p(\rho,\phi) = \left(\frac{1}{2\pi\lambda^2}\right)^{1/2} e^{i(R_p-N_p)\phi} \begin{pmatrix} \delta_{s_p,+1} I_{R_p,N_p}(t) \\ \delta_{s_p,-1} I_{R_p,N_p}(t) \\ 0 \\ 0 \end{pmatrix}. \tag{C6}$$

The energy for the proton is $E_p = (eB/m)(N_p + 1/2) - \mu_{Bp}Bs_p + p_{p,z}^2/2m$. When there is no subscript on the mass, we have approximated $m_n \simeq m_p \equiv m$. The spin projection along the magnetic field is $s_p = \pm 1$. The non-relativistic spinors differ from the relativistic spinors in that they are completely decoupled from each other. Furthermore, the different components of the spinors correspond to different energies due to the anomalous magnetic moment (recall that in the electron case, one could arrange the spinors so that the energy depends only on N_e). This complicates the calculation of the matrix element since the proton distribution function will now depend on spin (through the energy) and the matrix element cannot be directly summed over. Last, note that the order of N and R was switched going from the electron to proton case, since the electron has angular momentum $L_z(e) = N_e - R_e$ and the proton has angular momentum $L_z(p) = R_p - N_p$ due to the sign of the charge. The ground state of the proton is the state with $N_p = 0$ and $s_p = +1$. Both $s_p = \pm 1$ are allowed for $N_p = 0$.

The neutron and neutrino wave functions are the same as those used for the scattering calculation (Appendix A).

APPENDIX D: MATRIX ELEMENT FOR ABSORPTION

The S matrix is given by

$$S_{fi}=-i\int d^4x\mathcal{H}_{int} \tag{D1}$$

where the weak interaction, low energy effective Hamiltonian is [10,75]

$$\mathcal{H}_{\rm int}=\frac{G_F}{\sqrt{2}}\overline{\Psi}_p\gamma_\mu(c_V-c_A\gamma_5)\Psi_n\overline{\Psi}_e\gamma^\mu(1-\gamma_5)\Psi_\nu+\text{H.c.} \tag{D2}$$

where the neutral current coupling constants for the absorption process are [75] $c_V=1.00$ and $c_A=1.26$ (we shall use the same notation for these coupling constants for absorption and scattering, even though their values are different).

Plugging in the wave functions from Appendix C and the Hamiltonian in Eq. (D2) we find

$$\begin{aligned}S_{fi}=&-i\frac{G_F}{\sqrt{2}}L^{-1}V^{-1}2\pi\delta(E_e+E_p-k-E_n-Q)\\&\times 2\pi\delta(p_{e,z}+p_{p,z}-k_z-p_{n,z})\\&\times\int_0^\infty d\rho\rho\int_0^{2\pi}d\phi e^{i\mathbf{w}_\perp\cdot\mathbf{x}_\perp}\overline{U}_p(\rho,\phi)\\&\times\gamma_\mu(c_V-c_A\gamma_5)U_n\overline{U}_e(\rho,\phi)\gamma^\mu(1-\gamma_5)U_\nu\end{aligned} \tag{D3}$$

where $\mathbf{w}_\perp=(p_{n,x}+k_x)\mathbf{e}_x+(p_{n,y}+k_y)\mathbf{e}_y$, and $\mathbf{x}_\perp=x\mathbf{e}_x+y\mathbf{e}_y$. The transition rate (the square of S_{fi} divided by time) can be written in the form

$$\begin{aligned}W_{if}^{(\rm abs)}=&\frac{|S_{fi}|^2}{T}\\=&L^{-1}V^{-2}(2\pi)^2\delta(E_e+E_p-k-E_n-Q)\\&\times\delta(p_{e,z}+p_{p,z}-k_z-p_{n,z})|M|^2\end{aligned} \tag{D4}$$

where we have defined

$$|M|^2=\frac{G_F^2}{2}\left|\int_0^\infty d\rho\rho\int_0^{2\pi}d\phi e^{i\mathbf{w}_\perp\cdot\mathbf{x}_\perp}\overline{U}_p(\rho,\phi)\gamma_\mu(c_V-c_A\gamma_5)U_n\overline{U}_e(\rho,\phi)\gamma^\mu(1-\gamma_5)U_\nu\right|^2. \tag{D5}$$

The integrals over ρ and ϕ can be accomplished using Eqs. (4.6) and (4.7) in Ref. [79], which in our notation give

$$\begin{aligned}&\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(N_1-R_1)\phi}e^{i(R_2-N_2)\phi}e^{-i\mathbf{w}_\perp\cdot\mathbf{x}_\perp}\\&\quad=J_{(N_1-R_1)-(N_2-R_2)}(w_\perp\rho),\end{aligned} \tag{D6}$$

where $J_N(z)$ is the nth Bessel function, and

$$\begin{aligned}&\int_0^\infty\frac{d\rho\rho}{\lambda^2}I_{N_1R_1}(t)I_{R_2N_2}(t)J_{(N_1-R_1)-(N_2-R_2)}(\sqrt{2t}\lambda w_\perp)\\&\quad=(-1)^{N_2-R_2}I_{N_1N_2}(\lambda^2w_\perp^2/2)I_{R_1R_2}(\lambda^2w_\perp^2/2).\end{aligned} \tag{D7}$$

After performing these two integrals, the matrix element $|M|^2$ can be written as

$$\begin{aligned}|M|^2=&\frac{G_F^2}{2}I_{R_eR_p}^2(\lambda^2w_\perp^2/2)\\&\times|\overline{\tilde{U}}_p\gamma_\mu(c_V-c_A\gamma_5)U_n\overline{\tilde{U}}_e\gamma^\mu(1-\gamma_5)U_\nu|^2\end{aligned} \tag{D8}$$

where we have defined

$$\tilde{U}_p=\begin{pmatrix}\delta_{s_p,+1}\\\delta_{s_p,-1}\\0\\0\end{pmatrix}\quad\text{and}\quad\tilde{U}_e=\begin{pmatrix}C_1I_{N_e-1,N_p}(\lambda^2w_\perp^2/2)\\iC_2I_{N_e,N_p}(\lambda^2w_\perp^2/2)\\C_3I_{N_e-1,N_p}(\lambda^2w_\perp^2/2)\\iC_4I_{N_e,N_p}(\lambda^2w_\perp^2/2)\end{pmatrix}. \tag{D9}$$

At this point, the summation rule for the I_{ns} [Eq. (2.24) in Ref. [79]] may be used to sum over the guiding center coordinates with the result

$$\sum_{R_e=0}^{R_{max}}\sum_{R_p=0}^{R_{max}}I_{R_eR_p}^2=\sum_{R_e=0}^{R_{max}}1=A\frac{eB}{2\pi}. \tag{D10}$$

Since only the matrix element depends on σ_e, it may be directly summed over. As a result, the matrix element may be put into the form

$$\sum_{R_e=0}^{R_{max}}\sum_{R_p=0}^{R_{max}}\sum_{\sigma_e=\pm1}c(N_e,\sigma_e)|M|^2=\frac{G_F^2}{2}A\frac{eB}{2\pi}L_{\mu\nu}N^{\mu\nu} \tag{D11}$$

where the lepton piece is

$$L_{\mu\nu}=\sum_{\sigma_e=\pm 1} c(N_e,\sigma_e)\overline{\tilde{U}}_e\gamma_\mu(1-\gamma_5)U_\nu\overline{U}_\nu\gamma_\nu(1-\gamma_5)\tilde{U}_e$$
$$=\frac{1}{2k}\sum_{\sigma_e=\pm 1} c(N_e,\sigma_e)\mathrm{Tr}[\tilde{U}_e\overline{\tilde{U}}_e\gamma_\mu(1-\gamma_5)\not{K}\gamma_\nu(1-\gamma_5)] \tag{D12}$$

and the nucleon piece is

$$N^{\mu\nu}=\overline{\tilde{U}}_p\gamma^\mu(c_V-c_A\gamma_5)U_n\overline{U}_n\gamma^\nu(c_V-c_A\gamma_5)\tilde{U}_p. \tag{D13}$$

A moment of inspection shows that the form of the nucleon tensor is exactly the same as Eq. (A10) for the scattering problem if we replace the initial nucleon with the neutron and the final nucleon with the proton. The dependence on the shape of the proton wave functions is contained entirely in the I_{N_e,N_p} functions.

The lepton tensor is more complicated. In particular, notice that the $N_e=0$ electron Landau level is the only state for which there is only one polarization. Consequently, the matrix element will have quite a different structure for the electron ground Landau level and will contain the important electron contribution to the parity violation effect. The answers for $N_e=0$ and $N_e\geqslant 1$ will be given separately.

In the $N_e=0$ case, by representing $\tilde{U}_e\overline{\tilde{U}}_e$ in terms of gamma matrices, performing the traces, and then summing against $N^{\mu\nu}$, we find

$$\begin{aligned}L_{\mu\nu}N^{\mu\nu}(N_e=0)=&\Theta(p_{e,z})I^2_{0,N_p}(w_\perp^2/2eB)\\&\times[c_V^2+3c_A^2+(c_V^2-c_A^2)\mathbf{\Omega}\cdot\hat{\mathbf{B}}\\&+2c_A(c_A+c_V)(s_p+s_n\mathbf{\Omega}\cdot\hat{\mathbf{B}})\\&-2c_A(c_A-c_V)(s_n+s_p\mathbf{\Omega}\cdot\hat{\mathbf{B}})\\&+s_ns_p\{c_V^2-c_A^2+(c_V^2+3c_A^2)\mathbf{\Omega}\cdot\hat{\mathbf{B}}\}]\\&\rightarrow\Theta(p_{e,z})I^2_{0,N_p}(w_\perp^2/2eB)(c_V^2-c_A^2)\mathbf{\Omega}\cdot\hat{\mathbf{B}},\end{aligned} \tag{D14}$$

where we have taken the relativistic limit ($m_e\rightarrow 0$) for the electrons. For simplicity, several terms in this expression which do not affect the final result for the opacity have been discarded. First, we have thrown away terms which will only give corrections to the angle-independent piece of the opacity. Second, any terms with s_ns_p have been dropped since they will always give zero in the sum over s_n and s_p (since we are expanding the nucleon response function to linear order in the spin energies). Finally, we have dropped terms like $s_n\mathbf{\Omega}\cdot\hat{\mathbf{B}}$ and $s_p\mathbf{\Omega}\cdot\hat{\mathbf{B}}$ since the $N_e=0$ contribution is already proportional to B; these terms will yield an additional factor of B from the nucleon polarization which will be much smaller.

We shall need $L_{\mu\nu}N^{\mu\nu}(N_e=0)$ summed over all N_p. Using the summation rule for I_{NS},

$$\sum_{S=0}^{\infty} I^2_{NS}(x)=1, \tag{D15}$$

we find

$$\sum_{N_p=0}^{\infty} L_{\mu\nu}N^{\mu\nu}(N_e=0)=\Theta(p_{e,z})(c_V^2-c_A^2)\mathbf{\Omega}\cdot\hat{\mathbf{B}}, \tag{D16}$$

where Θ is the step function.

For the $N_e\geqslant 1$ case, all terms containing a factor s_ns_p, which gives zero in the sums over s_n and s_p, and corrections to the angle independent opacity will be dropped. Furthermore, we drop terms with no spin dependence which are proportional to $I_{N_e,N_p}I_{N_e-1,N_p}$. Since these terms have no spin dependence, they cannot couple to the nucleon polarization terms in the response function (see Appendix E), and hence they can be evaluated to lowest order in the inelasticity. In the sum over N_p, $I_{N_e,N_p}I_{N_e-1,N_p}$ will then give zero. For relativistic electrons we then find

$$\begin{aligned}L_{\mu\nu}N^{\mu\nu}(N_e\geqslant 1)=&\frac{1}{2}\left[I^2_{N_e-1,N_p}(w_\perp^2/2eB)\left(1-\frac{p_{e,z}}{|\Lambda|}\right)+I^2_{N_e,N_p}(w_\perp^2/2eB)\left(1+\frac{p_{e,z}}{|\Lambda|}\right)\right]\\&\times[c_V^2+3c_A^2+2c_A(c_A+c_V)s_n\mathbf{\Omega}\cdot\hat{\mathbf{B}}-2c_A(c_A-c_V)s_p\mathbf{\Omega}\cdot\hat{\mathbf{B}}]\\&+\frac{1}{2}\left[-I^2_{N_e-1,N_p}(w_\perp^2/2eB)\left(1-\frac{p_{e,z}}{|\Lambda|}\right)+I^2_{N_e,N_p}(w_\perp^2/2eB)\left(1+\frac{p_{e,z}}{|\Lambda|}\right)\right](c_V^2-c_A^2)\mathbf{\Omega}\cdot\hat{\mathbf{B}}.\end{aligned} \tag{D17}$$

Summing this expression over all N_p gives

$$\sum_{N_p=0}^{\infty} L_{\mu\nu}N^{\mu\nu}(N_e\geqslant 1) = c_V^2+3c_A^2+2c_A(c_A+c_V)s_n\mathbf{\Omega}\cdot\hat{\mathbf{B}} - 2c_A(c_A-c_V)s_p\mathbf{\Omega}\cdot\hat{\mathbf{B}}+\frac{p_{e,z}}{|\Lambda|}(c_V^2-c_A^2)\mathbf{\Omega}\cdot\hat{\mathbf{B}}. \tag{D18}$$

In addition, it will be necessary to sum the matrix element against nucleon recoil terms from the response function which contains

$$q_\perp^2=eB(2N_p+1)-p_{n,\perp}^2. \tag{D19}$$

The needed summation rule is [79]

$$\sum_{S=0}^{\infty} SI_{NS}^2(x)=N+x. \tag{D20}$$

In our case this gives

$$\sum_{N_p=0}^{\infty} N_p I_{N_eN_p}^2(w_\perp^2/2eB)=N_e+w_\perp^2/2eB, \tag{D21}$$

so that

$$\sum_{N_p=0}^{\infty} q_\perp^2 I_{N_eN_p}^2(w_\perp^2/2eB)=2eB(N_e+w_\perp^2/2eB)+eB-p_{n,\perp}^2 = eB(2N_e+1)+k_\perp^2+2\mathbf{k}_\perp\cdot\mathbf{p}_{n,\perp}. \tag{D22}$$

Note that the large neutron momentum terms ($p_{n,\perp}^2$) cancelled so that the ''averaged'' recoil momentum has the expected size. The final result needed is then

$$\sum_{N_p=0}^{\infty} q_\perp^2 L_{\mu\nu}N^{\mu\nu}(N_e\geqslant 1) = \left[2eBN_e+k_\perp^2+2\mathbf{k}_\perp\cdot\mathbf{p}_{n,\perp}+eB\frac{p_{e,z}}{|\Lambda|}\right][c_V^2+3c_A^2 + 2c_A(c_A+c_V)s_n\mathbf{\Omega}\cdot\hat{\mathbf{B}}-2c_A(c_A-c_V)s_p\mathbf{\Omega}\cdot\hat{\mathbf{B}}] + \left[eB+(2eBN_e+k_\perp^2+2\mathbf{k}_\perp\cdot\mathbf{p}_{n,\perp})\frac{p_{e,z}}{|\Lambda|}\right] \times(c_V^2-c_A^2)\mathbf{\Omega}\cdot\hat{\mathbf{B}}. \tag{D23}$$

APPENDIX E: NUCLEON RESPONSE FUNCTION FOR ABSORPTION

The response function for absorption is defined in Eq. (5.9). Using $p_{p,z}$ to integrate over the z-momentum delta function gives $p_{p,z}=p_{n,z}+k_z-p_{e,z}$. Using $p_{n,z}$ to integrate over the energy delta function gives

$$p_{n,z}=\frac{q_0+Q-q^2/2m+(\mu_{Bp}s_p-\mu_{Bn}s_n)B}{q_z/m}, \tag{E1}$$

where we have defined the energy transfer by $q_0\equiv k-E_e$ and the momentum transfer by $q_z\equiv k_z-p_{e,z}$, $q_\perp^2\equiv eB(2N_p+1)-p_{n,\perp}^2$, and $q^2\equiv q_\perp^2+q_z^2$. The result is

$$S_{s_ns_p}=\frac{m}{|q_z|}f_n(E_n)[1-f_p(E_p)], \tag{E2}$$

where the neutron energy is

$$E_n=\frac{p_{n,\perp}^2}{2m}+\frac{p_{n,z}^2}{2m}-\mu_{Bn}Bs_n=\frac{p_{n,\perp}^2}{2m} +\frac{[q_0+Q-q^2/2m+(\mu_{Bp}s_p-\mu_{Bn}s_n)B]^2}{4(q_z^2/2m)}-\mu_{Bn}s_nB, \tag{E3}$$

and the proton energy is $E_p=Q+E_n+q_0$. Defining dimensionless parameters y, z via

$$y\equiv\frac{E_n-\mu_n}{T}=-\frac{\mu_n}{T}+\frac{p_{n,\perp}^2}{2mT} +\frac{[q_0+Q-q^2/2m+(\mu_{Bp}s_p-\mu_{Bn}s_n)B]^2}{4T(q_z^2/2m)}-\frac{\mu_{Bn}s_nB}{T} \tag{E4}$$

and

$$z=\frac{\mu_n-\mu_p+q_0+Q}{T}, \tag{E5}$$

the response function can be written as

$$S_{s_ns_p}=\frac{m}{|q_z|}\left(\frac{1}{e^y+1}\right)\left(\frac{1}{1+e^{-y-z}}\right). \tag{E6}$$

The above expression is exact. We now consider the regime where the nucleons are nondegenerate. This is valid for the outer layers of the neutron star where asymmetric flux can develop. Since the nucleon spin energies are small, we shall expand $S_{s_ns_p}$ to first order in μ_BB. Using the nondegenerate nucleon conditions $f_p\ll 1$ and $e^{-y}\ll 1$, we find

$$S_{s_ns_p}=\frac{m}{|q_z|}e^{-y_0}(1-\delta y), \tag{E7}$$

where $y\simeq y_0+\delta y$, and

$$y_0=-\frac{\mu_n}{T}+\frac{p_{n,\perp}^2}{2mT}+\frac{(q_0+Q-q^2/2m)^2}{4T(q_z^2/2m)}$$

$$\delta y=-\frac{\mu_{Bn}s_nB}{2T}\left(1+\frac{q_0+Q-q_\perp^2/2m}{q_z^2/2m}\right)-\frac{\mu_{Bp}s_pB}{2T}\left(1-\frac{q_0+Q-q_\perp^2/2m}{q_z^2/2m}\right). \tag{E8}$$

The δy term now contains all the dependence on the nucleon spins. This term will give rise to the nucleon contribution to the asymmetric parity violation effect.

In evaluating the absorption opacity [Eq. (5.8)], it will be necessary to expand $S_{s_ns_p}$ for small "inelasticity." If the nucleon mass were infinite, energy conservation would give exactly $E_e=k+Q$. Thus we expect $S_{s_ns_p}$ to be sharply peaked about this electron energy, with a width proportional to $(T/m)^{1/2}$. We can expand the electron energy around the peak in a series in the small parameter $(T/m)^{1/2}$. There are two cases to consider: for the electron ground state ($N_e=0$), we shall want to define the dimensionless electron energy in terms of $p_{e,z}$, but for the case in which we are summing over a continuum of electron Landau levels it will be more convenient to define the dimensionless electron energy in terms of the perpendicular momentum $p_{e,\perp}^2\equiv 2eBN_e$.

In the $N_e=0$ case, $E_e\simeq|p_{e,z}|$ (neglecting m_e), we define the dimensionless electron energy u by

$$p_{e,z}=\pm(k+Q)(1+\epsilon u), \tag{E9}$$

where

$$\epsilon=\left(\frac{2T}{m}\right)^{1/2}\frac{|q_{z,0}|}{k+Q},\quad q_{z,0}=k_z\mp(k+Q) \tag{E10}$$

(since $\epsilon\ll 1$, we can set $|1+\epsilon u|=1+\epsilon u$ over the interesting range of u). As discussed in Sec. V B, we will only need the dominant term for the $N_e=0$ response function. Thus we can drop the nucleon polarization terms in δy and work to the lowest order in inelasticity. With these approximations, we find

$$S_{s_ns_p}=\frac{m}{|q_{z,0}|}\exp\left(\frac{\mu_n}{T}-\frac{p_{n,\perp}^2}{2mT}-u^2\right)\quad (N_e=0\ \text{state}), \tag{E11}$$

which takes the form of a simple Gaussian in u. Since $\epsilon\ll 1$, we can consider the range of u to extend from $-\infty$ to ∞ in the phase space integrals with exponentially small error.

When summing over N_e in Eq. (5.8), it will be more convenient to define u in terms of $p_{e,\perp}^2=2eBN_e$. Let

$$p_{e,\perp}^2=2eBN_e=E_\perp^2(1+\epsilon u), \tag{E12}$$

where

$$E_\perp^2=(k+Q)^2-p_{e,z}^2,\quad \epsilon=\left(\frac{8T}{m}\right)^{1/2}\frac{|q_z|(k+Q)}{E_\perp^2}. \tag{E13}$$

The allowed range of $p_{e,z}$ is now $p_{e,z}\in[-(k+Q),(k+Q)]$ in order to keep $E_\perp^2$ a positive number. The electron energy is

$$E_e\simeq(p_{e,\perp}^2+p_{e,z}^2)^{1/2}\simeq(k+Q)\left[1+\frac{\epsilon uE_\perp^2}{2(k+Q)^2}-\frac{\epsilon^2u^2E_\perp^4}{8(k+Q)^4}\right]. \tag{E14}$$

The expressions needed for Eq. (E7) are then

$$\exp(-y_0)\simeq\exp\left(\frac{\mu_n}{T}-\frac{p_{n,\perp}^2}{2mT}-u^2\right)\left[1+\frac{\epsilon u^3E_\perp^2}{2(k+Q)^2}-\frac{2u(k+Q)q^2}{\epsilon mE_\perp^2}\right] \tag{E15}$$

and

$$\delta y=-\frac{\mu_{Bn}s_nB}{2T}\left[1-\frac{q_\perp^2}{q_z^2}-\frac{\epsilon umE_\perp^2}{(k+Q)q_z^2}\left(1-\frac{\epsilon uE_\perp^2}{4(k+Q)^2}\right)\right]-\frac{\mu_{Bp}s_pB}{2T}\left[1+\frac{q_\perp^2}{q_z^2}+\frac{\epsilon umE_\perp^2}{(k+Q)q_z^2}\left(1-\frac{\epsilon uE_\perp^2}{4(k+Q)^2}\right)\right]. \tag{E16}$$

Note that, as in the scattering case, there is a term in δy with a coefficient which scales as

$$\frac{\epsilon mE_\perp^2}{(k+Q)q_z^2}\simeq\frac{(mT)^{1/2}}{|q_z|}\sim\left(\frac{m}{T}\right)^{1/2} \tag{E17}$$

for $k\sim T$, and this term can be much greater than unity.

APPENDIX F: REPLACING THE SUMS OVER N_e WITH INTEGRALS

In this appendix, we turn the sum over N_e in Eq. (5.8) into an integral. Let the sum be called

$$\mathcal{I}=\sum_{N_e=1}^{\infty}F(N_e) \tag{F1}$$

where the function F is given by

$$F(N_e)=(1-f_e)\sum_{N_p=0}^{\infty}\int\frac{d^2p_{n,\perp}}{(2\pi)^2}\sum_{s_n,s_p=\pm1}S_{s_ns_p}L_{\mu\nu}N^{\mu\nu}. \tag{F2}$$

Integrating the identity (see Ref. [81])

$$\sum_{n=-\infty}^{\infty}\delta(x-n)=\sum_{k=-\infty}^{\infty}\exp(2\pi ikx) \tag{F3}$$

against $F(x)$ over the region $x\in[1,\infty)$ gives the result

$$\sum_{N_e=1}^{\infty} F(N_e) = -\frac{1}{2}F(1) + \int_1^{\infty} dN_e F(N_e) + \text{oscillatory terms.} \tag{F4}$$

As discussed at the beginning of Sec. V, we shall ignore the oscillatory terms.

The $F(1)$ terms will give expressions smaller than the integral over N_e for both Eqs. (5.23) and (5.25). The reason is that the integral over N_e effectively divides by eB, so that the factor of eB is Eq. (5.8) is cancelled. For Eq. (5.24), the $F(1)$ term is odd in $p_{e,z}$ so that it integrates to zero (to lowest order in inelasticity) in the $p_{e,z}$ integral. The corrections involving inelasticity will make this term smaller by a factor of T/m than the $N_e=0$ term. Our result is then

$$\mathcal{I} = \int_1^{\infty} dN_e F(N_e). \tag{F5}$$

When changing the variable of integration from N_e to u using the expression $2eBN_e = E_\perp^2(1+\epsilon u)$ in Eq. (5.26), we can let u range from $-\infty$ to ∞ since $\epsilon \sim (T/m)^{1/2} \ll 1$.

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