Ginsparg-Wilson-Lüscher symmetry and ultralocality

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Important recent discoveries suggest that Ginsparg-Wilson-Lüscher (GWL) symmetry has analogous dynamical consequences for the theory on the lattice as chiral symmetry does in the continuum. While it is well known that an inherent property of lattice chiral symmetry is fermion doubling, we show here that an inherent property of GWL symmetry is that the infinitesimal symmetry transformation couples fermionic degrees of freedom at arbitrarily large lattice distances (non-ultralocality). The consequences of this result for the ultralocality of symmetric actions are discussed. [S0556-2821(99)05513-7]

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I. INTRODUCTION

One of the outstanding problems in theoretical particle physics is the question of a nonperturbative definition of the full standard model. Following Wilson's work on the renormalization group in late 1960s and early 1970s, it became an accepted practice to think of continuum field theory as a scaling limit of the appropriate model defined on the space-time lattice. Quite naturally, then, this approach became a primary candidate for achieving the goal of defining the theoretical framework of particle physics nonperturbatively.

However, lattice field theory became a useful tool in this respect only to the extent it was able to reflect the important symmetries encoded in the standard model. From the standpoint of principle, the only requirement for the lattice-regularized theory is that it possesses a critical point with the continuum limit, corresponding to the target field theory. While the presence of a particular symmetry of the target theory at the lattice level is not strictly required, it is desirable because it makes the lattice theory resemble its target more before the continuum limit is actually taken. Thus the fact that Wilson's formulation of lattice gauge theories [1] accommodates local gauge invariance exactly is arguably the single most important reason why the lattice approach took off in the context of high-energy physics.

Including gauge invariance on the lattice marked a nonperturbative formulation of QCD with proper gauge dynamics. However, at the same time, the persistent failure of accommodating chiral symmetries without fermion doubling kept lattice QCD severly impaired from both theoretical and practical points of view, and a lattice definition of the electroweak sector was not possible at all. Furthermore, there were serious reasons to believe that this is actually unavoidable [2].

Sufficiently new ideas with the potential of ending the ''chirally blind'' period in lattice field theory only appeared in the early 1990s. Starting with the influential paper of Kaplan [3], subsequent developments were variations on the idea that by assigning to every light degree of freedom additional heavy ones in an appropriate manner, it might be possible to enforce chiral dynamics on the low-energy lattice theory without a doubling of fermionic species. It became soon clear that to achieve strict chirality, the number of auxiliary degrees of freedom per single light one must be infinite. In this respect, *domain wall fermions* [4] represent a formulation with a finite total number of degrees of freedom, wherein the violations of chiral symmetry are viewed as a ''finite-volume effect.'' The auxiliary degrees of freedom are realized by an ''extra dimension,'' and the chiral limit at a fixed number of light degrees of freedom is achieved as the extension of the extra dimension becomes large. The domain wall fermion setup is quite natural for vectorlike theory such as lattice QCD, but its use for chiral gauge theories is not quite clear. Nevertheless, the variation on this approach proposed in [5] might represent a valid regularization of the standard model.

The *overlap formalism* [6] attempts to fully respect the infinity of additional degrees of freedom. Their effect is ''summed up'' into the overlap of ground states of the auxiliary finite many-body Hamiltonians. This setup is more flexible with respect to including chiral gauge theories than domain wall fermions, and it may represent a general way of defining these theories nonperturbatively. For the vectorlike case, Neuberger was able to express the fermionic partition function given by the overlap formula as the determinant of the new lattice Dirac operator (Neuberger operator) [7]. Thus, for vectorlike theory, the overlap prescription including auxiliary Hamiltonians can be turned into a standard fermionic path integral expression with a particular choice of lattice Dirac kernel.

Almost in parallel with the above developments, there has been a significant activity on developing further the old idea of perfect action for QCD [8]. Even though defined on the lattice, such action should be continuum-like in all dynamical respects, including the dynamical consequences of chiral symmetry [9]. What this formally implies for the perfect action is somewhat unclear, but as noted first by Hasenfratz [8], for fixed-point action (classically perfect action) the answer to that question was indirectly given long ago by Ginsparg and Wilson [10]. In particular, using renormalization group arguments, Ginsparg and Wilson suggested that the correct chiral dynamics can be ensured on the lattice by imposing the Ginsparg-Wilson (GW) relation for the lattice Dirac kernel, and Hasenfratz has shown that this condition is satisfied by the doubler-free fixed-point action.

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However, ''perfectness'' is not necessary for the GW relation to be satisfied. Indeed, in an interesting turn of events, Neuberger has shown that his lattice Dirac operator also represents an acceptable solution [11]. It thus turns out that the overlap and domain walls share with the fixed-point action the property of building in the Ginsparg-Wilson lattice chiral dynamics.

Lüscher put these intriguing developments on more solid formal (and also aesthetic) ground by identifying a symmetry principle behind the GW relation [12]. He proposed a modified chiral transformation of lattice fermionic variables, such that invariance with respect to this transformation is equivalent to imposing a GW relation. This meant that standard field-theoretical language and methods could suddenly be used to deal with chirality on the lattice. While the domain walls and overlap formalism seem rather mysterious and unnatural to many workers in the field, the new developments can be sumed up by saying that, instead of standard chiral symmetry, we need to demand Ginsparg-Wilson-Lüscher (GWL) symmetry and to study its field-theoretical consequences. The crucial element here is the fact that, while GWL symmetry ensures appropriate continuum-like chiral dynamics [10,13,14], fermion doubling is not a necessity. As expected and hoped for, it now appears that [at least U(1)] chiral gauge theories can also be constructed based on the fermionic actions with GWL symmetry [15].

The importance of the above formal developments also lies in the fact that we can now talk in general about the set of actions with GWL symmetry (GW actions), to study their common properties, to identify additional characteristics that could usefully differentiate between them, to identify new explicit solutions, and so on. It is possible that ultimately it will turn out that using domain wall fermions, the Neuberger operator, or some truncated perfect action will be the most practical way to include chiral dynamics in lattice QCD. Nevertheless, the field-theoretical language of GWL symmetry is very appealing and these are virtually unexplored territories with high potential for a surprising result.

In this paper we will study generalized version of the original Lüscher transformations [12] in the context of lattice Dirac kernels that are local, respect symmetries of the hypercubic lattice, are gauge invariant, and possess the correct classical continuum limit. An unconventional feature of Lüscher transformations is that their nature depends on the dynamics governing the fermionic theory under consideration. We show that if the dynamics is invariant, then the infinitesimal symmetry operation requires a rearrangement of infinitely many degrees of freedom for every fermionic variable on an unrestricted lattice. Stated equivalently, the transformation couples fermionic variables at arbitrarily large lattice distances (nonultralocality). This means that ensuring GWL symmetry requires a delicate collective process involving the cooperation of many (perhaps all) fermionic degrees of freedom contained in the system.

Note that this is the same kind of qualitative feature that is present when we enforce chiral dynamics through domain walls in extra dimensions. When the infinity of additional degrees of freedom that helped to arrange for chirality are integrated out and the Neuberger operator arises, that operator (and Lüscher symmetry transformation) couples variables at arbitrarily large lattice distances. Our result shows that this is an always-present property of GWL symmetry in the context of acceptable lattice Dirac operators.

The above conclusion has important implications for GW actions themselves. In particular, it implies non-ultralocality for the subset of GW operators, specified in Ref. [17] (see also footnote 3). While this subset is very relevant for practical purposes, the statement is most likely true in the most general case as well if one insists explicitly that the theory be doubler-free. From this point of view, we can refer to the theorem on the absence of ultralocal symmetry transformations presented here as the *weak theorem on nonultralocality*. The hypothesis about strict absence of ultralocal doubler-free GW actions (*strong theorem on nonultralocality*) still awaits its proof. These issues will be discussed in a separate subsection.

II. GENERALIZED LÜSCHER TRANSFORMATIONS

Our main interest in this paper is to study infinitesimal linear transformations of the type first proposed by Lüscher [12].

A. General algebraic structure

Consider a d-dimensional hypercubic lattice (finite or infinite), where d is an even integer. Let $\psi, \overline{\psi}^T$ be vectors of fermionic variables residing on lattice sites with the usual spin-gauge-flavor structure. Let further $\mathbf{D}$,$\mathbf{R}$ be arbitrary matrices acting in the corresponding linear space. To every such pair ($\mathbf{D}$,$\mathbf{R}$) we assign a one-parameter family of infinitesimal transformations,

$$\psi \rightarrow \psi + i\,\theta\gamma_5(\mathbb{I}-\mathbf{RD})\psi, \quad \overline{\psi}\rightarrow\overline{\psi}+\overline{\psi}i\,\theta(\mathbb{I}-\mathbf{DR})\gamma_5, \quad (1)$$

and call them generalized Lüscher transformations. They were considered, for example, in Ref. [16] for the case when $\mathbf{R}$ is trivial in spinor space and Hermitian. Here we will not make such a restriction.

An interesting subset of generalized Lüscher transformations is represented by those pairs ($\mathbf{D}$,$\mathbf{R}$) for which the transformation does not change the expression $\overline{\psi}\mathbf{D}\psi$ (''fermionic action''). The change is given by

$$\delta(\overline{\psi}\mathbf{D}\psi)=\overline{\psi}\mathbf{D}\,\delta\psi+\delta\overline{\psi}\mathbf{D}\,\psi=i\,\theta\overline{\psi}(\{\mathbf{D},\gamma_5\}-\mathbf{D}\{\mathbf{R},\gamma_5\}\mathbf{D})\,\psi,$$

where $\{,\}$ denotes the anticommutator and vanishes only if

$$\{\mathbf{D},\gamma_5\}=\mathbf{D}\{\mathbf{R},\gamma_5\}\mathbf{D} \quad \text{or} \quad \{\mathbf{D}^{-1},\gamma_5\}=\{\mathbf{R},\gamma_5\}. \quad (2)$$

We note that the first form of condition (2) is fundamental and the second one is equivalent to it if the inverse of $\mathbf{D}$ can be meaningfully defined. For such $\mathbf{D}$, it can also be written in equivalent explicit form

$$\mathbf{R}=\mathbf{D}^{-1}+\mathbf{F}, \quad \{\mathbf{F},\gamma_5\}=0, \quad (3)$$

with some arbitrary chirally symmetric $\mathbf{F}$.

B. Physically relevant restriction

We now specify three restrictions that will be used to define the subset of generalized Lüscher transformations relevant for GWL symmetry on the lattice.

(a) First of all, we assume that $\mathbf{D}$ represents some acceptable lattice Dirac operator. By ''acceptable'' we mean the following: (i) correct classical continuum limit, (ii) locality (exponential decay at large distances), (iii) invariance under symmetries of the hypercubic lattice (translations and symmetries of hypercube), and (iv) gauge invariance. We will define the corresponding concepts precisely as we will need them and denote the set of these acceptable operators as $\mathfrak{D}$. Note that we do not include the absence of doublers here, which is convenient to discuss separately.

Being composed of gauge fields, lattice Dirac operator actually represents a set of linear operators, one for every gauge configuration. We require invariance of the fermionic action in arbitrary gauge background, which results in the corresponding set of conditions (2). In this context, we will refer to them as GW relation. If $\mathbf{R}$ is trivial in spinor space, this reduces formally to the standard GW relation [10].

(b) The aim is to interpret Lüscher transformations as generalized chiral transformations. However, in view of relation (3), the corresponding symmetry of $\mathbf{D}$ neither poses a restriction on the set of acceptable operators nor is physically interesting unless further requirements are imposed on the matrix $\mathbf{R}$. Not surprisingly, the physically relevant restriction is given by the requirement that $\mathbf{R}$ be local [10,13]. The intuitive argument proceeds as follows: According to GW relation (2), $\mathbf{R}$ determines the character of the anticommutator of $\mathbf{D}^{-1}$ with γ_5. For $\mathbf{R}=0$, Lüscher transformations reduce to the usual chiral transformations and chiral symmetry requires the propagator to anticommute with γ_5. Since the inherent feature of such lattice Dirac operators is doubling [2], we have to consider a nonzero $\mathbf{R}$. If $\mathbf{R}$ decays sufficiently fast, then the propagator will anticommute with γ_5 at least at large distances, which might still result in essentially chiral dynamics. Indeed, as shown explicitly by Hasenfratz [13] in the context of the standard GW relation, this is indeed true if $\mathbf{R}$ is local. We therefore restrict ourselves to *local nonzero* $\mathbf{R}$.

In what follows, we will refer to $\mathbf{D}\in\mathfrak{D}$ for which there exist a local nonzero $\mathbf{R}$ such that the GW relation is satisfied as the operator with GWL symmetry (GW operator). To appreciate the power of this restriction, it is useful to consider the GW relation in the form (3) and to realize that $\mathbf{D}^{-1}$ is a nonlocal operator. For example, in the trivial gauge background $(U=1)$, the Fourier image of the propagator has the usual $1/p$ singularity. Such nonlocalities have to be canceled by $\mathbf{F}$ for arbitrary gauge configuration. Since $\mathbf{F}$ is chirally symmetric, this is possible if and only if the nonlocality of $\mathbf{D}^{-1}$ is entirely contained in its chirally symmetric part. This is very restrictive on $\mathbf{D}$, and physically it asserts that the chirally nonsymmetric portion of the propagator does not affect the long-distance physics at all. This is an essential property of GW operators that can be used as their alternative definition without any reference to operator $\mathbf{R}$: The set is defined by all $\mathbf{D}\in\mathfrak{D}$ such that chirally nonsymmetric part of $\mathbf{D}^{-1}$ is local in an arbitrary gauge background.

To make this explicit, we write $\mathbf{D}^{-1}$ in the relevant unique decomposition

$$\mathbf{D}^{-1}=(\mathbf{D}^{-1})_C+(\mathbf{D}^{-1})_N, \tag{4}$$

where $\{(\mathbf{D}^{-1})_C,\gamma_5\}=0$ and $[(\mathbf{D}^{-1})_N,\gamma_5]=0$. Then the above discussion requires that $\mathbf{F}$ in relation (3) be written in the form $\mathbf{F}=-(\mathbf{D}^{-1})_C+\tilde{\mathbf{F}}$, where $\tilde{\mathbf{F}}$ is an arbitrary *local* chirally symmetric matrix. Relation (3) then takes the form

$$\mathbf{R}=(\mathbf{D}^{-1})_N+\tilde{\mathbf{F}}, \quad \{\tilde{\mathbf{F}},\gamma_5\}=0 \quad (\tilde{\mathbf{F}} \text{ local}). \tag{5}$$

(c) The final restriction is motivated by noting that according to the fundamental GW relation (2), adding a chirally symmetric part to $\mathbf{R}$ has no effect on the dynamics dictated by the GWL symmetry. We will therefore not reduce the set of GW operators in any way if we only consider $\mathbf{R}$ whose chirally symmetric part is identically equal to zero, i.e.,

$$[\mathbf{R},\gamma_5]=0 \quad \text{or} \quad \mathbf{R}=\mathbf{R}_N. \tag{6}$$

Note that this restriction means setting $\tilde{\mathbf{F}}=0$ in relation (5).

In what follows, we will denote the set of all $\mathbf{R}$ that obey restrictions discussed in (b) and (c) as $\mathfrak{R}$. It is the set of nonzero local $\mathbf{R}$, satisfying (6). For $\mathbf{R}\in\mathfrak{R}$, the GW relation (2) can be written in the form

$$\{\mathbf{D},\gamma_5\}=2\mathbf{D}\mathbf{R}\gamma_5\mathbf{D} \quad \text{or} \quad \mathbf{R}=(\mathbf{D}^{-1})_N. \tag{7}$$

For future reference it is useful to assign to any $\mathbf{D}\in\mathfrak{D}$, $\mathbf{R}\in\mathfrak{R}$ an operator

$$\mathcal{D}\equiv 2\mathbf{R}\mathbf{D}, \tag{8}$$

which brings the GW relation to the canonical form

$$\{\mathcal{D},\gamma_5\}=\mathcal{D}\gamma_5\mathcal{D} \quad \text{or} \quad \tfrac{1}{2}\mathbb{I}=(\mathcal{D}^{-1})_N. \tag{9}$$

Here the first form is fundamental and the second one is equivalent to it if $\mathcal{D}^{-1}$ can be meaningfully defined.

To summarize, in this subsection we have restricted the set of pairs ($\mathbf{D}$,$\mathbf{R}$) representing generalized Lüscher transformations (1) to the subset where $\mathbf{D}\in\mathfrak{D}$, $\mathbf{R}\in\mathfrak{R}$ and the GW relation (7) is satisfied. We will denote the set of such transformations as $\mathfrak{T}$. By construction, set $\mathfrak{T}$ contains transformations physically relevant to the situation when GWL symmetry is present in the theory defined by an acceptable lattice Dirac operator.

C. Statement of the main result

The main result of this paper can be expressed by the following statement:

Transformations contained in $\mathfrak{T}$ couple infinitely many fermionic degrees of freedom on the infinite lattice. Stated equivalently, these transformations couple variables at arbitrarily large lattice distances, i.e., are non-ultralocal.

The above conclusion is based on the following considerations:

(a) Because of the form of the generalized Lüscher transformations, it is sufficient to show that the operator $\mathcal{D}$, assigned to arbitrary $(\mathbf{D},\mathbf{R})\in\mathfrak{T}$ in Eq. (8), couples infinitely many fermionic degrees of freedom.

(b) We will prove the property of $\mathcal{D}$ required in (a) *rigorously* for free fermions, i.e., for the subset of generalized Lüscher transformations, where the gauge field is set to unity and the gauge-flavor structure is ignored. The flavor structure of **D** is trivial from the start, and the gauge structure becomes so when $U=1$. GW relation (7) then enforces this also on **R** and hence $\mathcal{D}$.

(c) In gauge-invariant theory, lattice sites coupled in trivial gauge background will also be coupled in generic background. Hence, the same conclusion applies for this case too.

We stress that there are no physically interesting exceptions to the result formulated here.

III. TRANSFORMATIONS IN UNIT GAUGE BACKGROUND

In this section, we will consider the generalized Lüscher transformations for free fermions. However, we will keep all the notation of the previous section and the restriction will be implicitly understood. Since the gauge-flavor structure will be ignored, the operators considered here act on the vectors of $2^{d/2}$-component fermionic degrees of freedom residing on the sites of an infinite hypercubic Euclidean lattice in d even dimensions. Matrix **G** representing such operator can be uniquely expanded in the form

$$\mathbf{G}_{m,n}=\sum_{a=1}^{2^d}\mathbf{G}^a_{m,n}\Gamma^a, \tag{10}$$

where m,n label the lattice points, $\mathbf{G}^a$ denotes a matrix with space-time indices, and Γ^a is the element of the Clifford basis. The Clifford basis is built on gamma matrices satisfying $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu,\nu}\mathbb{I}$. For example, in four dimensions we have $\Gamma\equiv\{\mathbb{I},\gamma_\mu,\gamma_5,\gamma_5\gamma_\mu,\sigma_{\mu\nu,(\mu<\nu)}\}$, where $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$, $\sigma_{\mu\nu}\equiv(i/2)[\gamma_\mu,\gamma_\nu]$. Because of the completeness of the Clifford basis on the space of $2^{d/2}\times 2^{d/2}$ complex matrices, Eq. (10) describes arbitrary operator in question. In what follows we will refer to the operators $\mathbf{G}^a$ as the Clifford components of **G**.

A. Representation of local symmetric operators

Since locality and invariance under symmetries of the hypercubic lattice will play a crucial role in our discussion, we first define explicitly the Fourier representation for operators that satisfy these requirements. Hypercubic lattice structure is invariant under translations by an arbitrary lattice vector and under the subgroup of $O(d)$ transformations—hypercubic rotations and reflections. We refer to the former as *translation invariance* and to the latter as *hypercubic invariance*.

Definition 1. (Locality): Operator **G** is said to be local if there are positive real constants c, δ such that all its Clifford components $\mathbf{G}^a$ satisfy

$$|\mathbf{G}^a_{m,n}|<ce^{-\delta|m-n|}\quad\forall m,n.$$

Here $|m-n|$ denotes the Euclidean norm of $m-n$.

Definition 2. (Translation Invariance): Operator **G** is said to be translationally invariant if all its Clifford components $\mathbf{G}^a$ satisfy

$$\mathbf{G}^a_{m,n}=\mathbf{G}^a_{0,n-m}\equiv g^a_{n-m}\quad\forall m,n. \tag{11}$$

Definition 3. (Hypercubic Invariance): Let $\mathcal{H}$ be an element of the hypercubic group in defining representation and H the corresponding element of the representation induced on hypercubic group by spinorial representation of $O(d)$. Operator **G** is said to have hypercubic invariance if for arbitrary $\mathcal{H}$, m, n we have

$$\mathbf{G}_{n,m}=H^{-1}\mathbf{G}_{\mathcal{H}n,\mathcal{H}m}H.$$

The requirement of translation invariance and locality is equivalent to the existence of diagonal analytic Fourier images of space-time parts of **G**. In particular,

$$G^a(p)\equiv\sum_n g^a_n e^{ip\cdot n},\quad G(p)\equiv\sum_{a=1}^{2^d}G^a(p)\Gamma^a, \tag{12}$$

where functions $G^a(p)$ of lattice momenta $p\equiv(p_1,\ldots,p_d)$ are complex-valued, periodic, and analytic. Adding hypercubic symmetry as an additional constraint, we now define explicitly the Fourier representation of local symmetric operators that we will use.

Definition 4. (Set $\mathfrak{G}^{sl}$): Let $G^a(p)$, $a=1,2,\ldots,2^d$, be the complex-valued functions of real variables p_μ, and let $G(p)$ be the corresponding matrix function constructed as in Eq. (12). We say that $G(p)$ belongs to the set $\mathfrak{G}^{sl}$ if (a) every $G^a(p)$ is an anlytic function with period 2π in all p_μ and (b) for arbitrary hypercubic transformation $\mathcal{H}$ it is true identically that

$$G(p)=\sum_{a=1}^{2^d}G^a(p)\Gamma^a=\sum_{a=1}^{2^d}G^a(\mathcal{H}p)H^{-1}\Gamma^a H. \tag{13}$$

We emphasize that the set $\mathfrak{G}^{sl}$ is mathematically fully equivalent to the set of all local, translation invariant kernels **G**. We can therefore speak of $G(p)$ and **G** interchangeably, and indeed, we will frequently write $\mathbf{G}\in\mathfrak{G}^{sl}$.

We finally note that since any hypercubic transformation $\mathcal{H}$ can be decomposed into products of reflections of single axis ($\mathcal{R}_\mu$) and exchanges of two different axes ($\mathcal{X}_{\mu\nu}$), it is sufficient to require invariance under these operations. The transformation properties of all the elements of the Clifford basis are determined by the fact that γ_μ transforms as p_μ (vector). In particular,

$$R_\nu^{-1}\gamma_\mu R_\nu=\begin{cases}-\gamma_\mu & \text{if }\ \mu=\nu,\\ \gamma_\mu & \text{if }\ \mu\neq\nu,\end{cases}$$

and

$$X_{\rho\sigma}^{-1}\gamma_\mu X_{\rho\sigma}=\begin{cases}\gamma_\sigma & \text{if } \mu=\rho,\\ \gamma_\rho & \text{if } \mu=\sigma,\\ \gamma_\mu & \text{otherwise,}\end{cases}$$

where $R_\mu, X_{\mu\nu}$ are the spinorial representations of $\mathcal{R}_\mu, \mathcal{X}_{\mu\nu}$. The elements of the Clifford basis naturally split into groups with definite transformation properties, and the hypercubic symmetry thus translates into definite algebraic requirements on functions $G^a(p)$, which we will later exploit.

B. Sets $\mathfrak{D}$, $\mathfrak{R}$, and $\mathfrak{T}$

We now give the definition of fundamental sets that we introduced in Sec. II.

Definition 5. (Set $\mathfrak{D}$): Let $D(p)\in\mathfrak{G}^{sl}$ be a local symmetric operator, such that in the vicinity of $p=0$ its Clifford components $D^a(p)$ satisfy

$$D^a(p)=\begin{cases} ip_\mu+O(p^2) & \text{if } \Gamma^a=\gamma_\mu,\\ O(p^2) & \text{if } \Gamma^a\neq\gamma_\mu, \forall\mu.\end{cases} \tag{14}$$

Collection $\mathfrak{D}\subset\mathfrak{G}^{sl}$ of such elements $D(p)$ defines the set of acceptable lattice Dirac operators.

Definition 6. (Set $\mathfrak{R}$): Set $\mathfrak{R}$ consists of all nonzero local operators $\mathbf{R}$ such that condition (6) is satisfied.

Definition 7. (Set $\mathfrak{T}$): We define $\mathfrak{T}$ as the collection of pairs $(\mathbf{D},\mathbf{R})$ such that $\mathbf{D}\in\mathfrak{D}$, $\mathbf{R}\in\mathfrak{R}$, and GW relation (7) is satisfied.

The following simple auxiliary statement will be useful in what follows.

Lemma 1. If $(\mathbf{D},\mathbf{R})\in\mathfrak{T}$, then $\mathbf{R}\in\mathfrak{G}^{sl}$ and $\mathcal{D}\equiv 2\mathbf{RD}\in\mathfrak{G}^{sl}$.

Proof. Since $\mathbf{R}\in\mathfrak{R}$, it is local. GW relation (7) has to be satisfied, and from its second form it follows that $\mathbf{R}$ has to respect symmetries of $\mathbf{D}$. Hence $\mathbf{R}\in\mathfrak{G}^{sl}$ and consequently $\mathcal{D}\in\mathfrak{G}^{sl}$. ■

C. Ultralocality

We now give a precise meaning to ultralocality and to the notion that operator couples ''infinitely many degrees of freedom.'' By ultralocality we mean that the fermionic variables do not interact beyond some finite lattice distance:

Definition 8. (Ultralocality): Let $\mathcal{C}_N$ denote the set of all lattice sites contained in the hypercube of side $2N$, centered at $n=0$, i.e., $\mathcal{C}_N\equiv\{n:|n_\mu|\leq N,\mu=1,\dots,d\}$. Operator $\mathbf{G}$ is said to be ultralocal if there is a positive integer N, so that

$$\mathbf{G}^a_{m,n}=0,\quad \forall m,n:(m-n)\notin\mathcal{C}_N,\forall a.$$

When the ultralocal operator $\mathbf{G}$ acts on the vector of fermionic variables ψ, then every new $\psi'_m=\mathbf{G}_{m,n}\psi_n$ is a linear combination of finite number of variables residing in the corresponding hypercube $\mathcal{C}_N$ around point m. On the contrary, if the operator is non-ultralocal, then there exist a point m such that the ψ'_m is a combination of infinite number of old variables. When $\mathbf{G}$ is translationally invariant, this is true for arbitrary point m.

If operator $\mathbf{G}$ is translationally invariant and ultralocal, then $\mathbf{G}\in\mathfrak{G}^{sl}$ and for later reference it is useful to make explicit the following simple statement.

Lemma 2. Clifford components $G^a(p)$ coresponding to translation-invariant ultralocal operator $G(p)$ are functions with a finite number of Fourier terms.

Proof. This is a trivial consequence of ultralocality and the definition of Fourier image (12). In fact, ultralocality implies the existence of $\mathcal{C}_N$ such that in the notation of Eq. (11) we have

$$G^a(p)\equiv\sum_{n\in\mathcal{C}_N} g^a_n e^{ip\cdot n}. \tag{15}$$

■

D. Minimal periodic directions

Consider straight lines in momentum space passing through the origin. A special subset is defined by those lines for which all periodic functions $f(p)$ (periodic with 2π in all p_μ) will remain periodic when restricted to that line. Such lines run, in addition to origin, through other points p such that $p_\mu=2\pi k_\mu$, $k_\mu\in\mathbb{Z},\forall\mu$, and define so-called periodic directions in momentum space. Periodic directions are special from the point of view of hypercubic symmetry and symmetric functions simplify on them accordingly. In this subsection, we will consider the subset of periodic directions (minimal periodic directions), for which the structure of elements in $\mathfrak{G}^{sl}$ simplifies maximally when they are restricted to the corresponding lines.

Definition 9. (Restriction Δ^ρ): Let $\rho\in\{1,2,\dots,d\}$ and let $\overline{p}$ denote the restriction of the momentum variable p on the line defined through

$$\overline{p}_\mu=\begin{cases} q & \text{if } \mu=1,\dots,\rho,\\ 0 & \text{if } \mu=\rho+1,\dots,d.\end{cases}$$

Map Δ^ρ that assigns to arbitrary function $f(p)$ of d real variables a function $\overline{f}(q)$ of single real variable through

$$\Delta^\rho[f(p)]\equiv\overline{f}(q)\equiv f(\overline{p}) \tag{16}$$

will be referred to as restriction Δ^ρ.

The following auxiliary statement will be important in what follows:

Lemma 3. Let $\rho\in\{1,2,\dots,d\}$. Let further $G(p)\in\mathfrak{G}^{sl}$, and let $\overline{G}(q)$ be its restriction under Δ^ρ defined through

$$\overline{G}(q)=\sum_{a=1}^{2^d}\overline{G}^a(q)\Gamma^a,\quad \overline{G}^a(q)=\Delta^\rho[G^a(p)].$$

Then $\overline{G}(q)$ can be written in the form

$$\overline{G}(q)=X(q)\mathbb{I}+Y(q)\sum_{\mu=1}^{\rho}\gamma_\mu, \tag{17}$$

where $X(q)=X(-q)$, $Y(q)=-Y(-q)$ are analytic functions of one real variable, periodic with 2π.

Proof. Let us denote the following sets of indices for later convenience: $u\equiv\{1,2,\ldots,d\}$, $u^\rho\equiv\{1,2,\ldots,\rho\}$. It is useful to think of the Clifford basis as subdivided into nonintersecting subsets $\Gamma=\cup_j\Gamma_{(j)}$, where $\Gamma_{(j)}$, $j=0,1,\ldots,d$, contains the elements that can be written as the product of j gamma matrices. For example, $\Gamma_{(0)}=\{\mathbb{I}\}$, $\Gamma_{(1)}=\{\gamma_\mu,\mu\in u\}$, and so on. With the appropriate convention on ordering of gamma matrices in the definition of Γ^a, we can then rewrite the Clifford decomposition of $G(p)$ in the form

$$G(p)=\sum_{j=0}^{d}\sum_{\substack{\mu_1,\mu_2\cdots\mu_j\\ \mu_1<\mu_2\cdots<\mu_j}} F_{\mu_1,\mu_2,\ldots,\mu_j}(p)\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_j}, \tag{18}$$

where all $\mu_i\in u$. We will now consider contributions to $\overline{G}(q)$ originating from different subsets $\Gamma_{(j)}$.

(1) $j\geqslant 2$. Consider an arbitrary single term in decomposition (18), specified by the set of indices $\nu\equiv\{\mu_1,\mu_2,\ldots,\mu_j\}$. At least one of the following statements is true: (a) there exists an element $\mu\in\nu$, such that $\mu\notin u^\rho$; (b) there exist two elements $\mu,\nu\in\nu$ such that $\mu,\nu\in u^\rho$.

Indeed, assume that both of the above statements are false. Then, from (a) it follows that $\nu\subset u^\rho$. Since (b) is also false, this means that ν contains at most one element. This is the contradiction with the assumption that $j\geqslant 2$.

If then (a) is true for our particular ν, we can consider the reflection $\mathcal{R}_\mu$ through the corresponding axis μ. Since $R_\mu^{-1}\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_j}R_\mu=-\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_j}$, hypercubic symmetry of $G(p)$ requires $F_{\mu_1,\mu_2,\ldots,\mu_j}(\mathcal{R}_\mu p)=-F_{\mu_1,\mu_2,\ldots,\mu_j}(p)$. However, since $\mu\notin u^\rho$, the restricted variable $\overline{p}$ under Δ^ρ satisfies $\mathcal{R}_\mu\overline{p}=\overline{p}$, and hence

$$\overline{F}_{\mu_1,\mu_2,\ldots,\mu_j}(q)\equiv F_{\mu_1,\mu_2,\ldots,\mu_j}(\overline{p})=-F_{\mu_1,\mu_2,\ldots,\mu_j}(\overline{p})=0.$$

Similarly, if (b) is true, we can apply the exchange $\mathcal{X}_{\mu\nu}$ of the axes $\mu,\nu\in u^\rho$. Then, again, the Clifford element is odd, which forces this also on the corresponding function. However, $\overline{p}$ does not change under this operation and hence the restriction vanishes in this case too.

Consequently, $\overline{F}_{\mu_1,\mu_2,\ldots,\mu_j}(q)$ must vanish for any ν.

(2) $j<2$. After considerations of case (1), we can write $\overline{G}(q)$ in the form

$$\overline{G}(q)=X(q)\mathbb{I}+\sum_{\mu=1}^{d}Y_\mu(q)\gamma_\mu,$$

where $X(q),Y_\mu(q)$ are the restrictions of the corresponding Clifford elements. Invariance under reflection of the axis $\mu\notin u^\rho$ demands, however, that corresponding $Y_\mu(q)=0$. Moreover, if we exchange axes $\mu,\nu\in u^\rho$, then hypercubic symmetry implies

$$Y(q)\equiv Y_1(q)=Y_2(q)=\cdots=Y_\rho(q).$$

This gives the desired form (17) and the reflection properties of $X(q)$, $Y(q)$ follow from invariance under the product of reflections $\mathcal{R}_1\mathcal{R}_2\cdots\mathcal{R}_\rho$. Analyticity and periodicity are inherited from corresponding propereties of unrestricted operator. ■

E. Lemma

The most important ingredient in the proof of our main theorem will be the following auxiliary statement that was first formulated in Ref. [17].

Lemma 4. Let *K,L* be non-negative integers and ρ a positive real number. Consider the set $\mathcal{F}^{K,L}$ of all pairs of functions $[A(q),B(q)]$ that can be written in the forms

$$A(q)=\sum_{-L\leqslant n\leqslant K}a_n e^{iq\cdot n},\quad B(q)=\sum_{-L\leqslant n\leqslant K}b_n e^{iq\cdot n}, \tag{19}$$

where $q\in\mathbb{R}$, $n\in\mathbb{Z}$, and a_n, $b_n\in\mathbb{C}$ are such that a_K,b_K do not vanish simultaneously and a_{-L},b_{-L} do not vanish simultaneously. Further, let $\mathcal{F}_\rho^{K,L}\subset\mathcal{F}^{K,L}$ denote the set of all solutions on $\mathcal{F}^{K,L}$ of the equation

$$A(q)^2+\rho B(q)^2=1. \tag{20}$$

Then the following holds.

(a) If $K=L=0$, then $\mathcal{F}_\rho^{0,0}=\{(a_0,b_0):a_0^2+\rho b_0^2=1\}$.

(b) If $K=L>0$, then $\mathcal{F}_\rho^{K,K}=\{(A(q),B(q))\}$, such that

$$A(q)=a_{-K}e^{-iq\cdot K}+a_K e^{iq\cdot K},$$

$$B(q)=b_{-K}e^{-iq\cdot K}+b_K e^{iq\cdot K},$$

with

$$b_K\neq 0,\quad b_{-K}=\frac{1}{4\rho b_K},\quad a_K=ci\sqrt{\rho}b_K,\quad a_{-K}=\frac{c}{4i\sqrt{\rho}b_K},$$

where $\sqrt{\rho}>0$ and $c=\pm 1$.

(c) If $K\neq L$, then $\mathcal{F}_\rho^{K,L}=\emptyset$.

The usefulness of the above result lies in the fact that Eq. (20) arises as a GW condition (9) for $\mathcal{D}$ restricted by Δ^ρ. Lemma 4 provides us with classification of all solutions of this equation on the space of periodic functions with *finite* number of Fourier terms. Indeed, if both functions $A(q),B(q)$ have only strictly positive (negative) Fourier components, then the equation clearly cannot be satisfied.

All other cases are covered by lemma 4. Perhaps surprisingly, the Fourier structure of solutions of Eq. (20) is thus either very simple (essentially a single Fourier component) or very complicated (infinitely many of them).

Proof. Case (a) of constant functions $A(q),B(q)$ is obvious and so we concentrate on cases (b) and (c). Because of completeness and othogonality of the Fourier basis, Eq. (20) imposes the following set of conditions on coefficients a_n, b_n:

$$\sum_{\substack{-L\leq n\leq K\\ -L\leq k-n\leq K}} a_n a_{k-n} + \rho \sum_{\substack{-L\leq n\leq K\\ -L\leq k-n\leq K}} b_n b_{k-n} = \delta_{k,0}, \quad -2L\leq k\leq 2K. \tag{21}$$

Case (b). We split the set of equations (21) into groups that can be analyzed in sequence.

(I) $K\leq k\leq 2K$. Equations in this group involve the coefficients of nonnegative frequencies only. Starting from $k=2K$ and continuing down we have[1]

$$\begin{aligned}
0&=a_K^2 &&+\rho b_K^2,\\
0&=a_K a_{K-1} &&+\rho b_K b_{K-1},\\
0&=2a_K a_{K-2}+a_{K-1}^2 &&+\rho(2b_K b_{K-2}+b_{K-1}^2),\\
&\vdots\\
0&=2a_K a_0+2a_{K-1}a_1+\cdots+a_{K/2}^2 &&+\rho(2b_K b_0+2b_{K-1}b_1+\cdots+b_{K/2}^2).
\end{aligned} \tag{22}$$

The first equation is equivalent to

$$a_K = ci\sqrt{\rho}\, b_K, \quad \sqrt{\rho}>0, \quad c=\pm 1. \tag{23}$$

Since a_K, b_K are not simultaneously zero, it follows that they have to be both nonzero. Inserting this into the second equation of Eqs. (22) yields that also $a_{K-1}=ci\sqrt{\rho}\, b_{K-1}$.

This procedure can be repeated with the analogous result for other coefficients. Indeed, a generic equation in this sequence has the schematic form

$$2a_K a_{K-n} + f(a_{K-1},a_{K-2},\ldots,a_{K-n+1}) + \rho[2b_K b_{K-n} + f(b_{K-1},b_{K-2},\ldots,b_{K-n+1})] = 0,$$

where we have just grouped the variables conveniently. Since the relation $a_j = ci\sqrt{\rho}\, b_j$ already holds for $j=K,K-1,\ldots,K-n+1$, the variables grouped by function f will drop out of the equation and we are left with $a_{K-n}=ci\sqrt{\rho}\, b_{K-n}$ as claimed. By induction, we thus have that the set of equations (22) is equivalent to

$$a_n = ci\sqrt{\rho}\, b_n, \quad \sqrt{\rho}>0, \quad c=\pm 1, \quad n=0,1,\ldots,K. \tag{24}$$

(II) $-2K\leq k\leq -K$. We can use exactly the same reasoning for these equations as we did for group (I) and transform them into

$$a_{-n} = \bar{c}i\sqrt{\rho}\, b_{-n}, \quad \sqrt{\rho}>0, \quad \bar{c}=\pm 1, \quad n=0,1,\ldots,K. \tag{25}$$

The constants $c,\bar{c}$ are related. To see that, we examine the equation for $k=0$, namely,

$$\sum_{n=1}^{K} 2a_n a_{-n} + a_0^2 + \rho\left(\sum_{n=1}^{K} 2b_n b_{-n} + b_0^2\right) = 1.$$

Using Eqs. (24) and (25) this reduces to

$$(1-c\bar{c})2\rho\sum_{n=1}^{K} b_n b_{-n} = 1 \Rightarrow \bar{c} = -c. \tag{26}$$

Two useful implications of Eqs. (24), (25), (26) that we will use in examining the rest of the equations are

$$a_0 = b_0 = 0 \tag{27}$$

and

$$a_n a_m + \rho b_n b_m = \begin{cases} 0 & \text{if } nm\geq 0\\ 2\rho b_n b_m & \text{if } nm<0. \end{cases} \tag{28}$$

Note that if $K=1$, we have no other equations available. Equation (26) reduces to $b_1 b_{-1} = 1/4\rho$, which together with Eqs. (24), (25), and (27) implies the desired result. If $K>1$, then we have groups of equations that mix the coefficients of positive and negative frequencies.

(III) $1\leq k\leq K-1$. Because of constraints (28), only the monomials that are the products of one coefficient of positive frequency and one coefficient of negative frequency will contribute. Starting from $k=K-1$ the equations are

$$\begin{aligned}
0&=b_K b_{-1},\\
0&=b_K b_{-2}+b_{K-1}b_{-1},\\
0&=b_K b_{-3}+b_{K-1}b_{-2}+b_{K-2}b_{-1},\\
&\vdots\\
0&=b_K b_{-(K-1)}+b_{K-1}b_{-(K-2)}+\cdots+b_2 b_{-1}.
\end{aligned} \tag{29}$$

[1]For definiteness of notation, we assume implicitly that K is an even integer, but that distinction is only relevant for the notation, not the argument.

Since $b_K \neq 0$, it follows from the first equation that $b_{-1}=0$. Inserting this into the second equation, we have $b_{-2}=0$, and by trivial induction

$$b_{-n}=0=a_{-n}, \quad n=1,2,\ldots,K-1, \tag{30}$$

where we have already used the result (25).

(IV) $-K+1 \leqslant k \leqslant -1$. Analogously to group (III), this set of equations combined with result (24) is equivalent to

$$b_n=0=a_n, \quad n=1,2,\ldots,K-1. \tag{31}$$

(V) $k=0$. This is the only equation that is still available, and we have already put it in the form (26). Using Eqs. (30) and (31), this simplifies to

$$b_K b_{-K}=\frac{1}{4\rho}, \tag{32}$$

which together with Eqs. (24), (25) establishes the result (b).

Case (c). The strategy of splitting the total set of equations (21) into groups goes over to this case without any change (except for index ranges).

Assume first that $K>L$. If $L=0$, then all we have is a group (I) of equations and the equation (V). In particular, result (24) implies $a_0^2+\rho b_0^2=0$, while the equation for $k=0$ reduces to $a_0^2+\rho b_0^2=1$, thus leading to a contradiction. If $L>0$, then we will also have groups (II) and (III). However, since $L<K$, the result (30) implies that coefficients of all negative frequencies now vanish $a_{-n}=b_{-n}=0$, for $(n=1,2,\ldots,L)$. Consequently, the equation for $k=0$ again reduces to $a_0^2+\rho b_0^2=1$, which contradicts Eq. (24) and there is no solution.

For $K<L$ the same line of logic leads to the same conclusion, which completes the proof. ■

F. Theorem

The required tools are now in place to prove the following theorem.

Theorem 1. If $(\mathbf{D},\mathbf{R}) \in \mathfrak{T}$, then $\mathcal{D}=2\mathbf{RD}$ is not ultralocal.

Proof. We will proceed by contradiction. Let us therefore assume that there exist $(\mathbf{D},\mathbf{R}) \in \mathfrak{T}$ such that $\mathcal{D}$ actually is ultralocal and the following steps will lead us to contradiction.

(a) According to lemma 1, $\mathbf{R} \in \mathfrak{G}^{sl}$. Consequently, its Clifford components are analytic with well-defined Taylor series. In particular, let us for later convenience write explicitly

$$R^a(p)=\frac{r}{2}+O(p) \quad \text{if } \Gamma^a=\mathbb{I}. \tag{33}$$

(b) We now consider the restriction $\overline{\mathcal{D}}(q)=2\overline{R}(q)\overline{D}(q)$ under Δ^ρ. Taking into account lemma 3, the local properties (14) of $D(p)$, the fact that $[R,\gamma_5]=0$, and using notation of Eq. (33), we can conclude

$$\overline{\mathcal{D}}(q)=[1-A(q)]\mathbb{I}+iB(q)\sum_{\mu=1}^{\rho}\gamma_\mu, \tag{34}$$

where $A(q),B(q)$ are analytic functions periodic with 2π, such that following properties around $q=0$ are satisfied:[2]

$$A(q)=1+O(q^2), \quad B(q)=rq+O(q^2). \tag{35}$$

(c) The GW relation for $\overline{\mathcal{D}}$ given in Eq. (34) takes a simple form

$$A(q)^2+\rho B(q)^2=1, \tag{36}$$

and, according to lemma 2, ultralocality of $\mathcal{D}$ implies that $A(q),B(q)$ have Fourier series with finite number of terms.

(d) Because of (c), we can apply lemma 4 to conclude that functions $A(q),B(q)$ must either be the constants, or there is an integer $K_\rho>0$, such that

$$A(q)=a_{-K_\rho}e^{-iq\cdot K_\rho}+a_{K_\rho}e^{iq\cdot K_\rho},$$

$$B(q)=b_{-K_\rho}e^{-iq\cdot K_\rho}+b_{K_\rho}e^{iq\cdot K_\rho}.$$

The local properties (35) exclude the constants, while in the second case they dictate that the solutions are $A(q)=\cos(K_\rho q)$, $B(q)=r\sin(K_\rho q)/K_\rho$. For these functions we have

$$A^2+\rho B^2=\cos^2(K_\rho q)+\frac{r^2\rho}{K_\rho^2}\sin^2(K_\rho q). \tag{37}$$

(e) In view of Eqs. (36), (37) we have to distinguish two cases:

(i) If $r=0$, Eq. (36) cannot be identically satisfied and we already have a contradiction.

(ii) If $r\neq 0$, then to satisfy Eq. (36) we have to demand

$$r=c\frac{K_\rho}{\sqrt{\rho}}, \quad c=\pm 1, \quad \sqrt{\rho}>0.$$

This condition has to be satisfied for all $\rho \in \{1,2,\ldots,d\}$. In particular, if ρ is a square of another integer ($\rho=1$, for example), we have to conclude that r is a rational number. At the same time, if ρ is not a square of an integer ($\rho=2$, for example), we have to conclude that r is irational and we thus have a contradiction for $r\neq 0$ as well. This completes our proof. ■

The above result implies that every transformation (1) corresponding to $\mathbf{D} \in \mathfrak{D}$ with GWL symmetry couples variables at arbitrarily large lattice distances. Every transformed variable is a linear combination of infinitely many original ones. This establishes the *weak theorem on ultralocality* for GWL symmetry.

[2] Note that we are not strict about enforcing all the consequences of hypercubic symmetry because it is not necessary. For example, one can easily see that hypercubic symmetry requires the Taylor reminder in Eq. (33) be actually $O(p^2)$, and the reminder of $B(q)$ in Eq. (35) be $O(q^3)$.

G. Ultralocality of symmetric actions

Theorem 1 has the following useful immediate consequence.

Corollary 1. If $(\mathbf{D},\mathbf{R}) \in \mathfrak{T}$ and $\mathbf{R}$ is ultralocal then $\mathbf{D}$ must be non-ultralocal.

In other words, the lattice Dirac operator $\mathbf{D} \in \mathfrak{D}$, satisfying GW relation (7) with ultralocal $\mathbf{R}$ cannot be ultralocal [17].[3] This has some unfortunate drawbacks for practical use of actions in this category: It complicates perturbation theory, one loses obvious numerical advantages stemming from sparcity of the conventional operators, and the question of simulating them is nontrivial and widely open. Moreover, while locality can be ensured easily for the free case, it is usually not obvious in the presence of gauge fields if the action is not ultralocal. Studies such as [19] will probably be necessary for any individual operator that might be of interest.

We stress that no definite conclusion on ultralocality of $\mathbf{D}$ from Theorem 1 can be made if the corresponding $\mathbf{R}$ is not ultralocal. In fact, there exist elements $(\mathbf{D},\mathbf{R}) \in \mathfrak{T}$ such that $\mathbf{D}$ is ultralocal when $\mathbf{R}$ is not. For example, we can take

$$D(p) = \left(\sum_{\mu=1}^{d} \sin^2 p_\mu \right) \mathbb{I} + i \sum_{\mu=1}^{d} \sin p_\mu \gamma_\mu , \tag{38}$$

which satisfies the GW relation with $R(p) = \mathbb{I}/(1 + \Sigma_{\mu=1}^{d} \sin^2 p_\mu)$. The point is that set $\mathfrak{D}$ also contains operators with doublers, and the above example is one of them. Theorem 1 and Corollary 1 are valid regardless of whether the action is doubler-free or not. However, the hypothesis on the absence of ultralocal GW actions at the free level can only hold if operators with doublers are excluded as they should. In view of our discussion leading to GW relation (7), one could prove the *strong theorem on ultralocality* by proving the following hypothesis.

Hypothesis 1. There is no $D(p) \in \mathfrak{D}$ such that the following three requirements are satisfied simultaneously:

(a) $D(p)$ involves a finite number of Fourier terms.

(b) $(D^{-1}(p))_N$ is analytic.

(c) $(D^{-1}(p))_C$ is analytic except if $p_\mu = 0 \pmod{2\pi}, \forall \mu$.

Needless to say, it would be rather interesting to have a rigorous answer to whether the above hypothesis holds or not.

IV. CONCLUSION

The long-standing quest for incorporating chiral fermionic dynamics on the lattice properly culminated recently in the construction of a natural field-theoretical framework for studying questions related to this issue. The central building block of this framework is the notion of Lüscher transformations and corresponding GWL symmetry. While the standard chiral transformation appears to be a smooth limiting case of generalized Lüscher transformations (1), we argue here that there is a sharp discontinuity in the behavior of the two cases when the underlying fermionic dynamics exhibits the corresponding symmetry. While the chiral transformation only mixes variables on a single site, the infinitesimal GWL symmetry operation always requires a rearrangement of infinitely many degrees of freedom and couples variables at arbitrarily large lattice distances.

The above discontinuity is apparently at the heart of the fact that while fermion doubling is a definite property of chiral symmetry, it is an indefinite property of GWL symmetry. At the same time, luckily, dynamical consequences are not affected by this discontinuity. This appears to support the general picture which says that imposing a proper chiral dynamics without doubling requires a delicate cooperation of many fermionic degrees of freedom. These have to conspire to ensure that the chirally nonsymmetric part of the action does not affect the long-distance behavior of the propagator and that the would-be doublers from the chirally symmetric part become heavy.

Our discussion assumes that acceptable fermionic actions respect symmetries of the hypercubic lattice structure. This is reasonable since it guarantees the recovery of the corresponding Poincaré symmetries of Minkowski space in the continuum limit without tuning. We rely quite heavily on the consequences of hypercubic symmetry in particular, and so it would be interesting to know whether the picture changes if only the translation invariance is retained. At the free level, there indeed is a difference here for there exist ultralocal Lüscher transformations with symmetric lattice Dirac kernels. For example, in two dimensions we can consider the ultralocal operator

$$D(p) = (1 - \cos p_1 \cos p_2)\mathbb{I} + i \sin p_1 \cos p_2 \gamma_1 + i \sin p_2 \gamma_2 , \tag{39}$$

which does not respect hypercubic symmetry, satisfies GW relation (7) with $R(p) = 1/2$, and the Lüscher transformation is ultralocal. However, there is a doubler at $p = (\pi,\pi)$. Therefore, we can only hypothesize that if the requirement of hypercubic symmetry is lifted, there are no ultralocal Lüscher transformations involving doubler-free $D(p)$. It would be interesting to clarify whether that is indeed the case and also whether non-ultralocality of symmetric doubler-free actions holds.

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[3]For $\mathbf{R}$ trivial in spinor space, this result was stated in Ref. [17] as a simple extension of the canonical case by the techniques discussed there. The proof was presented for example at the VIELAT98 workshop. Shortly before this paper was ready for release, Bietenholz posted a note [18], where he uses these techniques in a similar fashion. Contrary to the original statement in that note, its revised version appears to claim the case identical to one discussed in Ref. [17] and here.

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